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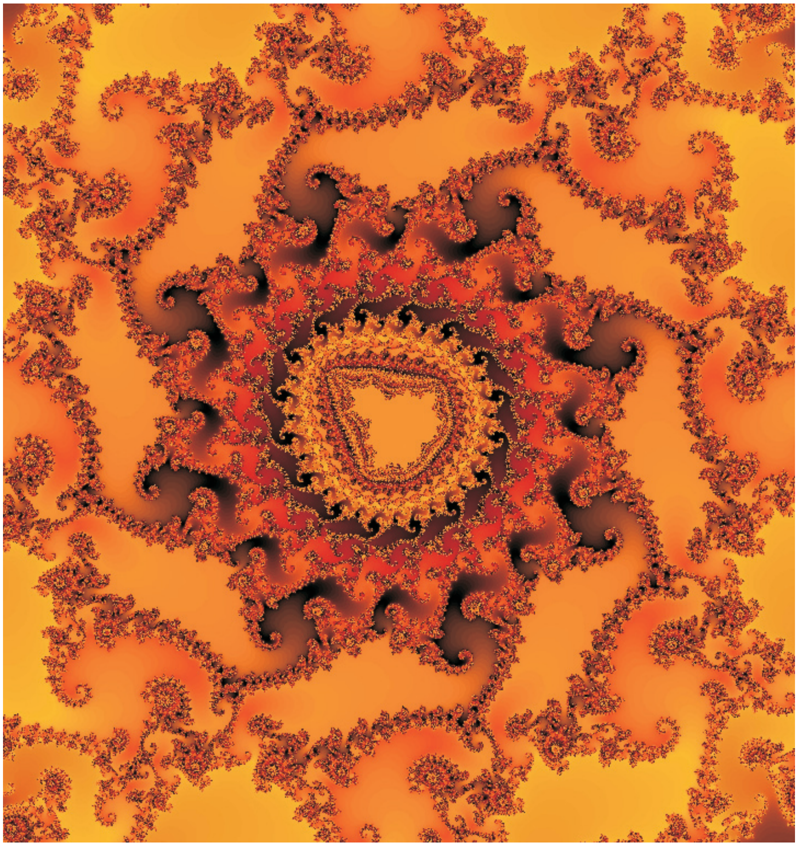

IGCSE® (0606)

O Level (4037)

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CAMBRIDGE ADDITIONAL MATHEMATICS (0606) (4037)

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FOREWORD

This book has been written to cover the '*Cambridge O Level Additional Mathematics (4037)*' and the '*Cambridge IGCSE® Additional Mathematics (0606)*' courses over a one-year period.

These syllabuses enable learners to extend the mathematics skills, knowledge, and understanding developed in the Cambridge IGCSE or O Level Mathematics courses, and use skills in the context of more advanced techniques.

The syllabuses have a Pure Mathematics only content which enables learners to acquire a suitable foundation in mathematics for further study in the subject. Knowledge of the content of the Cambridge IGCSE or O Level Mathematics syllabus (or an equivalent syllabus) is assumed.

Learners who successfully complete these courses gain lifelong skills, including:

- the further development of mathematical concepts and principles
- an ability to solve problems, present solutions logically, and interpret results.

This book is an attempt to cover, in one volume, the content outlined in the Cambridge O Level Additional Mathematics (4037) and Cambridge IGCSE Additional Mathematics (0606) syllabuses. The book can be used as a preparation for GCE Advanced Level Mathematics. The book has been endorsed by Cambridge.

To reflect the principles on which the course is based, we have attempted to produce a book and CD package that embraces understanding and problem solving in order to give students different learning experiences. Review exercises appear at the end of each chapter. Answers are given at the end of the book, followed by an index.

The interactive CD contains **Self Tutor** software (see p. 5), geometry and graphics software, demonstrations and simulations. The CD also contains the text of the book so that students can load it on a home computer and keep the textbook at school.

The examinations for Cambridge Additional Mathematics are in the form of two papers. Many of the problems in this textbook have been written to reflect the style of the examination questions. The questions, worked solutions and comments that appear in the book and CD were written by the authors.

The book can be used as a scheme of work but it is expected that the teacher will choose the order of topics. Exercises in the book range from routine practice and consolidation of basic skills, to problem solving exercises that are quite demanding.

In this changing world of mathematics education, we believe that the contextual approach shown in this book will enhance the students' understanding, knowledge and appreciation of mathematics, and its universal application.

We welcome your feedback.

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PMH, SHH, MH, CS

ABOUT THE AUTHORS

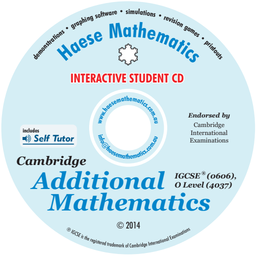
Michael Haese completed a BSc at the University of Adelaide, majoring in Infection and Immunity, and Applied Mathematics. He then completed Honours in Applied Mathematics, and a PhD in high speed fluid flows. He has a keen interest in education and a desire to see mathematics come alive in the classroom through its history and relationship with other subject areas. Michael has been the principal editor for Haese Mathematics since 2008.

Sandra Haese completed a BSc at the University of Adelaide, majoring in Pure Mathematics and Statistics. She taught mathematics at Underdale High School and later at Westminster School in Adelaide. In 1979, Sandra's husband Bob Haese began to write textbooks for mathematics students at high schools, and Sandra assumed the role of proof reader. She continues to work for Haese Mathematics as an editor and proof reader, and she produces much of the audio work for the Self Tutor software. In 2007 she was awarded Life Membership of the Mathematics Association of South Australia.

Mark Humphries completed an honours degree in Pure Mathematics and an Economics degree at the University of Adelaide. His mathematical interests include public key cryptography and number theory. He has been working at Haese Mathematics since 2006.

Chris Sangwin completed a BA in Mathematics at the University of Oxford, and an MSc and PhD in Mathematics at the University of Bath. He spent thirteen years in the Mathematics Department at the University of Birmingham, and from 2000 - 2011 was seconded half time to the UK Higher Education Academy "Maths Stats and OR Network" to promote learning and teaching of university mathematics. He was awarded a National Teaching Fellowship in 2006, and is now a Senior Lecturer in Mathematics Education in the Mathematics Education Centre at Loughborough University. His learning and teaching interests include automatic assessment of mathematics using computer algebra, and problem solving using the Moore method and similar student-centred approaches.

USING THE INTERACTIVE CD



The interactive Student CD that comes with this book is designed for those who want to utilise technology in teaching and learning Mathematics.

The CD icon that appears throughout the book denotes an active link on the CD. Simply click on the icon when running the CD to access a large range of interactive features that includes:

- printable worksheets
- graphing packages
- demonstrations
- simulations
- revision games
- SELF TUTOR

INTERACTIVE LINK

SELF TUTOR is an exciting feature of this book.

The **Self Tutor** icon on each worked example denotes an active link on the CD.

Simply 'click' on the **Self Tutor** (or anywhere in the example box) to access the worked example, with a teacher's voice explaining each step necessary to reach the answer.

Play any line as often as you like. See how the basic processes come alive using movement and colour on the screen.

Ideal for students who have missed lessons or need extra help.

Example 10

Self Tutor

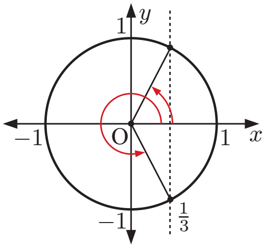
Find the two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\cos\theta = \frac{1}{3}$

b $\sin\theta = \frac{3}{4}$

c $\tan\theta = 2$

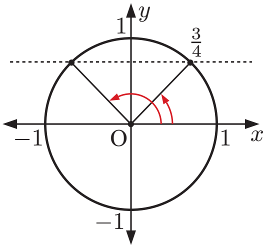
a $\cos^{-1}(\frac{1}{3}) \approx 1.23$



$\therefore \quad \theta \approx 1.23 \quad \text{or} \quad 2\pi - 1.23$

$\therefore \quad \theta \approx 1.23 \quad \text{or} \quad 5.05$

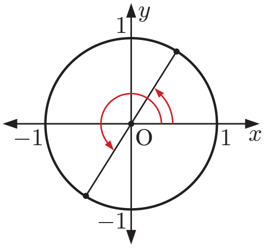
b $\sin^{-1}(\frac{3}{4}) \approx 0.848$



$\therefore \quad \theta \approx 0.848 \quad \text{or} \quad \pi - 0.848$

$\therefore \quad \theta \approx 0.848 \quad \text{or} \quad 2.29$

c $\tan^{-1}(2) \approx 1.11$



$\therefore \quad \theta \approx 1.11 \quad \text{or} \quad \pi + 1.11$

$\therefore \quad \theta \approx 1.11 \quad \text{or} \quad 4.25$

See **Chapter 8**, **The unit circle and radian measure**, page 209

SYMBOLS AND NOTATION USED IN THIS BOOK

$\mathbb{N}$	the set of natural numbers, $\{1, 2, 3, ...\}$
$\mathbb{Z}$	the set of integers, $\{0, \pm1, \pm2, \pm3, ...\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, ...\}$
$\mathbb{Q}$	the set of rational numbers
$\mathbb{Q}^+$	the set of positive rational numbers, $\{x \in \mathbb{Q},\ \ x > 0\}$
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^+$	the set of positive real numbers, $\{x \in \mathbb{R},\ \ x > 0\}$
$[a, b]$	the closed interval $\{x \in \mathbb{R} : a \leqslant x \leqslant b\}$
$[a, b)$	the interval $\{x \in \mathbb{R} : a \leqslant x < b\}$
$(a, b]$	the interval $\{x \in \mathbb{R} : a < x \leqslant b\}$
(a, b)	the open interval $\{x \in \mathbb{R} : a < x < b\}$
$\{x_1, x_2, ...\}$	the set with elements $x_1, x_2, ...$
$n(A)$	the number of elements in the finite set A
$\{x : ....$	the set of all x such that
$\in$	is an element of
$\notin$	is not an element of
$\varnothing$ or $\{\ \}$	the empty set
$\mathscr{E}$	the universal set
$\cup$	union
$\cap$	intersection
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\nsubseteq$	is not a subset of
$\not\subset$	is not a proper subset of
A'	the complement of the set A
$a^{\frac{1}{n}},\ \sqrt[n]{a}$	a to the power of $\frac{1}{n}$, nth root of a (if $a \geqslant 0$ then $\sqrt[n]{a} \geqslant 0$)
$a^{\frac{1}{2}},\ \sqrt{a}$	a to the power $\frac{1}{2}$, square root of a (if $a \geqslant 0$ then $\sqrt{a} \geqslant 0$)
$\lvert x \rvert$	the modulus or absolute value of x, that is $\begin{cases} x \text{ for } x \geqslant 0, & x \in \mathbb{R} \\ -x \text{ for } x < 0, & x \in \mathbb{R} \end{cases}$
$\equiv$	identity or is equivalent to
$\approx$	is approximately equal to
$n!$	n factorial for $n \in \mathbb{N}$ $(0! = 1)$
$\binom{n}{r}$	the binomial coefficient $\dfrac{n!}{r!(n-r)!}$ for $n, r \in \mathbb{N},\ 0 \leqslant r \leqslant n$
$>$	is greater than
$\geq$ or $\geqslant$	is greater than or equal to
$<$	is less than
$\leq$ or $\leqslant$	is less than or equal to

$\sum_{i=1}^{n} a_i$	$a_1 + a_2 + ... + a_n$
f	function f
$f : x \mapsto y$	f is a function under which x is mapped to y
$f(x)$	the image of x under the function f
f^{-1}	the inverse function of the function f
$g \circ f$, gf	the composite function of f and g
$\lim_{x \to a} f(x)$	the limit of $f(x)$ as x tends to a
$\frac{dy}{dx}$	the derivative of y with respect to x
$\frac{d^2y}{dx^2}$	the second derivative of y with respect to x
$f'(x)$	the derivative of $f(x)$ with respect to x
$f''(x)$	the second derivative of $f(x)$ with respect to x
$\int y \, dx$	the indefinite integral of y with respect to x
$\int_a^b y \, dx$	the definite integral of y with respect to x for values of x between a and b
e	base of natural logarithms
e^x	exponential function of x
$\lg x$	logarithm of x to base 10
$\ln x$	natural logarithm of x
$\log_a x$	logarithm to the base a of x
sin, cos, tan, cosec, sec, cot	the circular functions
A(x, y)	the point A in the plane with Cartesian coordinates x and y
AB	the line segment with endpoints A and B the distance from A to B the line containing points A and B
$\widehat{\mathrm{A}}$	the angle at A
$\mathrm{C\widehat{A}B}$	the angle between CA and AB
$\triangle\mathrm{ABC}$	the triangle whose vertices are A, B, and C
$\mathbf{a}$	the vector $\mathbf{a}$
$\overrightarrow{\mathrm{AB}}$	the vector represented in magnitude and direction by the directed line segment from A to B
$\lvert\mathbf{a}\rvert$	the magnitude of vector $\mathbf{a}$
$\lvert \overrightarrow{\mathrm{AB}} \rvert$	the magnitude of $\overrightarrow{\mathrm{AB}}$
$\mathbf{i}$, $\mathbf{j}$	unit vectors in the directions of the Cartesian coordinate axes
$\mathbf{M}$	a matrix $\mathbf{M}$
$\mathbf{M}^{-1}$	the inverse of the square matrix $\mathbf{M}$
$\det \mathbf{M}$	the determinant of the square matrix $\mathbf{M}$

TABLE OF CONTENTS

1

Sets and Venn diagrams

Contents:

Opening problem

A city has three football teams in the national league: A, B, and C.

In the last season, 20% of the city's population saw team A play, 24% saw team B, and 28% saw team C. Of these, 4% saw both A and B, 5% saw both A and C, and 6% saw both B and C. 1% saw all three teams play.

Things to think about:

a Writing out all of this information in sentences is very complicated. How can we represent this information more simply on a diagram?

b What percentage of the population:

  i saw only team A play
  ii saw team A *or* team B play but not team C
  iii did not see any of the teams play?

A SETS

SET NOTATION

A **set** is a collection of numbers or objects.

For example:

- the set of digits which we use to write numbers is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- if V is the set of all vowels, then $V = \{a, e, i, o, u\}$.

The numbers or objects in a set are called the **elements** or **members** of the set.

We use the symbol $\in$ to mean *is an element of* and $\notin$ to mean *is not an element of*.

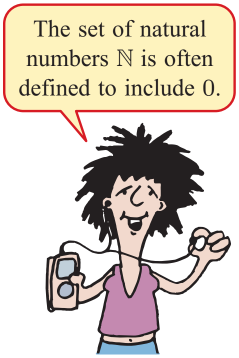
So, for the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ we can say $4 \in A$ but $9 \notin A$.

The set $\{\ \}$ or $\varnothing$ is called the **empty set** and contains no elements.

SPECIAL NUMBER SETS

The following is a list of some special number sets you should be familiar with:

- $\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, ...\}$ is the set of all **natural** or **counting numbers**.
- $\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \pm4, ...\}$ is the set of all **integers**.
- $\mathbb{Z}^+ = \{1, 2, 3, 4, 5, 6, 7, ...\}$ is the set of all **positive integers**.
- $\mathbb{Z}^- = \{-1, -2, -3, -4, -5, ...\}$ is the set of all **negative integers**.
- $\mathbb{Q}$ is the set of all **rational numbers**, or numbers which can be written in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$.
- $\mathbb{R}$ is the set of all **real numbers**, which are all numbers which can be placed on the number line.



COUNTING ELEMENTS OF SETS

The number of elements in set A is written $n(A)$.

For example, the set $A = \{2, 3, 5, 8, 13, 21\}$ has 6 elements, so we write $n(A) = 6$.

A set which has a finite number of elements is called a **finite set**.

For example: $A = \{2, 3, 5, 8, 13, 21\}$ is a finite set.
$\varnothing$ is also a finite set, since $n(\varnothing) = 0$.

Infinite sets are sets which have infinitely many elements.

For example, the set of positive integers $\{1, 2, 3, 4, ...\}$ does not have a largest element, but rather keeps on going forever. It is therefore an infinite set.

In fact, the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}^+$, $\mathbb{Z}^-$, $\mathbb{Q}$, and $\mathbb{R}$ are all infinite sets.

SUBSETS

Suppose A and B are two sets. A is a **subset** of B if every element of A is also an element of B. We write $A \subseteq B$.

For example, $\{2, 3, 5\} \subseteq \{1, 2, 3, 4, 5, 6\}$ as every element in the first set is also in the second set.

A is a **proper subset** of B if A is a subset of B but is *not equal to* B. We write $A \subset B$.

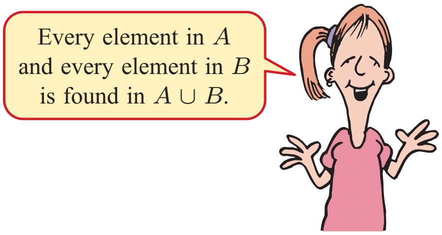
For example, $\mathbb{Z} \subset \mathbb{Q}$ since any integer $n = \frac{n}{1} \in \mathbb{Q}$. However, $\frac{1}{2} \in \mathbb{Q}$ but $\frac{1}{2} \notin \mathbb{Z}$.

We use $A \nsubseteq B$ to indicate that A is *not* a subset of B
and $A \not\subset B$ to indicate that A is *not* a proper subset of B.

UNION AND INTERSECTION

If A and B are two sets, then

- $A \cap B$ is the **intersection** of A and B, and consists of all elements which are in **both** A **and** B
- $A \cup B$ is the **union** of A and B, and consists of all elements which are in A **or** B.



For example:

- If $A = \{1, 3, 4\}$ and $B = \{2, 3, 5\}$ then $A \cap B = \{3\}$ and $A \cup B = \{1, 2, 3, 4, 5\}$.
- The set of integers is made up of the set of negative integers, zero, and the set of positive integers: $\mathbb{Z} = (\mathbb{Z}^- \cup \{0\} \cup \mathbb{Z}^+)$

DEMO

DISJOINT SETS

Two sets are **disjoint** or **mutually exclusive** if they have no elements in common.

If A and B are disjoint then $A \cap B = \varnothing$.

Example 1

Self Tutor

$M = \{2, 3, 5, 7, 8, 9\}$ and $N = \{3, 4, 6, 9, 10\}$

a True or false? **i** $4 \in M$ **ii** $6 \notin M$

b List the sets: **i** $M \cap N$ **ii** $M \cup N$

c Is **i** $M \subseteq N$ **ii** $\{9, 6, 3\} \subseteq N$?

a **i** 4 is not an element of M, so $4 \in M$ is false.

ii 6 is not an element of M, so $6 \notin M$ is true.

b **i** $M \cap N = \{3, 9\}$ since 3 and 9 are elements of both sets.

ii Every element which is in either M or N is in the union of M and N.

$\therefore \ M \cup N = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$

c **i** No. Not every element of M is an element of N.

ii Yes, as 9, 6, and 3 are also in N.

To write down $M \cup N$, start with M and add to it the elements of N which are not in M.

EXERCISE 1A

1 Write using set notation:

a 5 is an element of set D

b 6 is not an element of set G

c d is not an element of the set of all English vowels

d $\{2, 5\}$ is a subset of $\{1, 2, 3, 4, 5, 6\}$

e $\{3, 8, 6\}$ is not a subset of $\{1, 2, 3, 4, 5, 6\}$.

2 Find **i** $A \cap B$ **ii** $A \cup B$ for:

a $A = \{6, 7, 9, 11, 12\}$ and $B = \{5, 8, 10, 13, 9\}$

b $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7, 8\}$

c $A = \{1, 3, 5, 7\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

3 Suppose $A = \{0, 3, 5, 8, 14\}$ and $B = \{1, 4, 5, 8, 11, 13\}$. Write down the number of elements in:

a A **b** B **c** $A \cap B$ **d** $A \cup B$

4 True or false?

a $\mathbb{Z}^+ \subseteq \mathbb{N}$ **b** $\mathbb{N} \subset \mathbb{Z}$ **c** $\mathbb{N} = \mathbb{Z}^+$ **d** $\mathbb{Z}^- \subseteq \mathbb{Z}$

e $\mathbb{Q} \subset \mathbb{Z}$ **f** $\{0\} \subseteq \mathbb{Z}$ **g** $\mathbb{Z} \subseteq \mathbb{Q}$ **h** $\mathbb{Z}^+ \cup \mathbb{Z}^- = \mathbb{Z}$

5 Describe the following sets as either finite or infinite:

a the set of counting numbers between 10 and 20

b the set of counting numbers greater than 5

c the set of all rational numbers $\mathbb{Q}$

d the set of all rational numbers between 0 and 1.

6 True or false?

a $127 \in \mathbb{N}$ **b** $\frac{138}{279} \in \mathbb{Q}$ **c** $3\frac{1}{7} \notin \mathbb{Q}$ **d** $-\frac{4}{11} \in \mathbb{Q}$

7 Which of these pairs of sets are disjoint?

a $A = \{3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8\}$ **b** $P = \{3, 5, 6, 7, 8, 10\}$ and $Q = \{4, 9, 10\}$

8 True or false? If R and S are two non-empty sets and $R \cap S = \varnothing$, then R and S are disjoint.

9 **a** How many proper subsets does the set $\{a, b, c, d\}$ have?

b Copy and complete: "If a set has n elements then it has ...... proper subsets."

B INTERVAL NOTATION

To avoid having to list all members of a set, we often use a general description of its members. We often describe a set of all values of x with a particular property.

The notation $\{x : ......\}$ or $\{x \mid ......\}$ is used to describe "the set of all x such that ......".

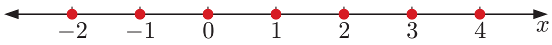
For example:

- $A = \{x \in \mathbb{Z} : -2 \leqslant x \leqslant 4\}$ reads "the set of all integers x such that x is between -2 and 4, including -2 and 4."

  such that

  the set of all

  We can represent A on a number line as:

  A is a finite set, and $n(A) = 7$.

- $B = \{x \in \mathbb{R} : -2 \leqslant x < 4\}$ reads "the set of all real x such that x is greater than or equal to -2 and less than 4."

  We represent B on a number line as:

  a filled in circle indicates −2 is included

  an open circle indicates 4 is not included

  B is an infinite set, and $n(B) = \infty$.

  We could also write $B = \{x : -2 \leqslant x < 4\}$, in which case we would *assume* that $x \in \mathbb{R}$.

Example 2

Self Tutor

Suppose $A = \{x \in \mathbb{Z} : 3 < x \leqslant 10\}$.

a Write down the meaning of the interval notation.

b List the elements of set A. **c** Find $n(A)$.

a The set of all integers x such that x is between 3 and 10, including 10.

b $A = \{4, 5, 6, 7, 8, 9, 10\}$ **c** There are 7 elements, so $n(A) = 7$.

CLOSED AND OPEN INTERVALS

An **interval** is a connected subset of the number line $\mathbb{R}$.

An interval is **closed** if *both* of its endpoints are included.

An interval is **open** if *both* of its endpoints are *not* included.

For $x \in \mathbb{R}$, we commonly use the following notation to concisely write intervals:

$[a, b]$ represents the closed interval $\{x \in \mathbb{R} : a \leqslant x \leqslant b\}$
$[a, b)$ represents the interval $\{x \in \mathbb{R} : a \leqslant x < b\}$
$(a, b]$ represents the interval $\{x \in \mathbb{R} : a < x \leqslant b\}$
(a, b) represents the open interval $\{x \in \mathbb{R} : a < x < b\}$

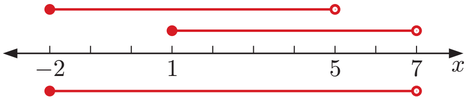
This shorter notation is not needed for the syllabus.

An interval which extends to infinity has no defined endpoint.
So, for $\{x \in \mathbb{R} : x \geqslant a\}$ we write $[a, \infty)$.

INTERVALS WHICH OVERLAP

When two intervals overlap, we consolidate them into a single interval.

For example: $[-2, 5) \cup [1, 7) = [-2, 7)$



EXERCISE 1B

1 Explain whether the following sets are finite or infinite:

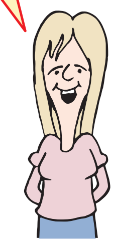
a $\{x \in \mathbb{Z} : -2 \leqslant x \leqslant 1\}$
b $\{x \in \mathbb{R} : -2 \leqslant x \leqslant 1\}$
c $\{x \in \mathbb{Z} : x \geqslant 5\}$
d $\{x \in \mathbb{Q} : 0 \leqslant x \leqslant 1\}$
e $(2, 4)$
f $[-3, 7]$
g $(-\infty, 0)$

2 For the following sets:

i Write down the meaning of the interval notation.
ii If possible, list the elements of A.
iii Find $n(A)$.
iv If possible, sketch A on a number line.

a $A = \{x \in \mathbb{Z} : -1 \leqslant x \leqslant 7\}$
b $A = \{x \in \mathbb{N} : -2 < x < 8\}$
c $A = \{x \in \mathbb{R} : 0 \leqslant x \leqslant 1\}$
d $A = \{x \in \mathbb{Q} : 5 \leqslant x \leqslant 6\}$
e $A = [-1, 5)$
f $A = \{x \in \mathbb{R} : 3 < x \leqslant 5 \ \cup \ x > 7\}$
g $A = (-\infty, 1] \cup (2, \infty)$
h $A = (-\infty, 2) \cup [1, \infty)$

In this course $0 \notin \mathbb{N}$.



3 Write in interval notation:

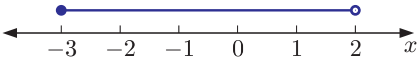
a the set of all integers between -100 and 100
b the set of all real numbers greater than 1000
c the set of all rational numbers between 2 and 3, including 2 and 3.

4 Write using interval notation:

a $\{-2, -1, 0, 1, 2, 3\}$

b $\{...., -6, -5, -4, -3\}$

c



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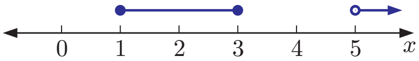
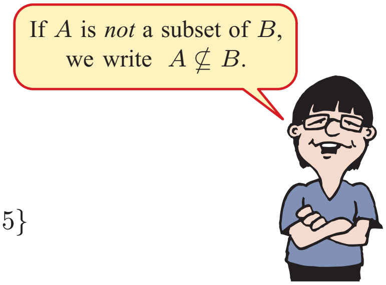


5 State whether $A \subseteq B$:



a $A = \varnothing, \quad B = \{2, 5, 7, 9\}$

b $A = \{2, 5, 8, 9\}, \quad B = \{8, 9\}$

c $A = \{x \in \mathbb{R} : 2 \leqslant x \leqslant 3\}, \quad B = \{x \in \mathbb{R}\}$

d $A = \{x \in \mathbb{Q} : 3 \leqslant x \leqslant 9\}, \quad B = \{x \in \mathbb{R} : 0 \leqslant x \leqslant 10\}$

e $A = \{x \in \mathbb{Z} : -10 \leqslant x \leqslant 10\}, \quad B = \{z \in \mathbb{Z} : 0 \leqslant z \leqslant 5\}$

f $A = \{x \in \mathbb{Q} : 0 \leqslant x \leqslant 1\}, \quad B = \{y \in \mathbb{Q} : 0 < y \leqslant 2\}$

6 For each of the following sets, determine whether the interval described is closed, open, or neither:

a $[2, 5)$ **b** $(-1, 3)$ **c** $(-\infty, -4]$ **d** $(4, \infty)$ **e** $[-2, 2]$ **f** $[0, 11)$

7 Given that $\mathbb{Q}$ is the set of rational numbers, we can define
$\mathbb{Q}^+$ as the set of positive rational numbers, $\{x \in \mathbb{Q} : x > 0\}$
and $\mathbb{Q}_0^+$ as the set of positive rational numbers and zero, $\{x \in \mathbb{Q} : x \geqslant 0\}$.

a Explain why the set $\mathbb{Q}$ cannot be illustrated on a number line.

b Describe in words, in interval notation, and using a number line, what would be meant by the set:

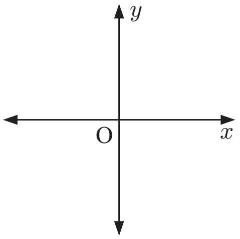
  i $\mathbb{R}^+$ **ii** $\mathbb{R}_0^+$

C RELATIONS

A **relation** is any set of points which connect two variables.

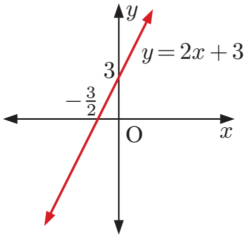
You should be familiar with points (x, y) in the Cartesian plane.

Any set of these points is a relation.

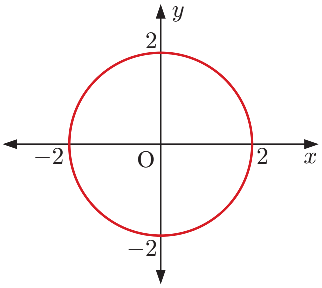


For example:

- $\{(x, y) : y = 2x + 3\}$ is the set of points which form a straight line with gradient 2 and y-intercept 3.



- $\{(x, y) : x^2 + y^2 = 4\}$ is the set of points which form a circle with radius 2 units centred at the origin.



EXERCISE 1C

GRAPHING PACKAGE

1 Illustrate the following sets in the Cartesian plane. In each case state whether the set is finite or infinite.

a $\{(x, y) : y = x\}$

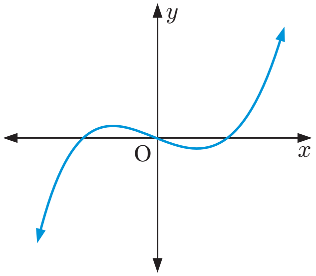
b $\{(x, y) : x + y = 1\}$

c $\{(x, y) : x > 0,\ y > 0\}$

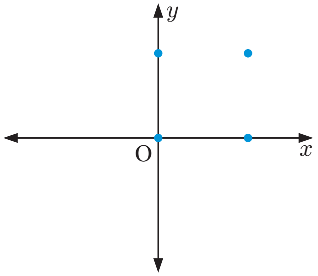
d $\{(x, y) : x + y > 1\}$

2 Let A be the set of points in each graph below. State whether A is finite or infinite.

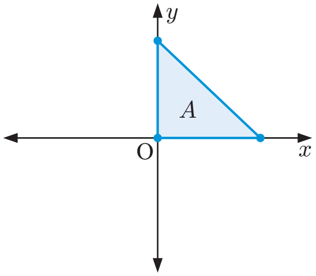
a



b



c



3 Suppose A is the set of points which define a straight line and B is the set of points which define a circle.

a Describe in words the meaning of: **i** $A \cap B$ **ii** $A \cup B$

b Describe, with illustration, what it means if $n(A \cap B)$ equals: **i** 2 **ii** 1 **iii** 0

D COMPLEMENTS OF SETS

UNIVERSAL SETS

Suppose we are only interested in the natural numbers from 1 to 20, and we want to consider subsets of this set. We say the set $\mathscr{E} = \{x \in \mathbb{N} : 1 \leqslant x \leqslant 20\}$ is the *universal set* in this situation.

The symbol $\mathscr{E}$ is used to represent the **universal set** under consideration.

COMPLEMENTARY SETS

The **complement** of A, denoted A', is the set of all elements of $\mathscr{E}$ which are *not* in A.

$$A' = \{x \in \mathscr{E} : x \notin A\}$$

For example, if the universal set $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8\}$, and the set $A = \{1, 3, 5, 7, 8\}$, then the complement of A is $A' = \{2, 4, 6\}$.

Three obvious relationships are observed connecting A and A'. These are:

- $A \cap A' = \varnothing$ as A' and A have no common members.
- $A \cup A' = \mathscr{E}$ as all elements of A and A' combined make up $\mathscr{E}$.
- $n(A) + n(A') = n(\mathscr{E})$

For example, $\mathbb{Q} \cap \mathbb{Q}' = \varnothing$ and $\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$.

Example 3

Self Tutor

Find C' given that:

a $\mathscr{E} = \{\text{all positive integers}\}$ and $C = \{\text{all even integers}\}$

b $C = \{x \in \mathbb{Z} : x \geqslant 2\}$ and $\mathscr{E} = \mathbb{Z}$

a $C' = \{\text{all odd integers}\}$ **b** $C' = \{x \in \mathbb{Z} : x \leqslant 1\}$

Example 4

Self Tutor

Suppose $\mathscr{E} = \{x \in \mathbb{Z} : -5 \leqslant x \leqslant 5\}$, $A = \{x \in \mathbb{Z} : 1 \leqslant x \leqslant 4\}$, and $B = \{x \in \mathbb{Z} : -3 \leqslant x < 2\}$. List the elements of:

a A **b** B **c** A' **d** B'

e $A \cap B$ **f** $A \cup B$ **g** $A' \cap B$ **h** $A' \cup B'$

a $A = \{1, 2, 3, 4\}$

b $B = \{-3, -2, -1, 0, 1\}$

c $A' = \{-5, -4, -3, -2, -1, 0, 5\}$

d $B' = \{-5, -4, 2, 3, 4, 5\}$

e $A \cap B = \{1\}$

f $A \cup B = \{-3, -2, -1, 0, 1, 2, 3, 4\}$

g $A' \cap B = \{-3, -2, -1, 0\}$

h $A' \cup B' = \{-5, -4, -3, -2, -1, 0, 2, 3, 4, 5\}$

EXERCISE 1D

1 Find the complement of C given that:

a $\mathscr{E} = \{\text{letters of the English alphabet}\}$ and $C = \{\text{vowels}\}$

b $\mathscr{E} = \{\text{integers}\}$ and $C = \{\text{negative integers}\}$

c $\mathscr{E} = \mathbb{Z}$ and $C = \{x \in \mathbb{Z} : x \leqslant -5\}$

d $\mathscr{E} = \mathbb{Q}$ and $C = \{x \in \mathbb{Q} : x \leqslant 2 \cup x \geqslant 8\}$

2 Suppose $\mathscr{E} = \{x \in \mathbb{Z} : 0 \leqslant x \leqslant 8\}$, $A = \{x \in \mathbb{Z} : 2 \leqslant x \leqslant 7\}$, and $B = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 8\}$. List the elements of:

a A **b** A' **c** B **d** B'

e $A \cap B$ **f** $A \cup B$ **g** $A \cap B'$ **h** $A' \cup B'$

3 Suppose P and Q' are subsets of $\mathscr{E}$. $n(\mathscr{E}) = 15$, $n(P) = 6$, and $n(Q') = 4$. Find:

a $n(P')$ **b** $n(Q)$

4 True or false?

a If $n(\mathscr{E}) = a$ and $n(A) = b$ where $A \subseteq \mathscr{E}$, then $n(A') = b - a$.

b If Q is a subset of $\mathscr{E}$ then $Q' = \{x \in \mathscr{E} : x \notin Q\}$.

5 Suppose $\mathscr{E} = \{x \in \mathbb{Z} : 0 < x \leqslant 12\}$, $A = \{x \in \mathbb{Z} : 2 \leqslant x \leqslant 7\}$,
$B = \{x \in \mathbb{Z} : 3 \leqslant x \leqslant 9\}$, and $C = \{x \in \mathbb{Z} : 5 \leqslant x \leqslant 11\}$.

List the elements of:

a B' **b** C' **c** A' **d** $A \cap B$

e $(A \cap B)'$ **f** $A' \cap C$ **g** $B' \cup C$ **h** $(A \cup C) \cap B'$

6 Consider the set of real numbers $\mathbb{R}$. Write down the complement of:

a $(-\infty, 0)$ **b** $[1, \infty)$ **c** $[-3, 2)$ **d** $(-5, 7]$

e $(-\infty, 1) \cup [3, \infty)$ **f** $[-5, 0) \cup (1, \infty)$

E PROPERTIES OF UNION AND INTERSECTION

In this section we will explore the number of elements in unions and intersections of sets.

Example 5

Self Tutor

Suppose $\mathscr{E} = \{\text{positive integers}\}$, $P = \{\text{multiples of 4 less than 50}\}$, and $Q = \{\text{multiples of 6 less than 50}\}$.

a List P and Q. **b** Find $P \cap Q$. **c** Find $P \cup Q$.

d Verify that $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$.

a $P = \{4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48\}$
$Q = \{6, 12, 18, 24, 30, 36, 42, 48\}$

b $P \cap Q = \{12, 24, 36, 48\}$

c $P \cup Q = \{4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 44, 48\}$

d $n(P \cup Q) = 16$ and $n(P) + n(Q) - n(P \cap Q) = 12 + 8 - 4 = 16$
So, $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$ is verified.

EXERCISE 1E

1 Suppose $\mathscr{E} = \mathbb{Z}^+$, $P = \{\text{prime numbers} < 25\}$, and $Q = \{2, 4, 5, 11, 12, 15\}$.

a List P. **b** Find $P \cap Q$. **c** Find $P \cup Q$.

d Verify that $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$.

2 Suppose $\mathscr{E} = \mathbb{Z}^+$, $P = \{\text{factors of 28}\}$, and $Q = \{\text{factors of 40}\}$.

a List P and Q. **b** Find $P \cap Q$. **c** Find $P \cup Q$.

d Verify that $n(P \cup Q) = n(P) + n(Q) - n(P \cap Q)$.

3 Suppose $\mathscr{E} = \mathbb{Z}^+$, $M = \{\text{multiples of 4 between 30 and 60}\}$, and $N = \{\text{multiples of 6 between 30 and 60}\}$.

a List M and N. **b** Find $M \cap N$. **c** Find $M \cup N$.

d Verify that $n(M \cup N) = n(M) + n(N) - n(M \cap N)$.

4 Suppose $\mathscr{E} = \mathbb{Z}$, $R = \{x \in \mathbb{Z} : -2 \leqslant x \leqslant 4\}$, and $S = \{x \in \mathbb{Z} : 0 \leqslant x < 7\}$.

a List R and S. **b** Find $R \cap S$. **c** Find $R \cup S$.

d Verify that $n(R \cup S) = n(R) + n(S) - n(R \cap S)$.

5 Suppose $\mathscr{E} = \mathbb{Z}$, $C = \{y \in \mathbb{Z} : -4 \leqslant y \leqslant -1\}$, and $D = \{y \in \mathbb{Z} : -7 \leqslant y < 0\}$.

a List C and D. **b** Find $C \cap D$. **c** Find $C \cup D$.

d Verify that $n(C \cup D) = n(C) + n(D) - n(C \cap D)$.

6 Suppose $\mathscr{E} = \mathbb{Z}^+$, $P = \{\text{factors of } 12\}$, $Q = \{\text{factors of } 18\}$, and $R = \{\text{factors of } 27\}$.

a List the sets P, Q, and R.

b Find: **i** $P \cap Q$ **ii** $P \cap R$ **iii** $Q \cap R$
iv $P \cup Q$ **v** $P \cup R$ **vi** $Q \cup R$

c Find: **i** $P \cap Q \cap R$ **ii** $P \cup Q \cup R$

7 Suppose $\mathscr{E} = \mathbb{Z}^+$, $A = \{\text{multiples of 4 less than } 40\}$, $B = \{\text{multiples of 6 less than } 40\}$, and $C = \{\text{multiples of 12 less than } 40\}$.

a List the sets A, B, and C.

b Find: **i** $A \cap B$ **ii** $B \cap C$ **iii** $A \cap C$
iv $A \cap B \cap C$ **v** $A \cup B \cup C$

c Verify that
$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C).$$

8 Suppose $\mathscr{E} = \mathbb{Z}^+$, $A = \{\text{multiples of 6 less than } 31\}$,
$B = \{\text{factors of } 30\}$, and $C = \{\text{primes } < 30\}$.

a List the sets A, B, and C.

b Find: **i** $A \cap B$ **ii** $B \cap C$ **iii** $A \cap C$
iv $A \cap B \cap C$ **v** $A \cup B \cup C$

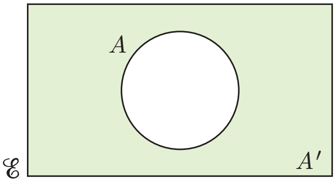
c Verify that
$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C).$$

F VENN DIAGRAMS

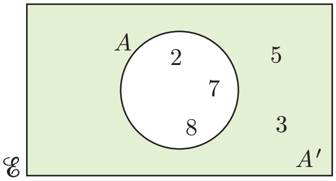
A **Venn diagram** consists of a universal set $\mathscr{E}$ represented by a rectangle. Sets within the universal set are usually represented by circles.

For example:

- This Venn diagram shows set A within the universal set $\mathscr{E}$. A', the complement of A, is the shaded region outside the circle.



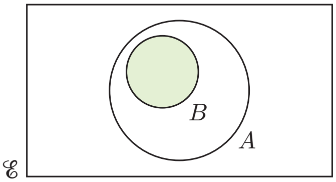
- The sets $\mathscr{E} = \{2, 3, 5, 7, 8\}$, $A = \{2, 7, 8\}$, and $A' = \{3, 5\}$ are represented by:



SUBSETS

If $B \subseteq A$ then every element of B is also in A.

The circle representing B is placed within the circle representing A.

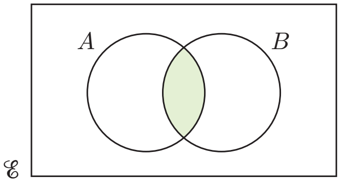


$B \subseteq A$

INTERSECTION

$A \cap B$ consists of all elements common to both A and B.

It is the shaded region where the circles representing A and B overlap.

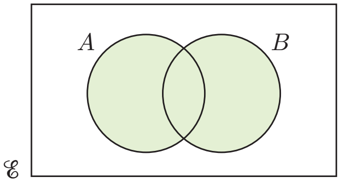


$A \cap B$

UNION

$A \cup B$ consists of all elements in A or B or both.

It is the shaded region which includes both circles.



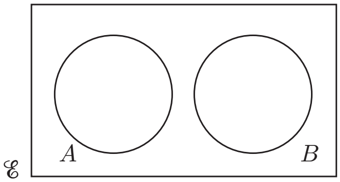
$A \cup B$

DISJOINT OR MUTUALLY EXCLUSIVE SETS

Disjoint sets do not have common elements.

They are represented by non-overlapping circles.

For example, if $A = \{2, 3, 8\}$ and $B = \{4, 5, 9\}$
then $A \cap B = \varnothing$.



$A \cap B = \varnothing$

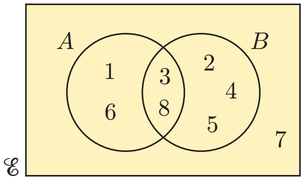
Example 6

Self Tutor

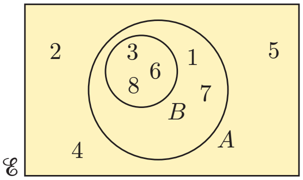
Suppose $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Illustrate on a Venn diagram the sets:

- **a** $A = \{1, 3, 6, 8\}$ and $B = \{2, 3, 4, 5, 8\}$
- **b** $A = \{1, 3, 6, 7, 8\}$ and $B = \{3, 6, 8\}$
- **c** $A = \{2, 4, 8\}$ and $B = \{3, 6, 7\}$.

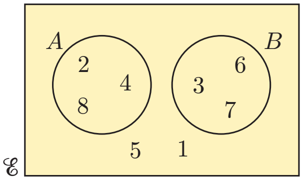
a $A \cap B = \{3, 8\}$



b $A \cap B = \{3, 6, 8\}$,
$B \subseteq A$



c $A \cap B = \varnothing$



EXERCISE 1F.1

1 Represent sets A and B on a Venn diagram, given:

- **a** $\mathscr{E} = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{2, 4, 5, 6\}$, and $B = \{1, 4, 6, 7\}$
- **b** $\mathscr{E} = \{2, 3, 4, 5, 6, 7\}$, $A = \{2, 4, 6\}$, and $B = \{5, 7\}$
- **c** $\mathscr{E} = \{2, 3, 4, 5, 6, 7\}$, $A = \{2, 4, 6\}$, and $B = \{3, 5, 7\}$
- **d** $\mathscr{E} = \{3, 4, 5, 7\}$, $A = \{3, 4, 5, 7\}$, and $B = \{3, 5\}$

2 Suppose $\mathscr{E} = \{x \in \mathbb{Z} : 1 \leqslant x \leqslant 10\}$, $A = \{\text{odd numbers} < 10\}$, and $B = \{\text{primes} < 10\}$.

- **a** List sets A and B.
- **b** Find $A \cap B$ and $A \cup B$.
- **c** Represent the sets A and B on a Venn diagram.

3 Suppose $\mathscr{E} = \{x \in \mathbb{Z} : 1 \leqslant x \leqslant 9\}$, $A = \{\text{factors of } 6\}$, and $B = \{\text{factors of } 9\}$.

- **a** List sets A and B.
- **b** Find $A \cap B$ and $A \cup B$.
- **c** Represent the sets A and B on a Venn diagram.

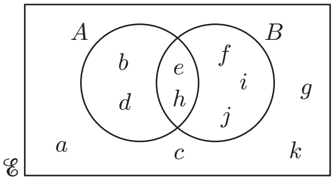
4 Suppose $\mathscr{E} = \{\text{even numbers between 0 and 30}\}$,
$P = \{\text{multiples of 4 less than 30}\}$, and
$Q = \{\text{multiples of 6 less than 30}\}$.

- **a** List sets P and Q.
- **b** Find $P \cap Q$ and $P \cup Q$.
- **c** Represent the sets P and Q on a Venn diagram.

5 Suppose $\mathscr{E} = \{x \in \mathbb{Z}^+ : x \leqslant 30\}$, $R = \{\text{primes less than 30}\}$, and
$S = \{\text{composites less than 30}\}$.

- **a** List sets R and S.
- **b** Find $R \cap S$ and $R \cup S$.
- **c** Represent the sets R and S on a Venn diagram.

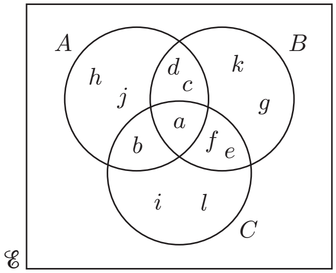
6



List the elements of set:

- **a** A
- **b** B
- **c** A'
- **d** B'
- **e** $A \cap B$
- **f** $A \cup B$
- **g** $(A \cup B)'$
- **h** $A' \cup B'$

7

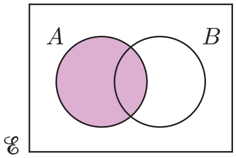


This Venn diagram consists of three overlapping circles A, B, and C.

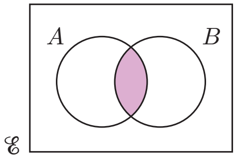
- **a** List the letters in set:
  - **i** A
  - **ii** B
  - **iii** C
  - **iv** $A \cap B$
  - **v** $A \cup B$
  - **vi** $B \cap C$
  - **vii** $A \cap B \cap C$
  - **viii** $A \cup B \cup C$
- **b** Find:
  - **i** $n(A \cup B \cup C)$
  - **ii** $n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$
- **c** What do you notice about your answers in **b**?

USING VENN DIAGRAMS TO ILLUSTRATE REGIONS

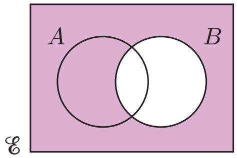
We can use shading to show various sets on a Venn diagram.
For example, for two intersecting sets A and B:



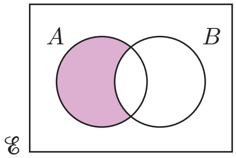
A is shaded



$A \cap B$ is shaded



B' is shaded



$A \cap B'$ is shaded

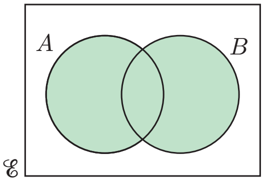
Example 7

Self Tutor

Shade the following regions for two intersecting sets A and B:

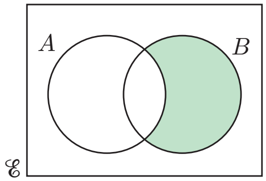
a $A \cup B$ **b** $A' \cap B$ **c** $(A \cap B)'$

a



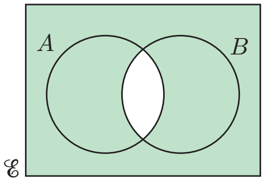
(in A, B, or both)

b



(outside A, intersected with B)

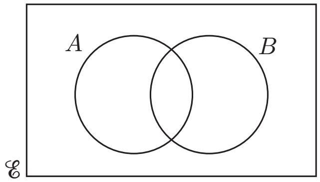
c



(outside $A \cap B$)

EXERCISE 1F.2

1

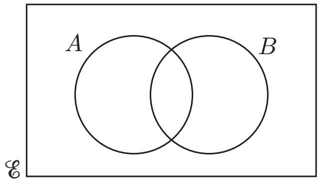


On separate Venn diagrams, shade regions for:

a $A \cap B$
b $A \cap B'$
c $A' \cup B$
d $A \cup B'$
e $A' \cap B$
f $A' \cap B'$

PRINTABLE VENN DIAGRAMS (OVERLAPPING)

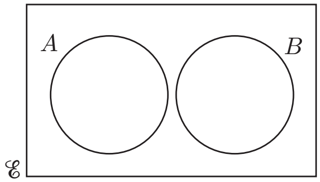
2



On separate Venn diagrams, shade regions for:

a $A \cup B$
b $(A \cup B)'$
c $(A \cap B)'$
d $A' \cup B'$
e $(A' \cup B')'$
f $(A \cup B')'$

3

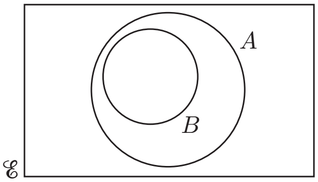


Suppose A and B are two disjoint sets. Shade on separate Venn diagrams:

a A
b B
c A'
d B'
e $A \cap B$
f $A \cup B$
g $A' \cap B$
h $A \cup B'$
i $(A \cap B)'$

PRINTABLE VENN DIAGRAMS (DISJOINT)

4

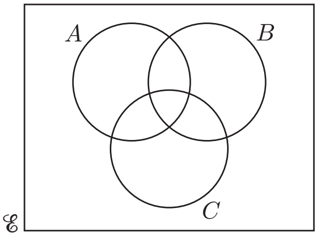


Suppose $B \subseteq A$, as shown in the given Venn diagram. Shade on separate Venn diagrams:

a A
b B
c A'
d B'
e $A \cap B$
f $A \cup B$
g $A' \cap B$
h $A \cup B'$
i $(A \cap B)'$

PRINTABLE VENN DIAGRAMS (SUBSET)

5



This Venn diagram consists of three intersecting sets. Shade on separate Venn diagrams:

a A
b B'
c $B \cap C$
d $A \cup B$
e $A \cap B \cap C$
f $A \cup B \cup C$
g $(A \cap B \cap C)'$
h $(B \cap C) \cup A$
i $(A \cup B) \cap C$
j $(A \cap C) \cup (B \cap C)$
k $(A \cap B) \cup C$
l $(A \cup C) \cap (B \cup C)$

PRINTABLE VENN DIAGRAMS (3 SETS)

Click on the icon to practise shading regions representing various subsets. You can practise with both two and three intersecting sets.

VENN DIAGRAMS

Discovery *The algebra of sets*

For the set of real numbers $\mathbb{R}$, we can write laws for the operations $+$ and $\times$:

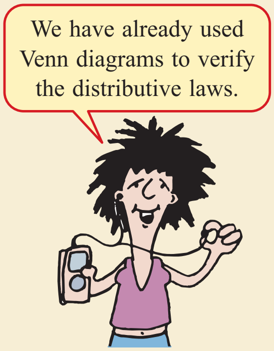
For any real numbers a, b, and c:

- **commutative** $a + b = b + a$ and $ab = ba$
- **identity** Identity elements 0 and 1 exist such that $a + 0 = 0 + a = a$ and $a \times 1 = 1 \times a = a$.
- **associativity** $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$
- **distributive** $a(b + c) = ab + ac$

The following are the **laws for the algebra of sets** under the operations $\cup$ and $\cap$:

For any subsets A, B, and C of the universal set $\mathscr{E}$:

- **commutative** $A \cap B = B \cap A$ and $A \cup B = B \cup A$
- **associativity** $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$
- **distributive** $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- **identity** $A \cup \varnothing = A$ and $A \cap \mathscr{E} = A$
- **complement** $A \cup A' = \mathscr{E}$ and $A \cap A' = \varnothing$
- **domination** $A \cup \mathscr{E} = \mathscr{E}$ and $A \cap \varnothing = \varnothing$
- **idempotent** $A \cap A = A$ and $A \cup A = A$
- **DeMorgan's** $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$
- **involution** $(A')' = A$



What to do:

1 With the aid of Venn diagrams, explain why the following laws are valid:

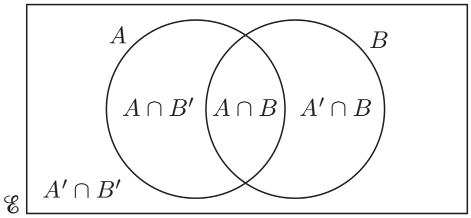
- **a** the *complement* law $(A')' = A$
- **b** the *commutative* laws $A \cap B = B \cap A$ and $A \cup B = B \cup A$
- **c** the *idempotent* laws $A \cap A = A$ and $A \cup A = A$
- **d** the *associative* laws $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \cup (B \cup C) = (A \cup B) \cup C$
- **e** the *distributive* laws $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

2 Use the laws for the algebra of sets to show that:

- **a** $A \cup (B \cup A') = \mathscr{E}$
- **b** $A \cap (B \cap A') = \varnothing$
- **c** $A \cup (B \cap A') = A \cup B$
- **d** $(A' \cup B')' = A \cap B$
- **e** $(A \cup B) \cap (A' \cap B') = \varnothing$
- **f** $(A \cup B) \cap (C \cup D) = (A \cap C) \cup (A \cap D) \cup (B \cap C) \cup (B \cap D)$.

G NUMBERS IN REGIONS

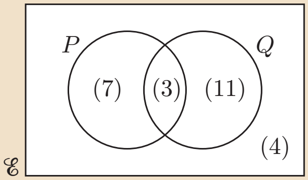
We have seen that there are four regions on a Venn diagram which contains two overlapping sets A and B.



There are many situations where we are only interested in the **number of elements** of $\mathscr{E}$ that are in each region. We do not need to show all the elements on the diagram, so instead we write the number of elements in each region in brackets.

Example 8

Self Tutor



In the Venn diagram given, (3) means that there are 3 elements in the set $P \cap Q$.

How many elements are there in:

- **a** P
- **b** Q'
- **c** $P \cup Q$
- **d** P, but not Q
- **e** Q, but not P
- **f** neither P nor Q?

a $n(P) = 7 + 3 = 10$

b $n(Q') = 7 + 4 = 11$

c $n(P \cup Q) = 7 + 3 + 11 = 21$

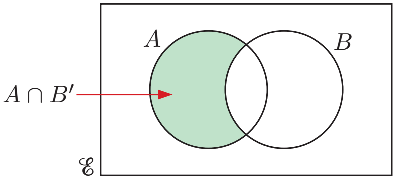
d $n(P$, but not $Q) = 7$

e $n(Q$, but not $P) = 11$

f $n(\text{neither } P \text{ nor } Q) = 4$

Venn diagrams allow us to easily visualise identities such as

$$n(A \cap B') = n(A) - n(A \cap B)$$
$$n(A' \cap B) = n(B) - n(A \cap B)$$

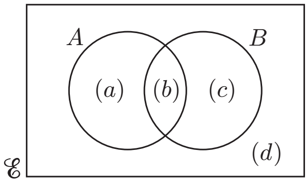


Example 9

Self Tutor

Given $n(\mathscr{E}) = 30$, $n(A) = 14$, $n(B) = 17$, and $n(A \cap B) = 6$, find:

a $n(A \cup B)$ **b** $n(A\text{, but not }B)$



We see that $b = 6$ $\{$as $n(A \cap B) = 6\}$

$a + b = 14$ $\{$as $n(A) = 14\}$

$b + c = 17$ $\{$as $n(B) = 17\}$

$a + b + c + d = 30$ $\{$as $n(\mathscr{E}) = 30\}$

$\therefore$ $b = 6$, $a = 8$, and $c = 11$

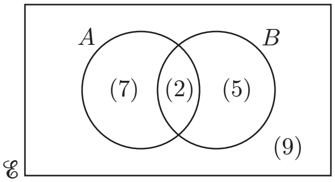
$\therefore$ $8 + 6 + 11 + d = 30$

$\therefore$ $d = 5$

a $n(A \cup B) = a + b + c = 25$ **b** $n(A\text{, but not }B) = a = 8$

EXERCISE 1G

1



In the Venn diagram given, (2) means that there are 2 elements in the set $A \cap B$.

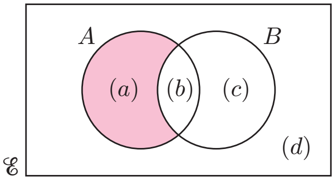
How many elements are there in:

a B **b** A'

c $A \cup B$ **d** A, but not B

e B, but not A **f** neither A nor B?

2



In the Venn diagram given, (a) means that there are a elements in the shaded region.

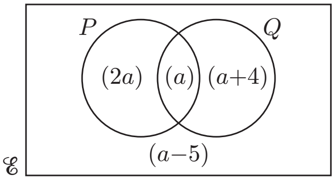
Notice that $n(A) = a + b$. Find:

a $n(B)$ **b** $n(A')$

c $n(A \cap B)$ **d** $n(A \cup B)$

e $n((A \cap B)')$ **f** $n((A \cup B)')$

3



The Venn diagram shows that $n(P \cap Q) = a$ and $n(P) = 3a$.

a Find:

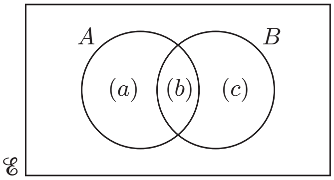
i $n(Q)$ **ii** $n(P \cup Q)$ **iii** $n(Q')$ **iv** $n(\mathscr{E})$

b Find a if:

i $n(\mathscr{E}) = 29$ **ii** $n(\mathscr{E}) = 31$

Comment on your results.

4



Use the Venn diagram to show that:
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

5 Given $n(\mathscr{E}) = 26$, $n(A) = 11$, $n(B) = 12$, and $n(A \cap B) = 8$, find:

- **a** $n(A \cup B)$
- **b** $n(B, \text{ but not } A)$

6 Given $n(\mathscr{E}) = 32$, $n(M) = 13$, $n(M \cap N) = 5$, and $n(M \cup N) = 26$, find:

- **a** $n(N)$
- **b** $n((M \cup N)')$

7 Given $n(\mathscr{E}) = 50$, $n(S) = 30$, $n(R) = 25$, and $n(R \cup S) = 48$, find:

- **a** $n(R \cap S)$
- **b** $n(S, \text{ but not } R)$

H PROBLEM SOLVING WITH VENN DIAGRAMS

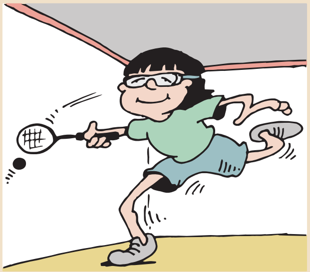
In this section we use Venn diagrams to illustrate real world situations. We can solve problems by considering the number of elements in each region.

Example 10

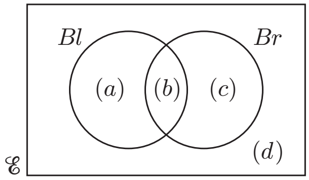
Self Tutor

A squash club has 27 members. 19 have black hair, 14 have brown eyes, and 11 have both black hair and brown eyes.

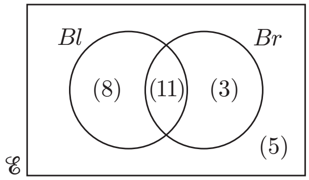
- **a** Place this information on a Venn diagram.
- **b** Hence find the number of members with:
  - **i** black hair or brown eyes
  - **ii** black hair, but not brown eyes.



a Let Bl represent the black hair set and Br represent the brown eyes set.



$a + b + c + d = 27$ {total members}
$a + b = 19$ {black hair}
$b + c = 14$ {brown eyes}
$b = 11$ {black hair and brown eyes}
$\therefore\ a = 8,\ c = 3,\ d = 5$



b **i** $n(Bl \cup Br) = 8 + 11 + 3 = 22$

22 members have black hair or brown eyes.

ii $n(Bl \cap Br') = 8$

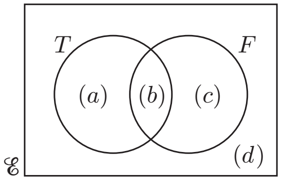
8 members have black hair, but not brown eyes.

Example 11

Self Tutor

A platform diving squad of 25 has 18 members who dive from 10 m and 17 who dive from 5 m. How many dive from both platforms?

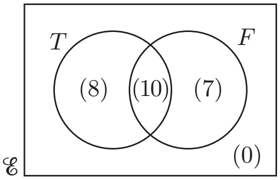
Let T represent those who dive from 10 m and F represent those who dive from 5 m.



$$d = 0 \quad \{\text{as all divers in the squad must dive from at least one of the platforms}\}$$

$$\begin{aligned} a + b &= 18 \\ b + c &= 17 \qquad \therefore\ a = 8,\ b = 10,\ c = 7 \\ a + b + c &= 25 \end{aligned}$$

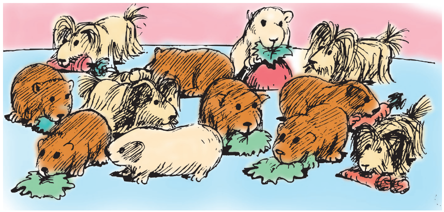
$$\begin{aligned} n(\text{both } T \text{ and } F) &= n(T \cap F) \\ &= 10 \end{aligned}$$



10 members dive from both platforms.

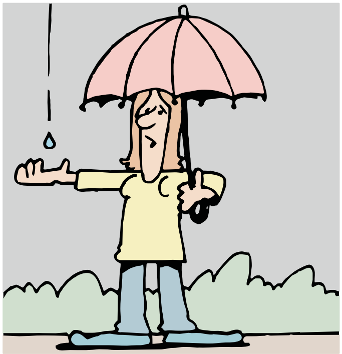
EXERCISE 1H

1 Pelé has 14 cavies as pets. Five have long hair and 8 are brown. Two are both brown and have long hair.



- **a** Place this information on a Venn diagram.
- **b** Hence find the number of cavies that:
  - **i** do not have long hair
  - **ii** have long hair and are not brown
  - **iii** are neither long-haired nor brown.

2



During a 2 week period, Murielle took her umbrella with her on 8 days. It rained on 9 days, and Murielle took her umbrella on five of the days when it rained.

- **a** Display this information on a Venn diagram.
- **b** Hence find the number of days that:
  - **i** Murielle did not take her umbrella and it rained
  - **ii** Murielle did not take her umbrella and it did not rain.

3 A badminton club has 31 playing members. 28 play singles and 16 play doubles. How many play both singles and doubles?

4 In a factory, 56 people work on the assembly line. 47 work day shifts and 29 work night shifts. How many work both day shifts and night shifts?

Example 12

Self Tutor

Consider the **Opening Problem** on page **12**:

A city has three football teams in the national league: A, B, and C.

In the last season, 20% of the city's population saw team A play, 24% saw team B, and 28% saw team C. Of these, 4% saw both A and B, 5% saw both A and C, and 6% saw both B and C. 1% saw all three teams play.

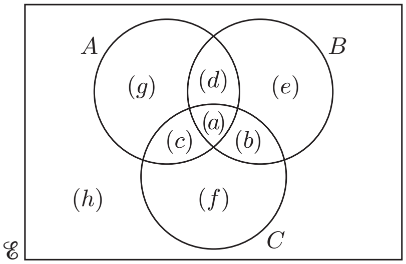
Using a Venn diagram, find the percentage of the city's population which:

a saw only team A play

b saw team A or team B play but not team C

c did not see any of the teams play.

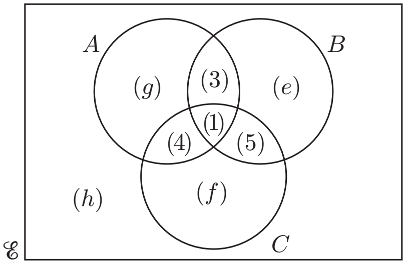
We construct the Venn diagram in terms of percentages.



Using the given information,

$$\begin{aligned} a &= 1 && \{1\% \text{ saw all three teams play}\} \\ a+d &= 4 && \{4\% \text{ saw } A \text{ and } B\} \\ a+b &= 6 && \{6\% \text{ saw } B \text{ and } C\} \\ a+c &= 5 && \{5\% \text{ saw } A \text{ and } C\} \end{aligned}$$

$\therefore \ d = 3, \ b = 5, \ \text{and} \ c = 4$

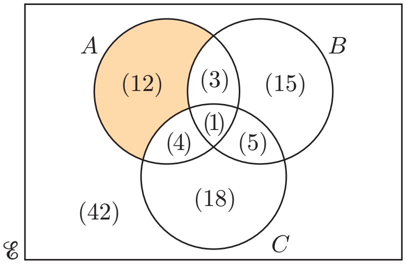


In total, 20% saw team A play,
so $g + 1 + 4 + 3 = 20 \quad \therefore \ g = 12$

24% saw team B play,
so $e + 1 + 5 + 3 = 24 \quad \therefore \ e = 15$

28% saw team C play,
so $f + 1 + 5 + 4 = 28 \quad \therefore \ f = 18$

In total we cover 100% of the population, so $h = 42$.



a $n(\text{saw } A \text{ only}) = 12\%$ {shaded}

b $n(A \text{ or } B, \text{ but not } C)$
$= 12\% + 3\% + 15\%$
$= 30\%$

c $n(\text{saw none of the teams}) = 42\%$

5 In a year group of 63 students, 22 study Biology, 26 study Chemistry, and 25 study Physics. 18 study both Physics and Chemistry, 4 study both Biology and Chemistry, and 3 study both Physics and Biology. 1 studies all three subjects.

a Display this information on a Venn diagram.

b How many students study:

i Biology only

ii Physics or Chemistry

iii none of Biology, Physics, or Chemistry

iv Physics but not Chemistry?

6 36 students participated in the mid-year adventure trip. 19 students went paragliding, 21 went abseiling, and 16 went white water rafting. 7 went abseiling and rafting, 8 went paragliding and rafting, and 11 went paragliding and abseiling. 5 students did all three activities.
Find the number of students who:

a went paragliding or abseiling
b only went white water rafting
c did not participate in any of the activities mentioned
d did exactly two of the activities mentioned.

7

There are 32 students in the woodwind section of the school orchestra. 11 students can play the flute, 15 can play the clarinet, and 12 can play the saxophone. 2 can play the flute and the saxophone, 2 can play the flute and the clarinet, and 6 can play the clarinet and the saxophone. 1 student can play all three instruments. Find the number of students who can play:

a none of the instruments mentioned
b only the saxophone
c the saxophone and the clarinet, but not the flute
d only one of the clarinet, saxophone, or flute.

8 In a particular region, most farms have livestock and crops. A survey of 21 farms showed that 15 grow crops, 9 have cattle, and 11 have sheep. 4 have sheep and cattle, 7 have cattle and crops, and 8 have sheep and crops. 3 have cattle, sheep, and crops. Find the number of farms with:

a only crops **b** only animals **c** exactly one type of animal, and crops.

Review set 1A

1 Suppose $S = \{x \in \mathbb{Z} : 2 < x \leqslant 7\}$.

a List the elements of S. **b** Find $n(S)$.
c How many proper subsets does S have?

2 Determine whether $A \subseteq B$ for the following sets:

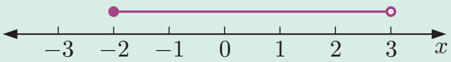
a $A = \{2, 4, 6, 8\}$ and $B = \{x \in \mathbb{Z} : 0 < x < 10\}$
b $A = \varnothing$ and $B = \{x \in (2, 3)\}$
c $A = \{x \in \mathbb{Q} : 2 < x \leqslant 4\}$ and $B = \{x \in \mathbb{R} : 0 \leqslant x < 4\}$
d $A = \{x \in (-\infty, 3)\}$ and $B = \{x \in (-\infty, 4]\}$

3 Find the complement of X given that:

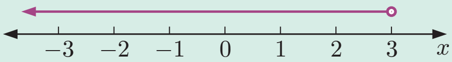
a $\mathscr{E} = \{\text{the 7 colours of the rainbow}\}$ and $X = \{\text{red, indigo, violet}\}$
b $\mathscr{E} = \{x \in \mathbb{Z} : -5 \leqslant x \leqslant 5\}$ and $X = \{-4, -1, 3, 4\}$
c $\mathscr{E} = \{x \in \mathbb{Q}\}$ and $X = \{x \in \mathbb{Q} : x < -8\}$
d $\mathscr{E} = \{x \in \mathbb{R}\}$ and $X = \{x \in [-3, 1) \cup (4, \infty)\}$

4 Write using interval notation, and state whether the interval is closed, open, or neither:

a



b

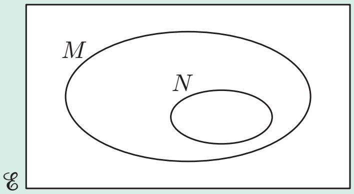


5 Illustrate in the Cartesian plane:

a $\{(x, y) : y = -2x\}$ **b** $\{(x, y) : x \leqslant y\}$

6 On separate Venn diagrams like the one alongside, shade:

a N' **b** $M \cap N$ **c** $M \cap N'$



7 Let $\mathscr{E} = \{$the letters in the English alphabet$\}$, $A = \{$the letters in "springbok"$\}$, and $B = \{$the letters in "waterbuck"$\}$.

a Find:

i $A \cup B$ **ii** $A \cap B$ **iii** $A \cap B'$

b Write a description for each of the sets in **a**.

c Show $\mathscr{E}$, A, and B on a Venn diagram.

8 Let $\mathscr{E} = \{x \in \mathbb{Z}^+ : x \leqslant 30\}$, $P = \{$factors of 24$\}$, and $Q = \{$factors of 30$\}$.

a List the elements of:

i P **ii** Q **iii** $P \cap Q$ **iv** $P \cup Q$

b Illustrate the sets P and Q on a Venn diagram.

9 A school has 564 students. During Term 1, 229 of them were absent for at least one day due to sickness, and 111 students missed some school because of family holidays. 296 students attended every day of Term 1.

a Display this information on a Venn diagram.

b Find the number of students who were away:

i for both sickness and holidays **ii** for holidays but not sickness

iii during Term 1 for any reason.

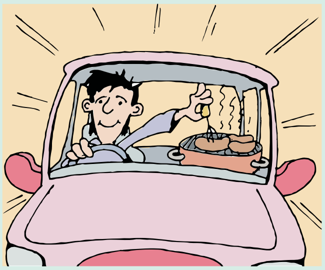
10 The main courses at a restaurant all contain rice or onion. Of the 23 choices, 17 contain onion and 14 contain rice. How many dishes contain both rice and onion?

11 38 students were asked what life skills they had. 15 could swim, 12 could drive, and 23 could cook. 9 could cook and swim, 5 could swim and drive, and 6 could drive and cook. There was 1 student who could do all three. Find the number of students who:

a could only cook

b could not do any of these things

c had exactly two life skills.



12 Consider the sets $\mathscr{E} = \{x \in \mathbb{Z}^+ : x \leqslant 10\}$, $P = \{\text{odd numbers less than } 10\}$, and $Q = \{\text{even numbers less than } 11\}$.

a List the sets P and Q. **b** What can be said about sets P and Q?

c Illustrate sets P and Q on a Venn diagram.

Review set 1B

1 True or false?

a $\mathbb{N} \subset \mathbb{Q}$ **b** $0 \in \mathbb{Z}^+$ **c** $0 \in \mathbb{Q}$

d $\mathbb{R} \subseteq \mathbb{Q}$ **e** $\mathbb{Z}^+ \cap \mathbb{Z}^- = \{0\}$

2 **a** Write using interval notation:

- **i** the real numbers between 5 and 12
- **ii** the integers between -4 and 7, including -4
- **iii** the natural numbers greater than 45.

b Which sets in **a** are finite and which are infinite?

3 List the subsets of $\{1, 3, 5\}$.

4 Let $\mathscr{E} = \{x \in \mathbb{Z} : 0 < x < 10\}$, $A = \{\text{the even integers between 0 and 9}\}$, and $B = \{\text{the factors of 8}\}$.

a List the elements of:

i A **ii** $A \cap B$ **iii** $(A \cup B)'$

b Represent this information on a Venn diagram.

5 S and T are disjoint sets. $n(S) = s$ and $n(T) = t$. Find:

a $S \cap T$ **b** $n(S \cup T)$

6 For each of the following sets, determine whether the interval described is closed, open, or neither:

a $x \in (-4, 3]$ **b** $x \in [-2, 2]$ **c** $x \in \mathbb{R}$

7 Suppose A and B are each sets of points which define straight lines.

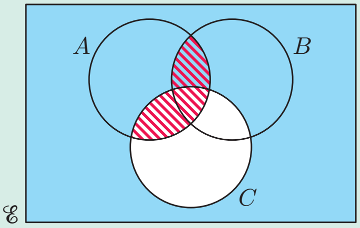
a Describe in words the meaning of:

i $A \cap B$ **ii** $A \cup B$

b Is $A \cap B$ necessarily finite? Explain your answer.

c If $A \cap B$ is finite, what possible values can $n(A \cap B)$ have?

8

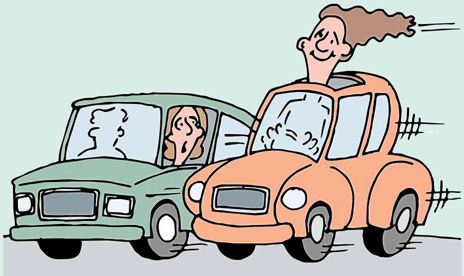


Give an expression for the region shaded in:

a blue **b** red.

9 In a car club, 13 members drive a manual and 15 members have a sunroof on their car. 5 have manual cars with a sunroof, and 4 have neither.

- **a** Display this information on a Venn diagram.
- **b** How many members:
  - **i** are in the club
  - **ii** drive a manual car without a sunroof
  - **iii** do not drive a manual car?



10 All attendees of a camp left something at home. 11 forgot to bring their towel, and 23 forgot their hat. Of the 30 campers, how many had neither a hat nor a towel?

11 Consider the sets $\mathscr{E} = \{x \in \mathbb{Z}^+ : x \leqslant 40\}$, $A = \{\text{factors of } 40\}$, and $B = \{\text{factors of } 20\}$.

- **a** List the sets A and B.
- **b** What can be said about sets A and B?
- **c** Illustrate sets A and B on a Venn diagram.

12 At a conference, the 58 delegates speak many different languages. 28 speak Arabic, 27 speak Chinese, and 39 speak English. 12 speak Arabic and Chinese, 16 speak both Chinese and English, and 17 speak Arabic and English. 2 speak all three languages. How many delegates speak:

- **a** Chinese only
- **b** none of these languages
- **c** neither Arabic nor Chinese?

2

Functions

Contents:

A RELATIONS AND FUNCTIONS

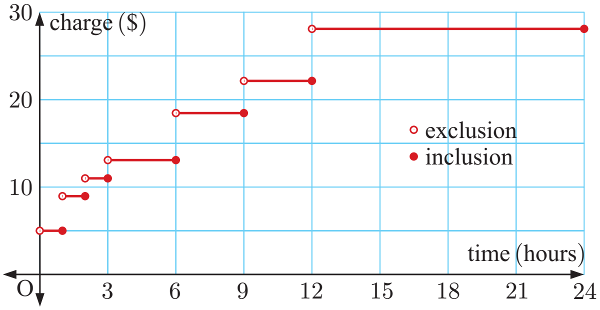
The charges for parking a car in a short-term car park at an airport are shown in the table below. The total charge is *dependent* on the length of time t the car is parked.

Car park charges	
Time t (hours)	*Charge*
0 - 1 hours	\$5.00
1 - 2 hours	\$9.00
2 - 3 hours	\$11.00
3 - 6 hours	\$13.00
6 - 9 hours	\$18.00
9 - 12 hours	\$22.00
12 - 24 hours	\$28.00

Looking at this table we might ask: How much would be charged for *exactly* one hour? Would it be $5 or $9?

To avoid confusion, we could adjust the table or draw a graph. We indicate that 2 - 3 hours really means a time over 2 hours up to and including 3 hours, by writing $2 < t \leqslant 3$ hours.

Car park charges	
Time t (hours)	*Charge*
$0 < t \leqslant 1$ hours	\$5.00
$1 < t \leqslant 2$ hours	\$9.00
$2 < t \leqslant 3$ hours	\$11.00
$3 < t \leqslant 6$ hours	\$13.00
$6 < t \leqslant 9$ hours	\$18.00
$9 < t \leqslant 12$ hours	\$22.00
$12 < t \leqslant 24$ hours	\$28.00



In mathematical terms, we have a relationship between two variables *time* and *charge*, so the schedule of charges is an example of a **relation**.

A relation may consist of a finite number of ordered pairs, such as $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$, or an infinite number of ordered pairs.

The parking charges example is clearly the latter, as every real value of time in the interval $0 < t \leqslant 24$ hours is represented.

The set of possible values of the variable on the horizontal axis is called the **domain** of the relation.

For example:
- the domain for the car park relation is $\{t \mid 0 < t \leqslant 24\}$
- the domain of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ is $\{-2, 1, 4\}$.

The set of possible values on the vertical axis is called the **range** of the relation.

For example:
- the range of the car park relation is $\{5, 9, 11, 13, 18, 22, 28\}$
- the range of $\{(1, 5), (-2, 3), (4, 3), (1, 6)\}$ is $\{3, 5, 6\}$.

We will now look at relations and functions more formally.

RELATIONS

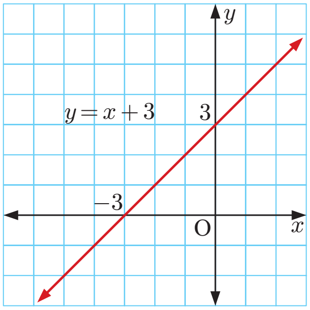
In **Chapter 1**, we saw that:

A **relation** is any set of points which connect two variables.

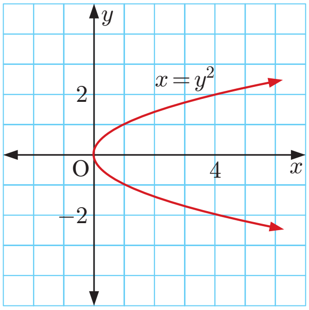
A relation is often expressed in the form of an **equation** connecting the **variables** x and y. The relation is a set of points (x, y) which can be viewed in the **Cartesian plane**.

For example, $y = x + 3$ and $x = y^2$ are the equations of two relations. Each equation generates a set of ordered pairs, which we can graph.

$y = x + 3$ is a set of points which lie in a straight line



$x = y^2$ is a set of points which lie in a smooth curve.



FUNCTIONS

A **function**, sometimes called a **mapping**, is a relation in which no two different ordered pairs have the same x-coordinate or first component.

We can see from the above definition that a function is a special type of relation.

Every function is a relation, but not every relation is a function.

TESTING FOR FUNCTIONS

Algebraic Test:

If a relation is given as an equation, and the substitution of any value for x results in one and only one value of y, then the relation is a function.

For example:

- $y = 3x - 1$ is a function, since for any value of x there is only one corresponding value of y
- $x = y^2$ is not a function, since if $x = 4$ then $y = \pm 2$.

Geometric Test or Vertical Line Test:

Suppose we draw all possible vertical lines on the graph of a relation.

- If each line cuts the graph at most once, then the relation is a function.
- If at least one line cuts the graph more than once, then the relation is *not* a function.

GRAPHICAL NOTE

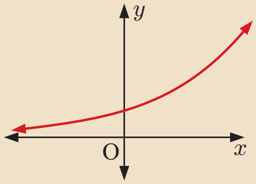
- If a graph contains a small **open circle** such as ——○—— , this point is **not included**.
- If a graph contains a small **filled-in circle** such as ——● , this point **is included**.
- If a graph contains an **arrow head** at an end such as ——► , then the graph continues indefinitely in that general direction, or the shape may repeat as it has done previously.

Example 1

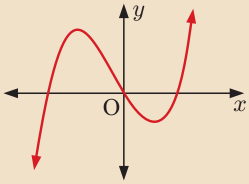
Self Tutor

Which of the following relations are functions?

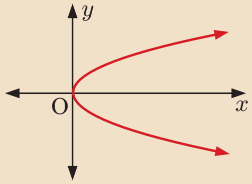
a



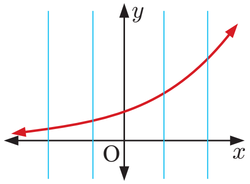
b



c

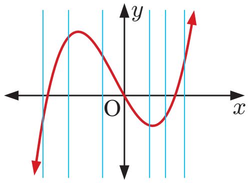


a



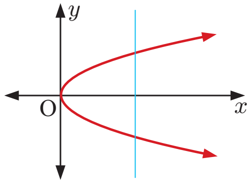
a function

b



a function

c



not a function

DEMO

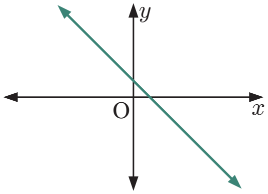
EXERCISE 2A.1

1 Which of the following sets of ordered pairs are functions? Give reasons for your answers.

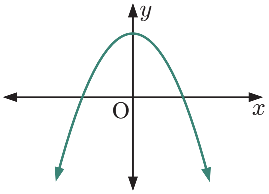
- **a** $\{(1, 3), (2, 4), (3, 5), (4, 6)\}$
- **b** $\{(1, 3), (3, 2), (1, 7), (-1, 4)\}$
- **c** $\{(2, -1), (2, 0), (2, 3), (2, 11)\}$
- **d** $\{(7, 6), (5, 6), (3, 6), (-4, 6)\}$
- **e** $\{(0, 0), (1, 0), (3, 0), (5, 0)\}$
- **f** $\{(0, 0), (0, -2), (0, 2), (0, 4)\}$

2 Use the vertical line test to determine which of the following relations are functions:

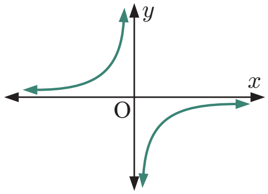
a



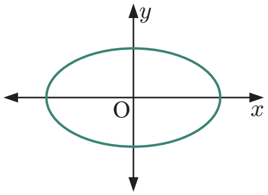
b



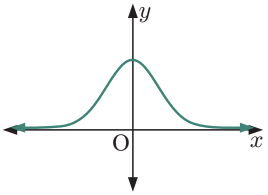
c



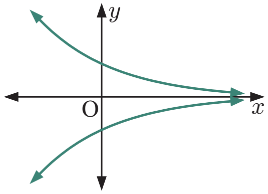
d



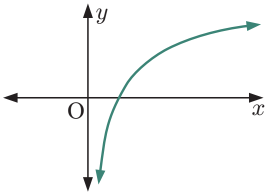
e



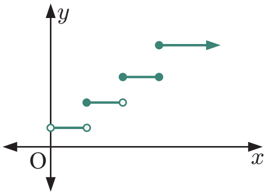
f



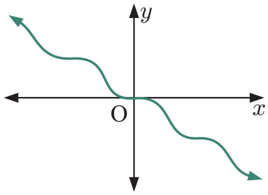
g



h



i

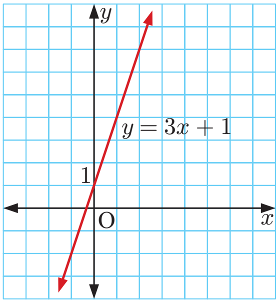


3 Give algebraic evidence to show that the relation $x^2 + y^2 = 16$ is not a function.

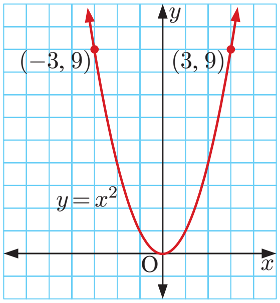
ONE-ONE FUNCTIONS

A **one-one function** is a function in which no two different ordered pairs have the same y-coordinate or second component.

For example:

- 

  The function $y = 3x + 1$ is one-one, since each distinct point on the graph has a different y-coordinate.

- 

  The function $y = x^2$ is *not* one-one, since the graph contains distinct points $(-3, 9)$ and $(3, 9)$ which have the same y-coordinate.

We can use the **horizontal line test** to determine whether a function is one-one:

Suppose we draw all possible horizontal lines on the graph of a function.

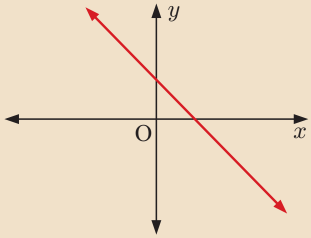
- If each line cuts the graph at most once, then the function is one-one.
- If at least one line cuts the graph more than once, then the function is *not* one-one.

Example 2

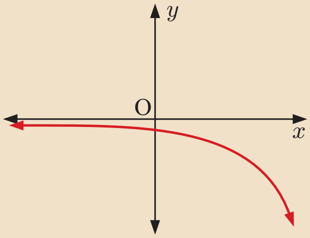
Self Tutor

Which of the following relations are one-one?

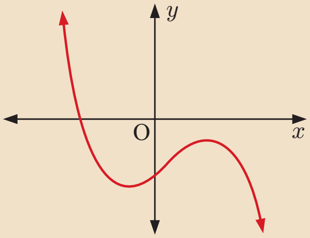
a



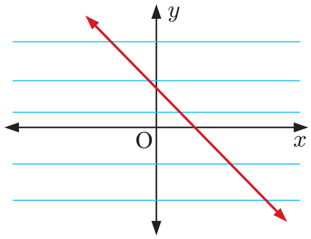
b



c

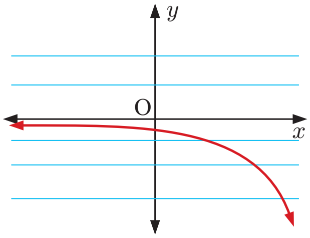


a



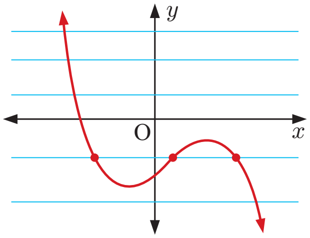
one-one

b



one-one

c

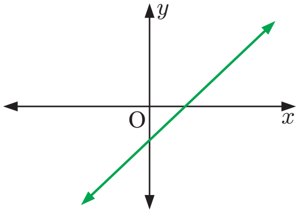


not one-one

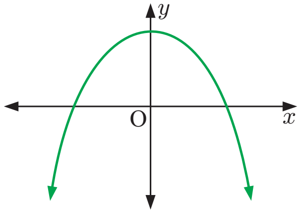
EXERCISE 2A.2

1 Which of the following functions are one-one?

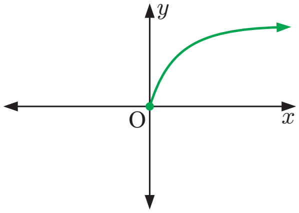
a



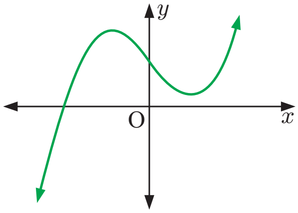
b



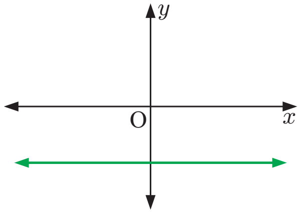
c



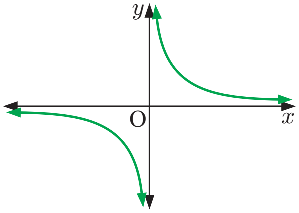
d



e

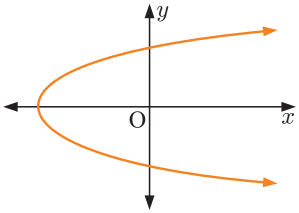


f

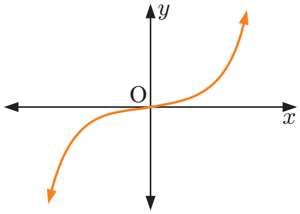


2 Determine whether the following relations are functions. If they are functions, determine whether they are one-one.

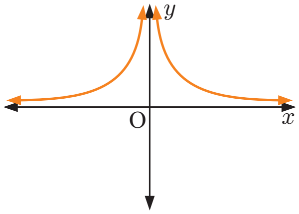
a



b



c

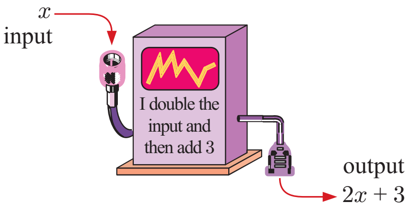


3 Consider the car park relation described on page **36**.

- **a** **i** Richard parked his car for 4 hours. How much did he pay?
  - **ii** Suppose Susie parked her car for t hours. If you know the value of t, can you uniquely determine how much she paid?
  - **iii** Is the car park relation a function?
- **b** **i** Janette paid \$18 for parking. Can you uniquely determine how long she parked for?
  - **ii** Is the car park function one-one?

B FUNCTION NOTATION

Function machines are sometimes used to illustrate how functions behave.



If 4 is the input fed into the machine, the output is $2(4) + 3 = 11$.

The above 'machine' has been programmed to perform a particular function. If we use f to represent that particular function, we can write:

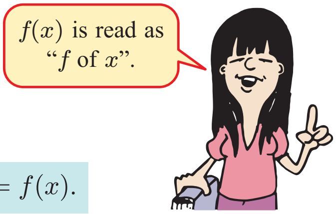
f is the function that will convert x into $2x + 3$.

So, f would convert 2 into $2(2)+3=7$ and
-4 into $2(-4)+3=-5$.

This function can be written as:

$$\boldsymbol{f : x \mapsto \underbrace{2x+3}}$$

function f such that x is converted into $2x+3$



Two other equivalent forms we use are $f(x)=2x+3$ and $y=2x+3$.

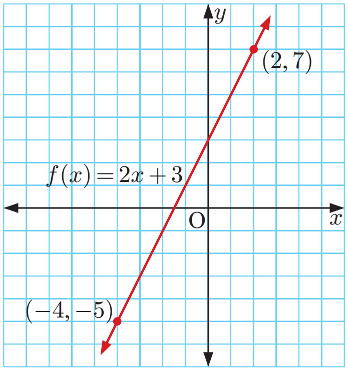
$f(x)$ is the value of y for a given value of x, so $y=f(x)$.

f is the function which converts x into $f(x)$, so we write $f : x \mapsto f(x)$.

$y=f(x)$ is sometimes called the **function value** or **image** of x.

For $f(x)=2x+3$:

- $f(2)=2(2)+3=7$.
  $\therefore$ the point (2, 7) lies on the graph of the function.
- $f(-4)=2(-4)+3=-5$.
  $\therefore$ the point $(-4, -5)$ also lies on the graph.



A **linear function** is a function of the form $f(x)=ax+b$ where a, b are real constants.
The graph of a linear function is a straight line.

Example 3

Self Tutor

If $f : x \mapsto 2x^2-3x$, find the value of: **a** $f(5)$ **b** $f(-4)$

$f(x)=2x^2-3x$

a $f(5)=2(5)^2-3(5)$ {replacing x with (5)}
$=2\times 25-15$
$=35$

b $f(-4)=2(-4)^2-3(-4)$ {replacing x with (-4)}
$=2(16)+12$
$=44$

EXERCISE 2B

1 If $f(x)=3x-x^2+2$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-3)$ **d** $f(-7)$ **e** $f(\frac{3}{2})$

2 If $g : x \mapsto x-\dfrac{4}{x}$, find the value of:

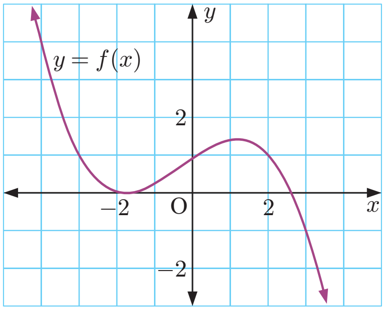
a $g(1)$ **b** $g(4)$ **c** $g(-1)$ **d** $g(-4)$ **e** $g(-\frac{1}{2})$

3 The graph of $y = f(x)$ is shown alongside.

a Find:

i $f(2)$ **ii** $f(3)$

b Find the value of x such that $f(x) = 4$.



Example 4

Self Tutor

If $f(x) = 5 - x - x^2$, find in simplest form: **a** $f(-x)$ **b** $f(x+2)$

a $f(-x) = 5 - (-x) - (-x)^2$ {replacing x with $(-x)$}

$= 5 + x - x^2$

b $f(x+2) = 5 - (x+2) - (x+2)^2$ {replacing x with $(x+2)$}

$= 5 - x - 2 - [x^2 + 4x + 4]$

$= 3 - x - x^2 - 4x - 4$

$= -x^2 - 5x - 1$

4 If $f(x) = 7 - 3x$, find in simplest form:

a $f(a)$ **b** $f(-a)$ **c** $f(a+3)$ **d** $f(b-1)$ **e** $f(x+2)$ **f** $f(x+h)$

5 If $F(x) = 2x^2 + 3x - 1$, find in simplest form:

a $F(x+4)$ **b** $F(2-x)$ **c** $F(-x)$ **d** $F(x^2)$ **e** $F(x^2-1)$ **f** $F(x+h)$

6 Suppose $G(x) = \dfrac{2x+3}{x-4}$.

a Evaluate: **i** $G(2)$ **ii** $G(0)$ **iii** $G(-\frac{1}{2})$

b Find a value of x such that $G(x)$ does not exist.

c Find $G(x+2)$ in simplest form.

d Find x if $G(x) = -3$.

7 f represents a function. What is the difference in meaning between f and $f(x)$?

8 The value of a photocopier t years after purchase is given by $V(t) = 9650 - 860t$ dollars.

a Find $V(4)$ and state what $V(4)$ means.

b Find t when $V(t) = 5780$ and explain what this represents.

c Find the original purchase price of the photocopier.

9 On the same set of axes draw the graphs of three different functions $f(x)$ such that $f(2) = 1$ and $f(5) = 3$.

10 Find a linear function $f(x) = ax + b$ for which $f(2) = 1$ and $f(-3) = 11$.

11 Given $f(x) = ax + \frac{b}{x}$, $f(1) = 1$, and $f(2) = 5$, find constants a and b.

12 Given $T(x) = ax^2 + bx + c$, $T(0) = -4$, $T(1) = -2$, and $T(2) = 6$, find a, b, and c.

C DOMAIN AND RANGE

The **domain** of a relation is the set of values of x in the relation.

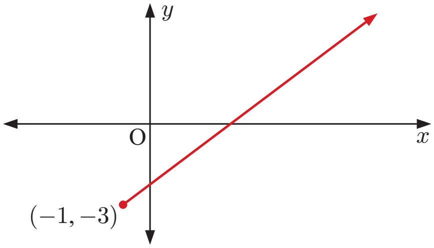
The **range** of a relation is the set of values of y in the relation.

The range is sometimes called the **image set**.

The domain and range of a relation are often described using **interval notation**.

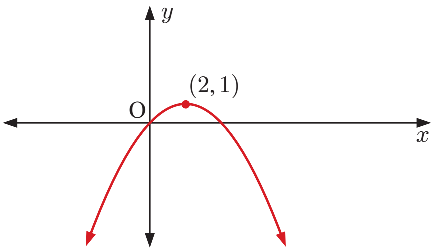
For example:

(1)



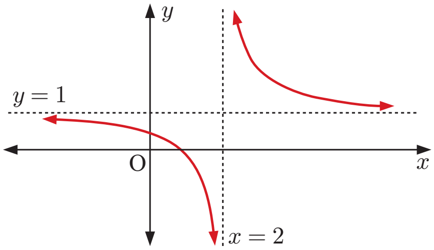
All values of $x \geqslant -1$ are included,
so the domain is $\{x : x \geqslant -1\}$.
All values of $y \geqslant -3$ are included,
so the range is $\{y : y \geqslant -3\}$.

(2)



x can take any value,
so the domain is $\{x : x \in \mathbb{R}\}$.
y cannot be > 1,
so the range is $\{y : y \leqslant 1\}$.

(3)



x can take all values except 2,
so the domain is $\{x : x \neq 2\}$.
y can take all values except 1,
so the range is $\{y : y \neq 1\}$.

DOMAIN AND RANGE OF FUNCTIONS

To fully describe a function, we need both a rule *and* a domain.

For example, we can specify $f(x) = x^2$ where $x \geqslant 0$.

If a domain is not specified, we use the **natural domain**, which is the largest part of $\mathbb{R}$ for which $f(x)$ is defined.

For example, consider the domains in the table opposite:

$f(x)$	*Natural domain*
x^2	$x \in \mathbb{R}$
$\sqrt{x}$	$x \geqslant 0$
$\frac{1}{x}$	$x \neq 0$
$\frac{1}{\sqrt{x}}$	$x > 0$

Click on the icon to obtain software for finding the domain and range of different functions.

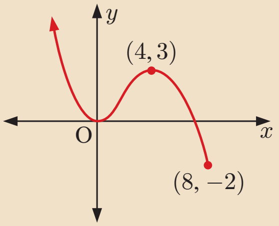
DOMAIN AND RANGE

Example 5

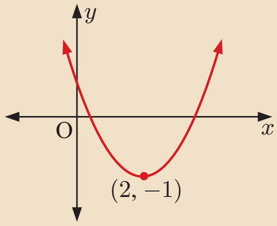
Self Tutor

For each of the following graphs, state the domain and range:

a



b



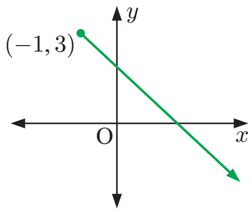
a Domain is $\{x : x \leqslant 8\}$
Range is $\{y : y \geqslant -2\}$

b Domain is $\{x : x \in \mathbb{R}\}$
Range is $\{y : y \geqslant -1\}$

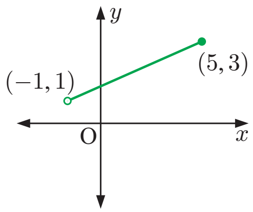
EXERCISE 2C

1 For each of the following graphs, state the domain and range:

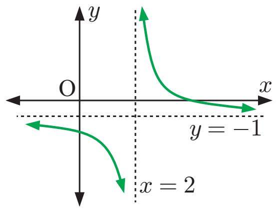
a



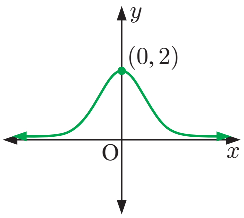
b



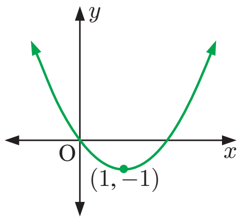
c



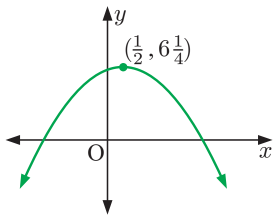
d



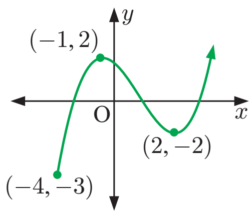
e



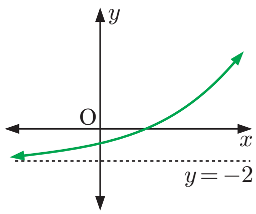
f



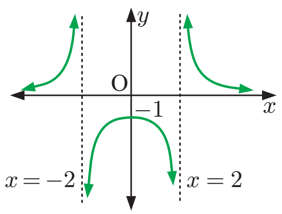
g



h



i



Example 6

Self Tutor

State the domain and range of each of the following functions:

a $f(x)=\sqrt{x-5}$ **b** $f(x)=\dfrac{1}{x-5}$ **c** $f(x)=\dfrac{1}{\sqrt{x-5}}$

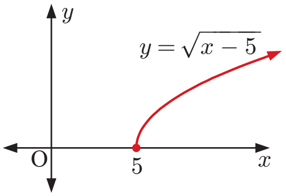
a $\sqrt{x-5}$ is defined when $x-5\geqslant 0$

$\therefore\ x\geqslant 5$

$\therefore$ the domain is $\{x : x\geqslant 5\}$.

A square root cannot be negative.

$\therefore$ the range is $\{y : y\geqslant 0\}$.



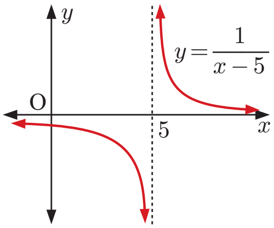
b $\dfrac{1}{x-5}$ is defined when $x-5\neq 0$

$\therefore\ x\neq 5$

$\therefore$ the domain is $\{x : x\neq 5\}$.

No matter how large or small x is, $y=f(x)$ is never zero.

$\therefore$ the range is $\{y : y\neq 0\}$.



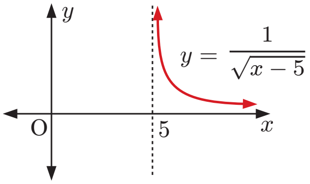
c $\dfrac{1}{\sqrt{x-5}}$ is defined when $x-5>0$

$\therefore\ x>5$

$\therefore$ the domain is $\{x : x>5\}$.

$y=f(x)$ is always positive and never zero.

$\therefore$ the range is $\{y : y>0\}$.



2 State the values of x for which $f(x)$ is defined, and hence state the domain of the function.

a $f(x)=\sqrt{x+6}$ **b** $f : x\mapsto\dfrac{1}{x^2}$ **c** $f(x)=\dfrac{-7}{\sqrt{3-2x}}$

3 Find the domain and range of each of the following functions:

a $f : x\mapsto 2x-1$ **b** $f(x)=3$ **c** $f : x\mapsto\sqrt{x}$

d $f(x)=\dfrac{1}{x+1}$ **e** $f(x)=-\dfrac{1}{\sqrt{x}}$ **f** $f : x\mapsto\dfrac{1}{3-x}$

4 Use technology to help sketch graphs of the following functions. Find the domain and range of each.

a $f(x)=\sqrt{x-2}$ **b** $f : x\mapsto\dfrac{1}{x^2}$ **c** $f : x\mapsto\sqrt{4-x}$

d $y=x^2-7x+10$ **e** $f(x)=\sqrt{x^2+4}$ **f** $f(x)=\sqrt{x^2-4}$

g $f : x\mapsto 5x-3x^2$ **h** $f : x\mapsto x+\dfrac{1}{x}$ **i** $y=\dfrac{x+4}{x-2}$

j $y=x^3-3x^2-9x+10$ **k** $f : x\mapsto\dfrac{3x-9}{x^2-x-2}$ **l** $y=x^2+x^{-2}$

m $y=x^3+\dfrac{1}{x^3}$ **n** $f : x\mapsto x^4+4x^3-16x+3$

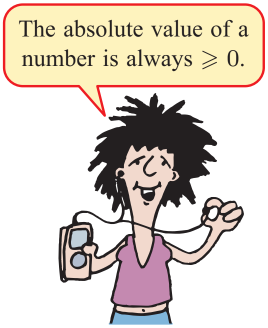
DOMAIN AND RANGE

D THE MODULUS FUNCTION

The **modulus** or **absolute value** of a real number is its size, ignoring its sign.

We denote the absolute value of x by $|x|$.

For example, the modulus of 4 is 4, and the modulus of -9 is 9. We write $|4| = 4$ and $|-9| = 9$.



Example 7

Self Tutor

If $x = -3$, find the value of:

a $|x|$ **b** $x\,|x|$ **c** $\left|x^2+x\right|$ **d** $\left|\dfrac{7x-1}{2}\right|$

a $|x|$
$= |-3|$
$= 3$

b $x\,|x|$
$= (-3)\,|-3|$
$= -3 \times 3$
$= -9$

c $\left|x^2+x\right|$
$= \left|(-3)^2+(-3)\right|$
$= |6|$
$= 6$

d $\left|\dfrac{7x-1}{2}\right|$
$= \left|\dfrac{7(-3)-1}{2}\right|$
$= |-11|$
$= 11$

EXERCISE 2D.1

1 Find the value of:

a $|5|$ **b** $|-5|$ **c** $|7-3|$ **d** $|3-7|$

e $\left|2^2-10\right|$ **f** $|15-3\times 5|$ **g** $\left|\dfrac{3-1}{5+2}\right|$ **h** $\left|\dfrac{2^3}{(-3)^3}\right|$

2 If $x = 4$, find the value of:

a $|x-5|$ **b** $|10-x|$ **c** $\left|3x-x^2\right|$ **d** $\left|\dfrac{2x+1}{x-1}\right|$

3 If $x = -2$, find the value of:

a $|x|$ **b** $x\,|x|$ **c** $-\left|x-x^2\right|$ **d** $\dfrac{|1+3x|}{x+1}$

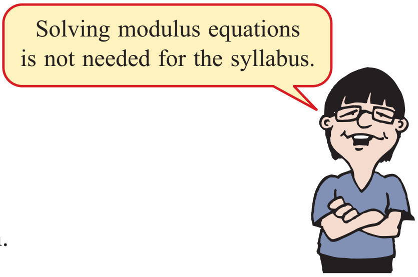
MODULUS EQUATIONS

The equation $|x| = 2$ has two solutions: $x = 2$ and $x = -2$.

If $|x| = a$ where $a > 0$, then $x = \pm a$.

If $|x| = |b|$ then $x = \pm b$.

We use these rules to solve equations involving the modulus function.



Example 8

Self Tutor

Solve for x: **a** $|2x+3|=7$ **b** $|3-2x|=-1$

a $|2x+3|=7$

$\therefore\ 2x+3=\pm 7$

$\therefore\ 2x+3=7$ **or** $2x+3=-7$

$\therefore\ 2x=4$ $\qquad\therefore\ 2x=-10$

$\therefore\ x=2$ $\qquad\therefore\ x=-5$

So, $x=2$ or -5

b $|3-2x|=-1$

has no solution as LHS is never negative.

Example 9

Self Tutor

Solve for x: $|x+1|=|2x-3|$

If $|x+1|=|2x-3|$, then $x+1=\pm(2x-3)$

$\therefore\ x+1=2x-3$ **or** $x+1=-(2x-3)$

$\therefore\ 4=x$ $\qquad\therefore\ x+1=-2x+3$

$\therefore\ 3x=2$

$\therefore\ x=\frac{2}{3}$

So, $x=\frac{2}{3}$ or 4.

EXERCISE 2D.2

1 Solve for x:

a $|x|=3$ **b** $|x|=-5$ **c** $|x|=0$

d $|x-1|=3$ **e** $|3-x|=4$ **f** $|x+5|=-1$

g $|3x-2|=1$ **h** $|3-2x|=3$ **i** $|2-5x|=12$

2 Solve for x:

a $|3x-1|=|x+2|$ **b** $|2x+5|=|1-x|$ **c** $|x+1|=|2-x|$

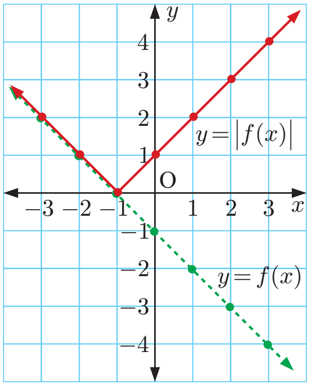
d $|x|=|5-x|$ **e** $|1-4x|=|x-1|$ **f** $|3x+2|=|2-x|$

THE GRAPH OF $y=|f(x)|$

Consider the function $f(x)=-x-1$.

In the table below, we calculate the values of $f(x)$ and $|f(x)|$ for $x=-3,-2,-1,0,1,2,3$.

x	-3	-2	-1	0	1	2	3
$f(x)$	2	1	0	-1	-2	-3	-4
$\|f(x)\|$	2	1	0	1	2	3	4



Using these values, we can plot $y=f(x)$ and $y=|f(x)|$ on the same set of axes.

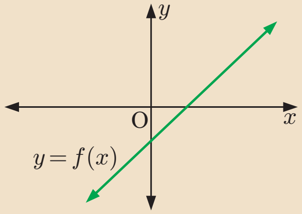
To draw the graph of $y = |f(x)|$, any parts of $y = f(x)$ that are below the x-axis are reflected in the x-axis.

Example 10

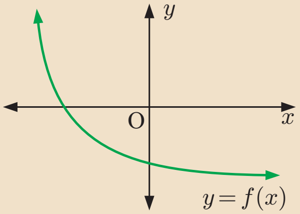
Self Tutor

For the following graphs, sketch the graph of $y = |f(x)|$:

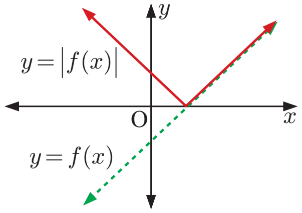
a



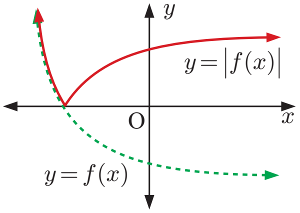
b



a



b

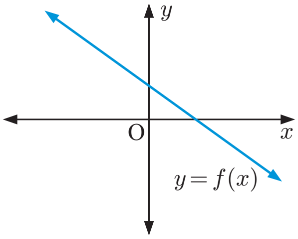


EXERCISE 2D.3

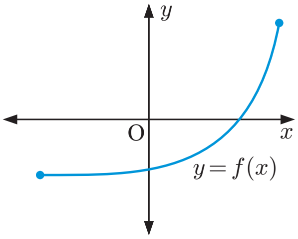
PRINTABLE DIAGRAMS

1 For the following graphs, sketch the graph of $y = |f(x)|$:

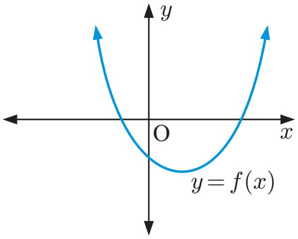
a



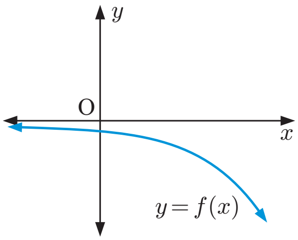
b



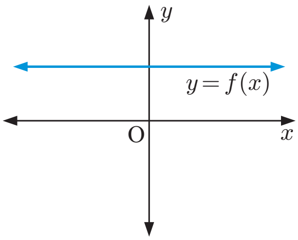
c



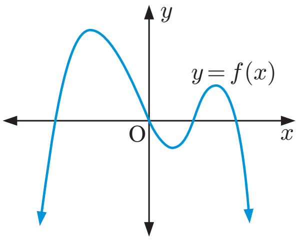
d



e



f



2 Which of the functions $y = |f(x)|$ in question **1** are one-one?

3 Suppose the range of $y = f(x)$ is $\{y : -6 \leqslant y \leqslant 2\}$. Write down the range of $y = |f(x)|$.

4 Determine whether the following statements are true or false:

- **a** If $y = f(x)$ is one-one, then $y = |f(x)|$ is one-one.
- **b** If $y = f(x)$ is not one-one, then $y = |f(x)|$ is not one-one.
- **c** The graphs of $y = f(x)$ and $y = |f(x)|$ always meet the x-axis at the same point(s).
- **d** The graphs of $y = f(x)$ and $y = |f(x)|$ always meet the y-axis at the same point.

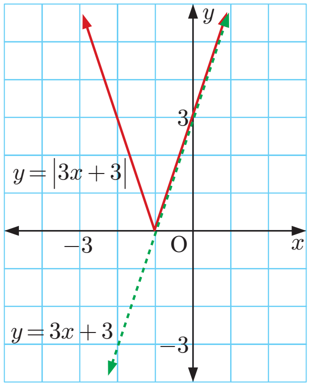
Example 11

Self Tutor

Draw the graph of $y = |3x + 3|$.

We first draw the graph of $y = 3x + 3$.

The part of the graph that is below the x-axis is then reflected in the x-axis to produce $y = |3x + 3|$.



5 Draw the graph of:

- **a** $y = |x|$
- **b** $y = |x + 3|$
- **c** $y = |6 - 2x|$
- **d** $y = |3x + 1|$
- **e** $y = |10 - 4x|$
- **f** $y = \left|\frac{1}{2}x + 2\right|$

E COMPOSITE FUNCTIONS

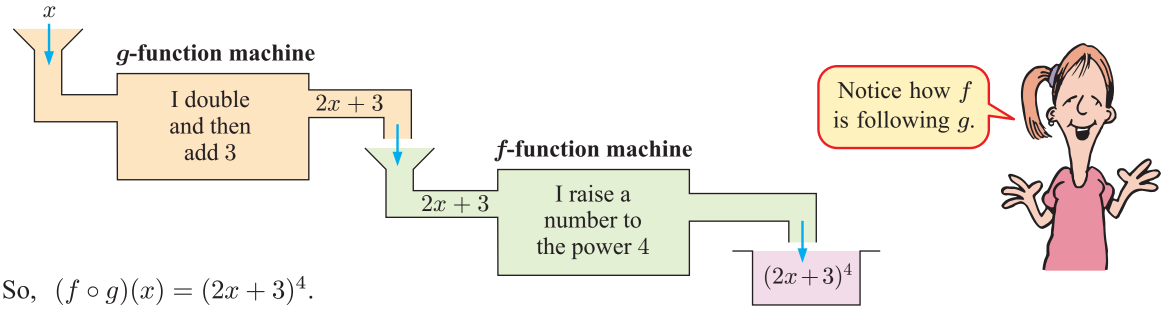
Given $f : x \mapsto f(x)$ and $g : x \mapsto g(x)$, the **composite function** of f and g will convert x into $f(g(x))$.

$f \circ g$ or fg is used to represent the composite function of f and g. It means "f following g".

$$(\boldsymbol{f \circ g})(\boldsymbol{x}) \quad \text{or} \quad \boldsymbol{fg(x) = f(g(x))}$$

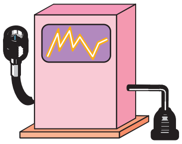
Consider $f : x \mapsto x^4$ and $g : x \mapsto 2x + 3$.

$f \circ g$ means that g converts x to $2x + 3$ and then f converts $(2x + 3)$ to $(2x + 3)^4$.



So, $(f \circ g)(x) = (2x + 3)^4$.

Algebraically, if $f(x)=x^4$ and $g(x)=2x+3$ then



$$\begin{aligned}(f\circ g)(x) &= f(g(x))\\ &= f(2x+3) && \{g \text{ operates on } x \text{ first}\}\\ &= (2x+3)^4 && \{f \text{ operates on } g(x) \text{ next}\}\end{aligned}$$

and

$$\begin{aligned}(g\circ f)(x) &= g(f(x))\\ &= g(x^4) && \{f \text{ operates on } x \text{ first}\}\\ &= 2(x^4)+3 && \{g \text{ operates on } f(x) \text{ next}\}\\ &= 2x^4+3\end{aligned}$$

So, $f(g(x)) \neq g(f(x))$.

In general, $(f\circ g)(x) \neq (g\circ f)(x)$.

We can also compose a function f with itself. The resulting function is $(f\circ f)(x)$ or $f^2(x)$.

In general, $(f\circ f)(x) \neq (f(x))^2$.

Example 12

Self Tutor

Given $f: x \mapsto 2x+1$ and $g: x\mapsto 3-4x$, find in simplest form:

a $(f\circ g)(x)$ **b** $gf(x)$ **c** $f^2(x)$

$f(x)=2x+1$ and $g(x)=3-4x$

a
$$\begin{aligned}&(f\circ g)(x)\\ &= f(g(x))\\ &= f(3-4x)\\ &= 2(3-4x)+1\\ &= 6-8x+1\\ &= 7-8x\end{aligned}$$

b
$$\begin{aligned}&gf(x)\\ &= g(f(x))\\ &= g(2x+1)\\ &= 3-4(2x+1)\\ &= 3-8x-4\\ &= -8x-1\end{aligned}$$

c
$$\begin{aligned}&f^2(x)\\ &= f(f(x))\\ &= f(2x+1)\\ &= 2(2x+1)+1\\ &= 4x+2+1\\ &= 4x+3\end{aligned}$$

In the previous **Example** you should have observed how we can substitute an expression into a function.

If $f(x)=2x+1$ then

$$\begin{aligned}f(\Delta) &= 2(\Delta)+1\\ \therefore\quad f(3-4x) &= 2(3-4x)+1.\end{aligned}$$

Example 13

Self Tutor

Given $f(x)=6x-5$ and $g(x)=x^2+x$, determine:

a $(g\circ f)(-1)$ **b** $(f\circ f)(0)$

a $(g\circ f)(-1) = g(f(-1))$

Now $f(-1) = 6(-1)-5$
$= -11$

$$\begin{aligned}\therefore\quad (g\circ f)(-1) &= g(-11)\\ &= (-11)^2+(-11)\\ &= 110\end{aligned}$$

b $(f\circ f)(0) = f(f(0))$

Now $f(0) = 6(0)-5$
$= -5$

$$\begin{aligned}\therefore\quad (f\circ f)(0) &= f(-5)\\ &= 6(-5)-5\\ &= -35\end{aligned}$$

The domain of the composite of two functions depends on the domain of the original functions.

For example, consider $f(x)=x^2$ with domain $x \in \mathbb{R}$ and $g(x)=\sqrt{x}$ with domain $x \geqslant 0$.

$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= (\sqrt{x})^2\\ &= x\end{aligned}$$

The domain of $(f \circ g)(x)$ is $x \geqslant 0$, not $\mathbb{R}$, since $(f \circ g)(x)$ is defined using function $g(x)$.

EXERCISE 2E

1 Given $f : x \mapsto 2x+3$ and $g : x \mapsto 1-x$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$ **c** $(f \circ g)(-3)$

2 Given $f(x)=2+x$ and $g(x)=3-x$, find:

a $fg(x)$ **b** $gf(x)$ **c** $f^2(x)$

3 Given $f(x)=\sqrt{6-x}$ and $g(x)=5x-7$, find:

a $(g \circ g)(x)$ **b** $(f \circ g)(1)$ **c** $(g \circ f)(6)$

4 Given $f : x \mapsto x^2$ and $g : x \mapsto 2-x$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.
Find also the domain and range of $f \circ g$ and $g \circ f$.

5 Suppose $f(x)=3x+5$ and $g(x)=2x-3$.

a Find $(f \circ g)(x)$. **b** Solve $(f \circ g)(x)=g(x-2)$.

6 Suppose $f : x \mapsto x^2+1$ and $g : x \mapsto 3-x$.

a Find in simplest form: **i** $fg(x)$ **ii** $gf(x)$

b Find the value(s) of x such that $gf(x)=f(x)$.

7 **a** If $ax+b=cx+d$ for all values of x, show that $a=c$ and $b=d$.
Hint: If it is true for all x, it is true for $x=0$ and $x=1$.

b Given $f(x)=2x+3$ and $g(x)=ax+b$ and that $(f \circ g)(x)=x$ for all values of x, deduce that $a=\frac{1}{2}$ and $b=-\frac{3}{2}$.

c Is the result in **b** true if $(g \circ f)(x)=x$ for all x?

8 Given $f(x)=\sqrt{1-x}$ and $g(x)=x^2$, find:

a $(f \circ g)(x)$ **b** the domain and range of $(f \circ g)(x)$.

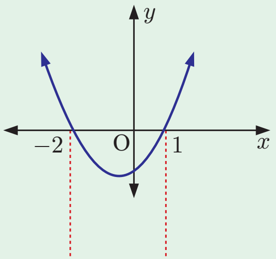
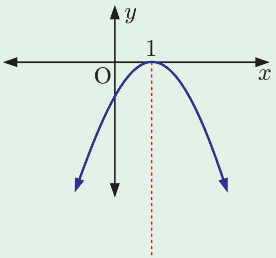
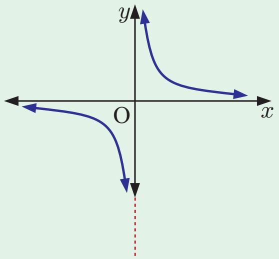
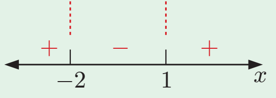
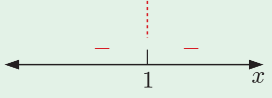
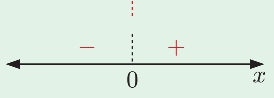
F SIGN DIAGRAMS

Sometimes we do not wish to draw a time-consuming graph of a function but wish to know when the function is positive, negative, zero, or undefined. A **sign diagram** enables us to do this and is relatively easy to construct.

For the function $f(x)$, the sign diagram consists of:

- a **horizontal line** which is really the x-axis
- **positive** $(+)$ and **negative** $(-)$ signs indicating that the graph is **above** and **below** the x-axis respectively
- the **zeros** of the function, which are the x-intercepts of the graph of $y=f(x)$, and the **roots** of the equation $f(x)=0$
- values of x where the graph is undefined.

Consider the three functions given below.

Function	$y = (x+2)(x-1)$	$y = -2(x-1)^2$	$y = \frac{4}{x}$
Graph			
Sign diagram			

You should notice that:

- A sign change occurs about a zero of the function for single linear factors such as $(x+2)$ and $(x-1)$. This indicates **cutting** of the x-axis.
- No sign change occurs about a zero of the function for squared linear factors such as $(x-1)^2$. This indicates **touching** of the x-axis.
-  indicates that a function is **undefined** at $x = 0$.

In general:

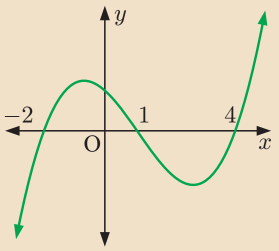
- when a linear factor has an **odd power** there is a change of sign about that zero
- when a linear factor has an **even power** there is no sign change about that zero.

Example 14

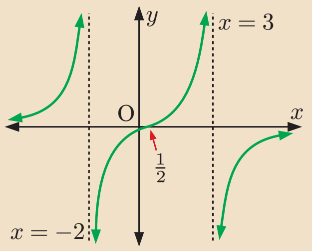
Self Tutor

Draw sign diagrams for:

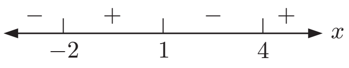
a



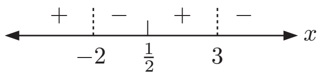
b



a



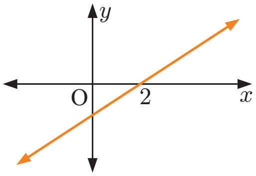
b



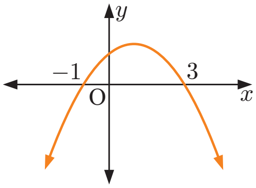
EXERCISE 2F

1 Draw sign diagrams for these graphs:

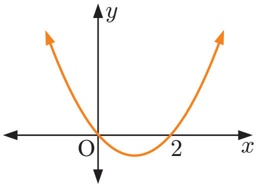
a



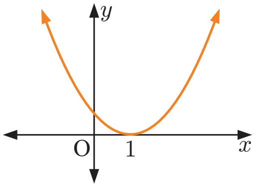
b



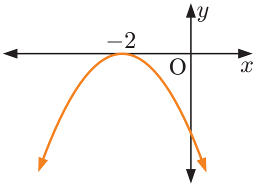
c



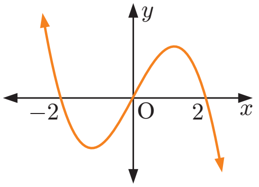
d



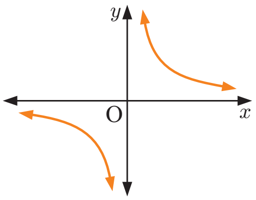
e



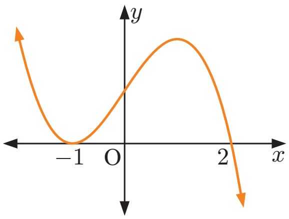
f



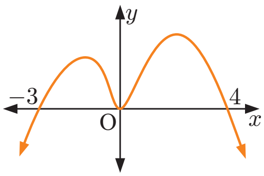
g



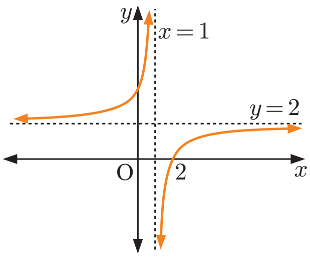
h



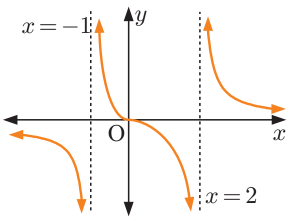
i



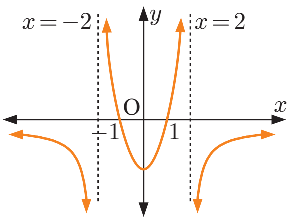
j



k



l



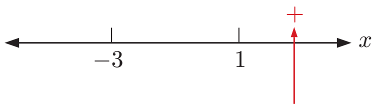
Example 15 Self Tutor

Draw a sign diagram for:

a $(x+3)(x-1)$

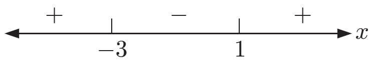
b $-4(x-3)^2$

a $(x+3)(x-1)$ has zeros -3 and 1.

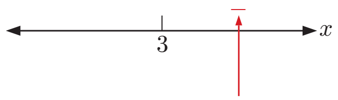


We substitute any number > 1.
When $x = 2$ we have $(5)(1) > 0$,
so we put a $+$ sign here.

As the factors are single, the signs alternate.

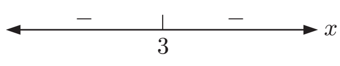


b $-4(x-3)^2$ has zero 3.



We substitute any number > 3.
When $x = 4$ we have $-4(1)^2 < 0$,
so we put a $-$ sign here.

As the factor is squared, the signs do not change.



2 Draw sign diagrams for:

a $(x+4)(x-2)$ **b** $x(x-3)$ **c** $x(x+2)$

d $-(x+1)(x-3)$ **e** $(2x-1)(3-x)$ **f** $(5-x)(1-2x)$

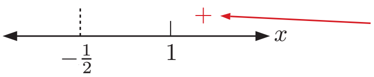
g $(x+2)^2$ **h** $2(x-3)^2$ **i** $-3(x+4)^2$

Example 16

Self Tutor

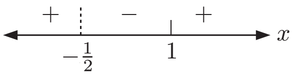
Draw a sign diagram for $\dfrac{x-1}{2x+1}$.

$\dfrac{x-1}{2x+1}$ is zero when $x=1$ and undefined when $x=-\frac{1}{2}$.



When $x=10$, $\dfrac{x-1}{2x+1}=\dfrac{9}{21}>0$

Since $(x-1)$ and $(2x+1)$ are single factors, the signs alternate.



3 Draw sign diagrams for:

a $\dfrac{x+2}{x-1}$ **b** $\dfrac{x}{x+3}$ **c** $\dfrac{2x+3}{4-x}$

d $\dfrac{4x-1}{2-x}$ **e** $\dfrac{3x}{x-2}$ **f** $\dfrac{-8x}{3-x}$

g $\dfrac{(x-1)^2}{x}$ **h** $\dfrac{4x}{(x+1)^2}$ **i** $\dfrac{(x+2)(x-1)}{3-x}$

j $\dfrac{x(x-1)}{2-x}$ **k** $\dfrac{(x+2)(x-2)}{-x}$ **l** $\dfrac{3-x}{(2x+3)(x-2)}$

4 Draw sign diagrams for:

a $1+\dfrac{3}{x+1}$ **b** $x-\dfrac{1}{x}$ **c** $x-\dfrac{1}{x^2}$

G INVERSE FUNCTIONS

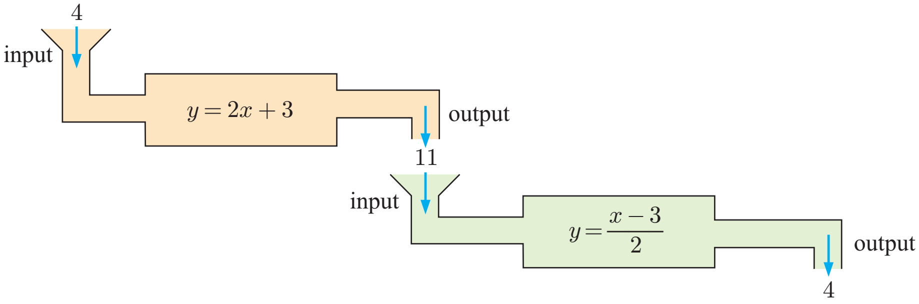
The operations of + and −, × and ÷, are **inverse operations** as one undoes what the other does.

For example, $x+3-3=x$ and $x\times 3\div 3=x$.

The function $y=2x+3$ can be "undone" by its **inverse function** $y=\dfrac{x-3}{2}$.

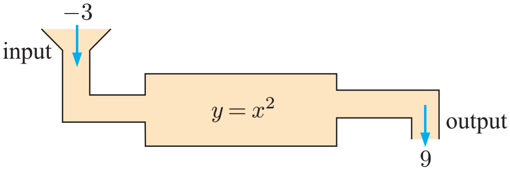
We can think of this as two machines. If the machines are inverses then the second machine *undoes* what the first machine does.

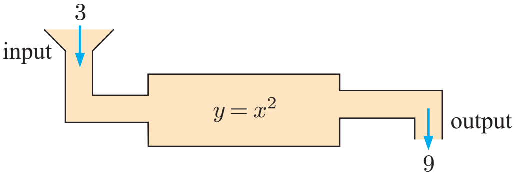
No matter what value of x enters the first machine, it is returned as the output from the second machine.



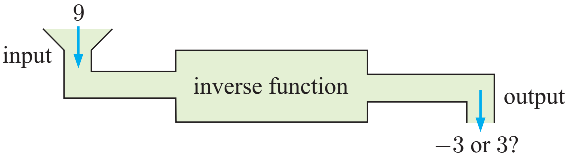
However, not all functions have an inverse function.

For example, consider the function $y = x^2$. The inputs -3 and 3 both produce an output of 9.





So, if we gave an inverse function the input 9, how would it know whether the output should be -3 or 3? It cannot answer both, since the inverse function would fail the vertical line test.



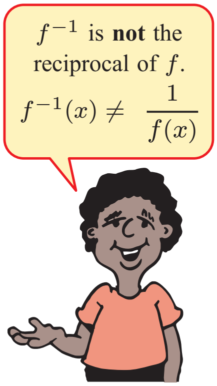
So, if a function has two inputs which produce the same output, then the function does not have an inverse function.

For a function to have an **inverse**, the function must be **one-one**. It must pass the horizontal line test.

If $y = f(x)$ has an **inverse function**, this new function:

- is denoted $f^{-1}(x)$
- is the reflection of $y = f(x)$ in the line $y = x$
- satisfies $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$.

The function $y = x$, defined as $f : x \mapsto x$, is the **identity function**.

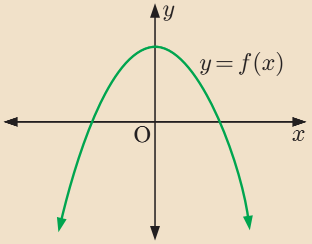


Example 17

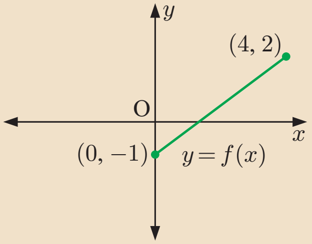
Self Tutor

If $y = f(x)$ has an inverse function, sketch $y = f^{-1}(x)$, and state the domain and range of $f(x)$ and $f^{-1}(x)$.

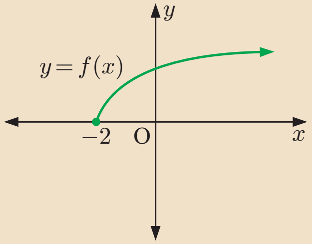
a



b

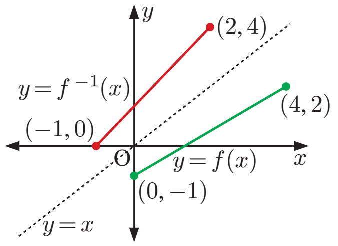


c



a The function fails the horizontal line test, so it is not one-one. The function does not have an inverse function.

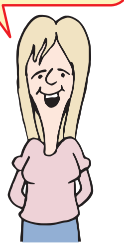
b



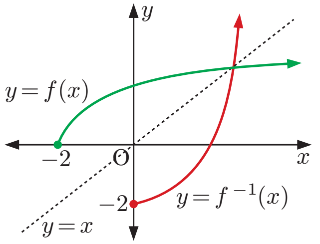
$f(x)$ has domain $\{x : 0 \leqslant x \leqslant 4\}$
and range $\{y : -1 \leqslant y \leqslant 2\}$.

$f^{-1}(x)$ has domain $\{x : -1 \leqslant x \leqslant 2\}$
and range $\{y : 0 \leqslant y \leqslant 4\}$.

$y = f^{-1}(x)$ is the reflection of $y = f(x)$ in the line $y = x$.



c



$f(x)$ has domain $\{x : x \geqslant -2\}$
and range $\{y : y \geqslant 0\}$.

$f^{-1}(x)$ has domain $\{x : x \geqslant 0\}$
and range $\{y : y \geqslant -2\}$.

From **Example 17**, we can see that:

The domain of f^{-1} is equal to the range of f.

The range of f^{-1} is equal to the domain of f.

If (x, y) lies on f, then (y, x) lies on f^{-1}. Reflecting the function in the line $y = x$ has the algebraic effect of interchanging x and y.

So, if the function is given as an equation, then we interchange the variables to find the equation of the inverse function.

For example, if f is given by $y = 5x + 2$ then f^{-1} is given by $x = 5y + 2$.

Example 18

Self Tutor

Consider $f : x \mapsto 2x+3$.

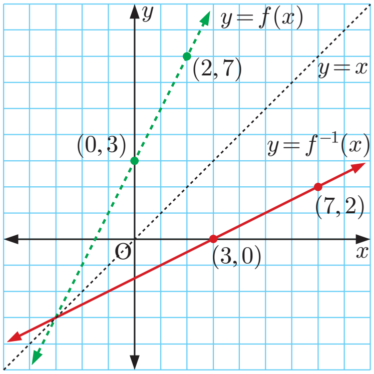
a On the same axes, graph f and its inverse function f^{-1}.

b Find $f^{-1}(x)$ using variable interchange.

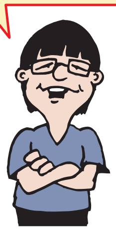
c Check that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$.

a $f(x) = 2x+3$ passes through $(0, 3)$ and $(2, 7)$.

$\therefore$ $f^{-1}(x)$ passes through $(3, 0)$ and $(7, 2)$.



If f includes point (a, b), then f^{-1} includes point (b, a).



b

$$\begin{aligned} f \text{ is } \quad & y = 2x+3, \\ \therefore\ f^{-1} \text{ is } \quad & x = 2y+3 \\ \therefore\ & x-3 = 2y \\ \therefore\ & \frac{x-3}{2} = y \\ \therefore\ & f^{-1}(x) = \frac{x-3}{2} \end{aligned}$$

c

$$\begin{aligned} & (f \circ f^{-1})(x) \\ &= f(f^{-1}(x)) \\ &= f\left(\frac{x-3}{2}\right) \\ &= 2\left(\frac{x-3}{2}\right)+3 \\ &= x \end{aligned}$$

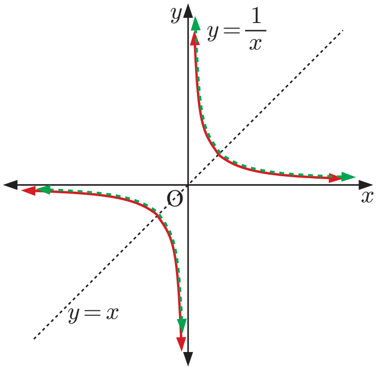
and

$$\begin{aligned} & (f^{-1} \circ f)(x) \\ &= f^{-1}(f(x)) \\ &= f^{-1}(2x+3) \\ &= \frac{(2x+3)-3}{2} \\ &= \frac{2x}{2} \\ &= x \end{aligned}$$

Any function which has an inverse, and whose graph is symmetrical about the line $y = x$, is a **self-inverse function**.

If f is a self-inverse function then $f^{-1} = f$.

For example, the function $f(x) = \frac{1}{x}$, $x \neq 0$, is said to be self-inverse, as $f = f^{-1}$.

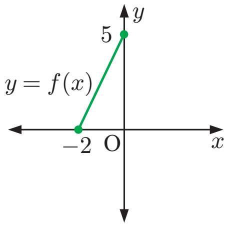


EXERCISE 2G

PRINTABLE GRAPHS

1 If $y = f(x)$ has an inverse function, sketch $y = f^{-1}(x)$, and state the domain and range of $f(x)$ and $f^{-1}(x)$.

a



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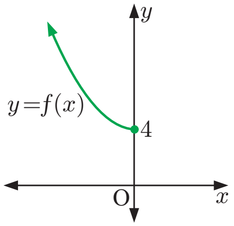

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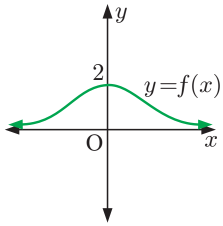

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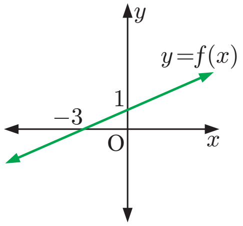
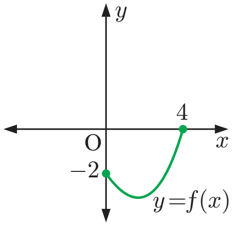
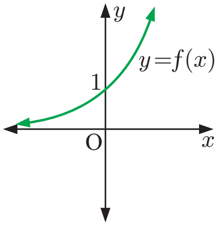


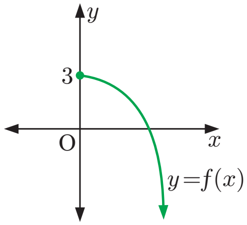
e



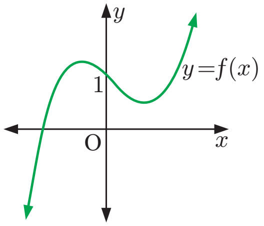
f



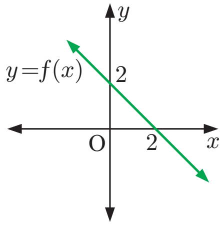
g



h



i



2 Which of the functions in **1** is a self-inverse function?

3 If the domain of $H(x)$ is $\{x : -2 \leqslant x < 3\}$, state the range of $H^{-1}(x)$.

4 For each of the following functions f:

- **i** On the same set of axes, sketch $y = x$, $y = f(x)$, and $y = f^{-1}(x)$.
- **ii** Find $f^{-1}(x)$ using variable interchange.

a $f : x \mapsto 3x + 1$ **b** $f : x \mapsto \dfrac{x+2}{4}$

5 For each of the following functions f:

- **i** Find $f^{-1}(x)$.
- **ii** Sketch $y = f(x)$, $y = f^{-1}(x)$, and $y = x$ on the same set of axes.
- **iii** Show that $(f^{-1} \circ f)(x) = (f \circ f^{-1})(x) = x$, the identity function.

a $f : x \mapsto 2x + 5$ **b** $f : x \mapsto x + 3$ **c** $f : x \mapsto \dfrac{x+6}{2}$

6 Given $f(x) = 2x - 5$, find $(f^{-1})^{-1}(x)$. What do you notice?

7 Sketch the graph of $f : x \mapsto x^3$ and its inverse function $f^{-1}(x)$.

8 Given $f : x \mapsto \dfrac{1}{x}$, $x \neq 0$, find f^{-1} algebraically and show that f is a self-inverse function.

9 Consider the function $f(x) = \frac{1}{2}x - 1$.

- **a** Find $f^{-1}(x)$.
- **b** Find: **i** $(f \circ f^{-1})(x)$ **ii** $(f^{-1} \circ f)(x)$.

10 Consider the functions $f : x \mapsto 2x + 5$ and $g : x \mapsto \dfrac{8-x}{2}$.

- **a** Find $g^{-1}(-1)$.
- **b** Show that $f^{-1}(-3) - g^{-1}(6) = 0$.
- **c** Find x such that $(f \circ g^{-1})(x) = 9$.

11 Consider the functions $f : x \mapsto 5^x$ and $g : x \mapsto \sqrt{x}$.

- **a** Find: **i** $f(2)$ **ii** $g^{-1}(4)$
- **b** Solve the equation $(g^{-1} \circ f)(x) = 25$.

12 Which of these functions is a self-inverse function?

- **a** $f(x) = 2x$
- **b** $f(x) = x$
- **c** $f(x) = -x$
- **d** $f(x) = \dfrac{2}{x}$
- **e** $f(x) = -\dfrac{6}{x}$
- **f** $f(x) = \dfrac{x}{3}$

13 Given $f : x \mapsto 2x$ and $g : x \mapsto 4x - 3$, show that $(f^{-1} \circ g^{-1})(x) = (g \circ f)^{-1}(x)$.

Discovery — Functions and form

We already know that numbers have equivalent forms. For example, $\frac{1}{2}$, $\frac{3}{6}$, $\frac{5}{10}$, and 0.5 all represent the same number.

Similarly, a function might have different, but equivalent, algebraic representations.

Choosing a particular form for an expression helps us understand the behaviour of the function better. By anticipating what you are going to do with your function you can choose a form which will make the task easier.

For example, you will have seen in previous years that the equation of a straight line can be written in:

- gradient-intercept form $y = mx + c$ where m is the gradient and the y-intercept is c
- point-gradient form $y - b = m(x - a)$ where the line goes through (a, b) and has gradient m
- general-form $Ax + By = D$.

A given straight line can be converted between these forms easily, but each emphasises different features of the straight line.

What to do:

1 What different forms have you seen for a quadratic function $y = ax^2 + bx + c$?

2 Two expressions $f(x)$ and $g(x)$ are **equivalent** on the domain D if $f(x) = g(x)$ for all $x \in D$.

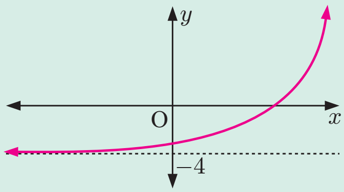
- **a** Discuss whether: $f(x) = \dfrac{x^2-1}{x-1}$ and $g(x) = x + 1$ are equivalent on:
  - **i** $x \in \mathbb{R}$ **ii** $x \in \mathbb{R}^-$ **iii** $\{x : x > 1\}$ **iv** $\{x \in \mathbb{R} : x \neq 1\}$
- **b** When considering algebraically whether two functions are equivalent, what things do we need to be careful about?

  Hint: $\dfrac{x^2-1}{x-1} = \dfrac{(x+1)(x-1)}{x-1} = x + 1$ only if $x \neq 1$.

Review set 2A

1 Determine whether the following relations are functions:

a



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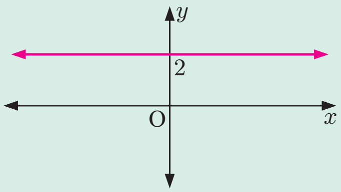
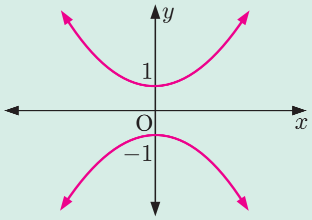
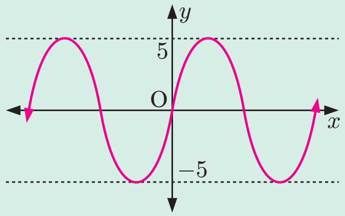


c



d



2 Suppose $f(x) = ax + b$ where a and b are constants. If $f(1) = 7$ and $f(3) = -5$, find a and b.

3 Solve for x:

a $|x - 5| = 7$

b $|2x + 1| = |x - 4|$

4 If $g(x) = x^2 - 3x$, find in simplest form:

a $g(-2)$

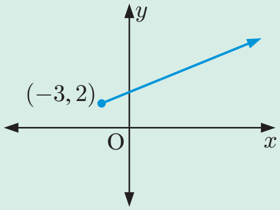
b $g(x + 1)$

5 For each of the following functions:

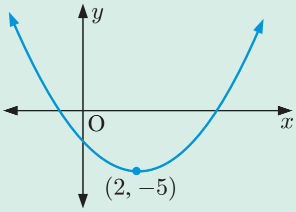
i find the domain and range

ii determine whether the function is one-one.

a



b



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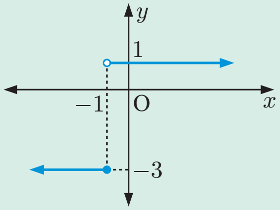
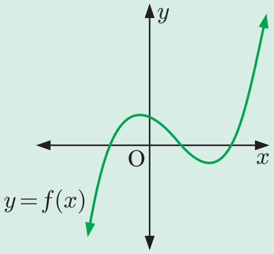


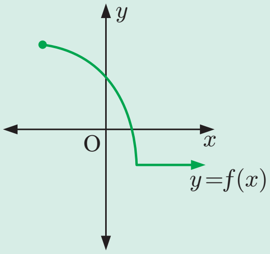
6 Draw the graph of $y = |f(x)|$ for:

PRINTABLE GRAPHS

a



b



7 Draw the graph of $y = |2x - 1|$.

8 Draw a sign diagram for:

a $(3x+2)(4-x)$ **b** $\dfrac{x-3}{(x+2)^2}$

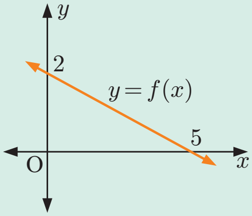
9 If $f(x) = 2x - 3$ and $g(x) = x^2 + 2$, find in simplest form:

a $fg(x)$ **b** $gf(x)$

10 If $y = f(x)$ has an inverse, sketch the graph of $y = f^{-1}(x)$.

PRINTABLE GRAPHS

a



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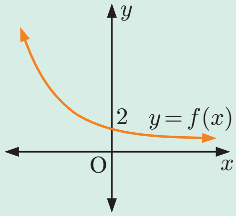
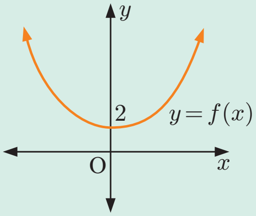


c



11 Find $f^{-1}(x)$ given that $f(x)$ is:

a $4x + 2$ **b** $\dfrac{3-5x}{4}$

12 Consider $f(x) = x^2$ and $g(x) = 1 - 6x$.

a Show that $f(-3) = g(-\frac{4}{3})$. **b** Find $(f \circ g)(-2)$.

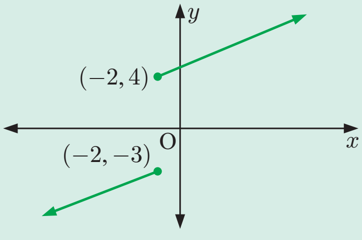
c Find x such that $g(x) = f(5)$.

13 Given $f : x \mapsto 3x + 6$ and $h : x \mapsto \dfrac{x}{3}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

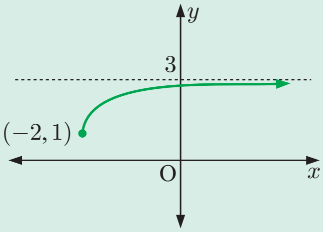
Review set 2B

1 Determine whether the following relations are functions. If they are functions, determine whether they are one-one.

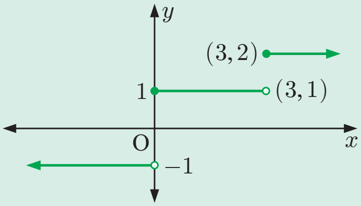
a



b



c



2 Given $f(x) = x^2 + 3$, find:

a $f(-3)$

b x such that $f(x) = 4$.

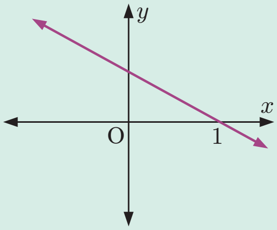
3 Solve for x:

a $|1 - 2x| = 11$

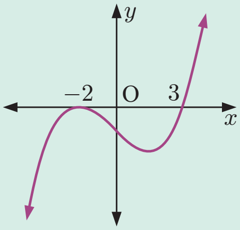
b $|5x - 1| = |9x - 13|$

4 Draw a sign diagram for each graph:

a



b



5 Given $h(x) = 7 - 3x$, find:

a $h(2x - 1)$

b $h^2(x)$

c $h^2(-1)$

6 Suppose the range of $y = f(x)$ is $\{y : -7 \leqslant y \leqslant -3\}$. Write down the range of $y = |f(x)|$.

7 Draw the graph of $y = \left|1 - \frac{1}{3}x\right|$.

8 Suppose $f(x) = 1 - 2x$ and $g(x) = 5x$.

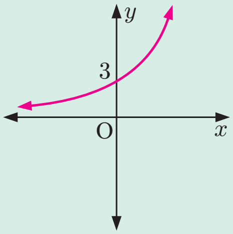
a Find in simplest form: **i** $fg(x)$ **ii** $gf(x)$.

b Solve $fg(x) = g(x + 2)$.

9 Suppose $f(x) = ax^2 + bx + c$, $f(0) = 5$, $f(-2) = 21$, and $f(3) = -4$. Find a, b, and c.

10 If $y = f(x)$ has an inverse, sketch the graph of $y = f^{-1}(x)$:

a



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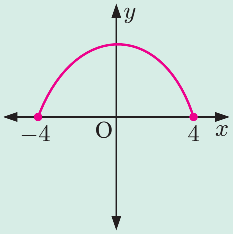

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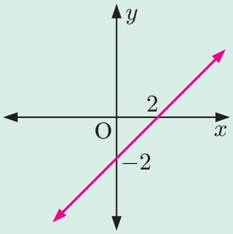


PRINTABLE DIAGRAMS

11 Find the inverse function $f^{-1}(x)$ for:

a $f(x) = 7 - 4x$

b $f(x) = \dfrac{3 + 2x}{5}$

12 Given $f : x \mapsto 5x - 2$ and $h : x \mapsto \dfrac{3x}{4}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

13 Given $f(x) = 2x + 11$ and $g(x) = x^2$, find $(g \circ f^{-1})(3)$.

3

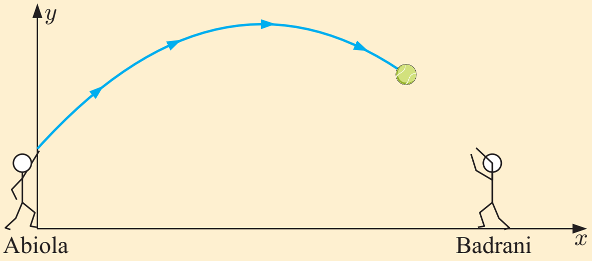
Quadratics

Contents:

Opening problem

Abiola and Badrani are standing 40 metres apart, throwing a ball between them. When Abiola throws the ball, it travels in a smooth arc. At the time when the ball has travelled x metres horizontally towards Badrani, its height is y metres.

SIMULATION



x (m)	0	5	10	15	20	25	30
y (m)	1.25	10	16.25	20	21.25	20	16.25

Things to think about:

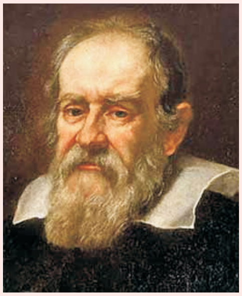
- **a** Use technology to plot these points.
- **b** What *shape* is the graph of y against x?
- **c** What is the maximum height reached by the ball?
- **d** What *formula* gives the height of the ball when it has travelled x metres horizontally towards Badrani?
- **e** Will the ball reach Badrani before it bounces?

Historical note

Galileo Galilei (1564 - 1642) was born in Pisa, Tuscany. He was a philosopher who played a significant role in the scientific revolution of that time.

Within his research he conducted a series of experiments on the paths of projectiles, attempting to find a mathematical description of falling bodies.

Two of Galileo's experiments consisted of rolling a ball down a grooved ramp that was placed at a fixed height above the floor and inclined at a fixed angle to the horizontal. In one experiment the ball left the end of the ramp and descended to the floor. In the second, a horizontal shelf was placed at the end of the ramp, and the ball travelled along this shelf before descending to the floor.


Galileo

In each experiment Galileo altered the release height h of the ball and measured the distance d the ball travelled before landing. The units of measurement were called 'punti' (points).

In both experiments Galileo found that once the ball left the ramp or shelf, its path was parabolic and could therefore be modelled by a quadratic function.

QUADRATICS

A **quadratic equation** is an equation of the form $ax^2 + bx + c = 0$ where a, b, and c are constants, $a \neq 0$.

A **quadratic function** is a function of the form $y = ax^2 + bx + c$, $a \neq 0$.

Quadratic functions are members of the family of **polynomials**. The first few members of this family are shown in the table.

Polynomial function	*Type*
$y = ax + b,\ \ a \neq 0$	linear
$y = ax^2 + bx + c,\ \ a \neq 0$	quadratic
$y = ax^3 + bx^2 + cx + d,\ \ a \neq 0$	cubic
$y = ax^4 + bx^3 + cx^2 + dx + e,\ \ a \neq 0$	quartic

A QUADRATIC EQUATIONS

Acme Leather Jacket Co. makes and sells x leather jackets each week. Their profit function is given by $P = -12.5x^2 + 550x - 2125$ dollars.

How many jackets must be made and sold each week in order to obtain a weekly profit of $3000?

Clearly we need to solve the equation:

$$-12.5x^2 + 550x - 2125 = 3000$$

We can rearrange the equation to give

$$12.5x^2 - 550x + 5125 = 0,$$

which is of the form $ax^2 + bx + c = 0$ and is thus a quadratic equation.

SOLVING QUADRATIC EQUATIONS

To solve quadratic equations we have the following methods to choose from:

- **factorise** the quadratic and use the rule:

  If $ab = 0$ then $a = 0$ or $b = 0$.

- **complete the square**
- use the **quadratic formula**
- use **technology**.

The **roots** or **solutions** of $ax^2 + bx + c = 0$ are the values of x which satisfy the equation, or make it true.

For example: Consider $x^2 - 3x + 2 = 0$.

When $x = 2$,
$$\begin{aligned} x^2 - 3x + 2 &= (2)^2 - 3(2) + 2 \\ &= 4 - 6 + 2 \\ &= 0 \quad \checkmark \end{aligned}$$

So, $x = 2$ is a root of the equation $x^2 - 3x + 2 = 0$.

SOLVING BY FACTORISATION

Step 1: If necessary, rearrange the equation so one side is zero.

Step 2: Fully factorise the other side.

Step 3: Apply the rule: If $ab = 0$ then $a = 0$ or $b = 0$.

Step 4: Solve the resulting linear equations.

Example 1

Self Tutor

Solve for x:

a $3x^2 + 5x = 0$

b $x^2 = 5x + 6$

a
$$\begin{aligned} 3x^2 + 5x &= 0 \\ \therefore\ x(3x+5) &= 0 \\ \therefore\ x = 0 \text{ or } 3x+5 &= 0 \\ \therefore\ x = 0 \text{ or } x &= -\tfrac{5}{3} \end{aligned}$$

b
$$\begin{aligned} x^2 &= 5x + 6 \\ \therefore\ x^2 - 5x - 6 &= 0 \\ \therefore\ (x-6)(x+1) &= 0 \\ \therefore\ x &= 6 \text{ or } -1 \end{aligned}$$

Example 2

Self Tutor

Solve for x:

a $4x^2 + 1 = 4x$

b $6x^2 = 11x + 10$

a
$$\begin{aligned} 4x^2 + 1 &= 4x \\ \therefore\ 4x^2 - 4x + 1 &= 0 \\ \therefore\ (2x-1)^2 &= 0 \\ \therefore\ x &= \tfrac{1}{2} \end{aligned}$$

b
$$\begin{aligned} 6x^2 &= 11x + 10 \\ \therefore\ 6x^2 - 11x - 10 &= 0 \\ \therefore\ (2x-5)(3x+2) &= 0 \\ \therefore\ x &= \tfrac{5}{2} \text{ or } -\tfrac{2}{3} \end{aligned}$$

Caution:

- Do not be tempted to divide both sides by an expression involving x.
  If you do this then you may lose one of the solutions.

  For example, consider $x^2 = 5x$.

  Correct solution
  $$\begin{aligned} x^2 &= 5x \\ \therefore\ x^2 - 5x &= 0 \\ \therefore\ x(x-5) &= 0 \\ \therefore\ x &= 0 \text{ or } 5 \end{aligned}$$

  Incorrect solution
  $$\begin{aligned} x^2 &= 5x \\ \therefore\ \frac{x^2}{x} &= \frac{5x}{x} \\ \therefore\ x &= 5 \end{aligned}$$

  By dividing both sides by x, we lose the solution $x = 0$.

- Be careful when taking square roots of both sides of an equation. You may otherwise lose solutions.
  For example:
  - Consider $x^2 = 25$.

    Correct solution
    $$\begin{aligned} x^2 &= 25 \\ \therefore\ x &= \pm\sqrt{25} \\ \therefore\ x &= \pm 5 \end{aligned}$$

    Incorrect solution
    $$\begin{aligned} x^2 &= 25 \\ \therefore\ x &= \sqrt{25} \\ \therefore\ x &= 5 \end{aligned}$$

- Consider $(2x-7)^2=(x+1)^2$.

Correct solution

$$\begin{aligned} (2x-7)^2 &= (x+1)^2 \\ \therefore\ (2x-7)^2-(x+1)^2 &= 0 \\ \therefore\ (2x-7+x+1)(2x-7-x-1) &= 0 \\ \therefore\ (3x-6)(x-8) &= 0 \\ \therefore\ x &= 2 \text{ or } 8 \end{aligned}$$

Incorrect solution

$$\begin{aligned} (2x-7)^2 &= (x+1)^2 \\ \therefore\ 2x-7 &= x+1 \\ \therefore\ x &= 8 \end{aligned}$$

EXERCISE 3A.1

1 Solve the following by factorisation:

a $4x^2+7x=0$
b $6x^2+2x=0$
c $3x^2-7x=0$
d $2x^2-11x=0$
e $3x^2=8x$
f $9x=6x^2$
g $x^2-5x+6=0$
h $x^2=2x+8$
i $x^2+21=10x$
j $9+x^2=6x$
k $x^2+x=12$
l $x^2+8x=33$

2 Solve the following by factorisation:

a $9x^2-12x+4=0$
b $2x^2-13x-7=0$
c $3x^2=16x+12$
d $3x^2+5x=2$
e $2x^2+3=5x$
f $3x^2+8x+4=0$
g $3x^2=10x+8$
h $4x^2+4x=3$
i $4x^2=11x+3$
j $12x^2=11x+15$
k $7x^2+6x=1$
l $15x^2+2x=56$

Example 3 Self Tutor

Solve for x: $3x+\frac{2}{x}=-7$

$$\begin{aligned} 3x+\frac{2}{x} &= -7 \\ \therefore\ x\left(3x+\frac{2}{x}\right) &= -7x && \{\text{multiplying both sides by } x\} \\ \therefore\ 3x^2+2 &= -7x && \{\text{expanding the brackets}\} \\ \therefore\ 3x^2+7x+2 &= 0 && \{\text{making the RHS } 0\} \\ \therefore\ (x+2)(3x+1) &= 0 && \{\text{factorising}\} \\ \therefore\ x &= -2 \text{ or } -\tfrac{1}{3} \end{aligned}$$

3 Solve for x:

a $(x+1)^2=2x^2-5x+11$
b $(x+2)(1-x)=-4$
c $5-4x^2=3(2x+1)+2$
d $x+\frac{2}{x}=3$
e $2x-\frac{1}{x}=-1$
f $\frac{x+3}{1-x}=-\frac{9}{x}$

SOLVING BY 'COMPLETING THE SQUARE'

As you would be aware by now, not all quadratics factorise easily. For example, x^2+4x+1 cannot be factorised by simple factorisation. In other words, we cannot write x^2+4x+1 in the form $(x-a)(x-b)$ where a, b are rational.

An alternative way to solve equations like $x^2 + 4x + 1 = 0$ is by 'completing the square'.

Equations of the form $ax^2 + bx + c = 0$ can be converted to the form $(x + p)^2 = q$, from which the solutions are easy to obtain.

Example 4

Self Tutor

Solve exactly for x:

a $(x + 2)^2 = 7$

b $(x - 1)^2 = -5$

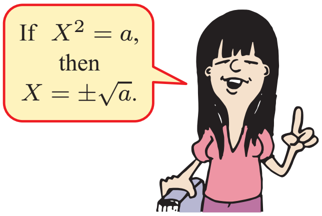
a $(x + 2)^2 = 7$

$\therefore \quad x + 2 = \pm\sqrt{7}$

$\therefore \quad x = -2 \pm \sqrt{7}$

b $(x - 1)^2 = -5$

has no real solutions since the square $(x - 1)^2$ cannot be negative.



The completed square form of an equation is $(x + p)^2 = q$.

If we expand this out, $x^2 + 2px + p^2 = q$.

Notice that the *coefficient of* x equals $2p$. Therefore, p is half the coefficient of x in the expanded form.

If we have $x^2 + 2px = q$, then we "complete the square" by adding in p^2 to both sides of the equation.

Example 5

Self Tutor

Solve for exact values of x: $x^2 + 4x + 1 = 0$

$x^2 + 4x + 1 = 0$

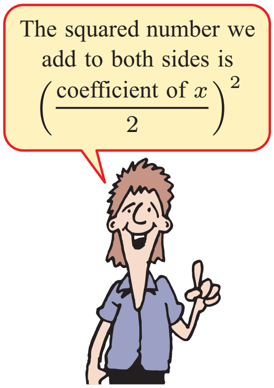
$\therefore \quad x^2 + 4x = -1$ {put the constant on the RHS}

$\therefore \quad x^2 + 4x + 2^2 = -1 + 2^2$ {completing the square}

$\therefore \quad (x + 2)^2 = 3$ {factorising LHS}

$\therefore \quad x + 2 = \pm\sqrt{3}$

$\therefore \quad x = -2 \pm \sqrt{3}$



Example 6

Self Tutor

Solve exactly for x: $-3x^2 + 12x + 5 = 0$

$-3x^2 + 12x + 5 = 0$

$\therefore \quad x^2 - 4x - \frac{5}{3} = 0$ {dividing both sides by -3}

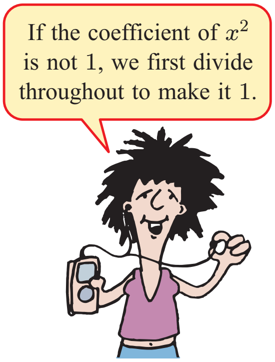
$\therefore \quad x^2 - 4x = \frac{5}{3}$ {putting the constant on the RHS}

$\therefore \quad x^2 - 4x + 2^2 = \frac{5}{3} + 2^2$ {completing the square}

$\therefore \quad (x - 2)^2 = \frac{17}{3}$ {factorising LHS}

$\therefore \quad x - 2 = \pm\sqrt{\frac{17}{3}}$

$\therefore \quad x = 2 \pm \sqrt{\frac{17}{3}}$



EXERCISE 3A.2

1 Solve exactly for x:

a $(x+5)^2 = 2$ **b** $(x+6)^2 = -11$ **c** $(x-4)^2 = 8$

d $(x-8)^2 = 7$ **e** $2(x+3)^2 = 10$ **f** $3(x-2)^2 = 18$

g $(x+1)^2 + 1 = 11$ **h** $(2x+1)^2 = 3$ **i** $(1-3x)^2 - 7 = 0$

2 Solve exactly by completing the square:

a $x^2 - 4x + 1 = 0$ **b** $x^2 + 6x + 2 = 0$ **c** $x^2 - 14x + 46 = 0$

d $x^2 = 4x + 3$ **e** $x^2 + 6x + 7 = 0$ **f** $x^2 = 2x + 6$

g $x^2 + 6x = 2$ **h** $x^2 + 10 = 8x$ **i** $x^2 + 6x = -11$

3 Solve exactly by completing the square:

a $2x^2 + 4x + 1 = 0$ **b** $2x^2 - 10x + 3 = 0$ **c** $3x^2 + 12x + 5 = 0$

d $3x^2 = 6x + 4$ **e** $5x^2 - 15x + 2 = 0$ **f** $4x^2 + 4x = 5$

4 Solve for x:

a $3x - \frac{2}{x} = 4$ **b** $1 - \frac{1}{x} = -5x$ **c** $3 + \frac{1}{x^2} = -\frac{5}{x}$

5 Suppose $ax^2 + bx + c = 0$ where a, b, and c are constants, $a \neq 0$.
Solve for x by completing the square.

THE QUADRATIC FORMULA

Historical note — The quadratic formula

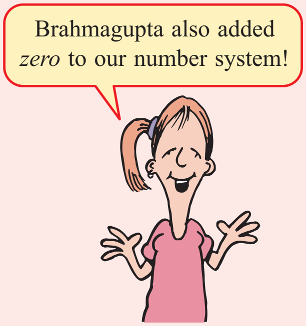
Thousands of years ago, people knew how to calculate the area of a shape given its side lengths. When they wanted to find the side lengths necessary to give a certain area, however, they ended up with a quadratic equation which they needed to solve.

The first known solution of a quadratic equation is written on the Berlin Papyrus from the Middle Kingdom (2160 - 1700 BC) in Egypt. By 400 BC, the Babylonians were using the method of 'completing the square'.

Pythagoras and **Euclid** both used geometric methods to explore the problem. Pythagoras noted that the square root was not always an integer, but he refused to accept that irrational solutions existed. Euclid also discovered that the square root was not always rational, but concluded that irrational numbers *did* exist.

A major jump forward was made in India around 700 AD, when Hindu mathematician **Brahmagupta** devised a general (but incomplete) solution for the quadratic equation $ax^2 + bx = c$ which was equivalent to $x = \dfrac{\sqrt{4ac+b^2} - b}{2a}$. Taking into account the sign of c, this is one of the two solutions we know today.

The final, complete solution as we know it today first came around 1100 AD, by another Hindu mathematician called **Baskhara**. He was the first to recognise that any positive number has two square roots, which could be negative or irrational. In fact, the quadratic formula is known in some countries today as 'Baskhara's Formula'.



While the Indians had knowledge of the quadratic formula even at this early stage, it took somewhat longer for the quadratic formula to arrive in Europe.

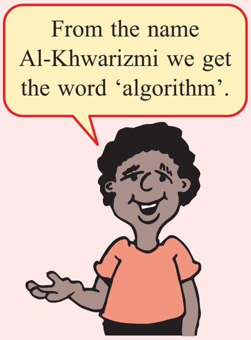
Muhammad Al-Khwarizmi

Around 820 AD, the Islamic mathematician **Muhammad bin Musa Al-Khwarizmi**, who was familiar with the work of Brahmagupta, recognised that for a quadratic equation to have real solutions, the value $b^2 - 4ac$ could not be negative. Al-Khwarizmi's work was brought to Europe by the Jewish mathematician and astronomer **Abraham bar Hiyya** (also known as Savasorda) who lived in Barcelona around 1100.

By 1545, **Girolamo Cardano** had blended the algebra of Al-Khwarizmi with the Euclidean geometry. His work allowed for the existence of complex or imaginary roots, as well as negative and irrational roots.

At the end of the 16th Century the mathematical notation and symbolism was introduced by **François Viète** in France.

In 1637, when **René Descartes** published *La Géométrie*, the quadratic formula adopted the form we see today.



If $\boldsymbol{ax^2 + bx + c = 0}$, $\boldsymbol{a \neq 0}$, then $\boldsymbol{x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$.

Proof:

$$\text{If} \quad ax^2 + bx + c = 0, \quad a \neq 0$$

$$\text{then} \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \{\text{dividing each term by } a, \text{ as } a \neq 0\}$$

$$\therefore \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\therefore \quad x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \qquad \{\text{completing the square on LHS}\}$$

$$\therefore \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \{\text{factorising}\}$$

$$\therefore \quad x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

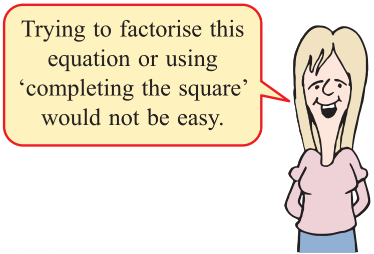
For example, consider the Acme Leather Jacket Co. equation from page **65**.

We need to solve: $12.5x^2 - 550x + 5125 = 0$
so in this case $a = 12.5$, $b = -550$, $c = 5125$

$$\therefore \quad x = \frac{550 \pm \sqrt{(-550)^2 - 4(12.5)(5125)}}{2(12.5)}$$

$$= \frac{550 \pm \sqrt{46\,250}}{25}$$

$$\approx 30.60 \text{ or } 13.40$$



However, for this application the number of jackets x needs to be a whole number, so $x = 13$ or 31 would produce a profit of around \$3000 each week.

Example 7

Self Tutor

Solve for x:

a $x^2 - 2x - 6 = 0$

b $2x^2 + 3x - 6 = 0$

a $x^2 - 2x - 6 = 0$ has
$a = 1$, $b = -2$, $c = -6$

$$\therefore \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-6)}}{2(1)}$$

$$\therefore \quad x = \frac{2 \pm \sqrt{4 + 24}}{2}$$

$$\therefore \quad x = \frac{2 \pm \sqrt{28}}{2}$$

$$\therefore \quad x = \frac{2 \pm 2\sqrt{7}}{2}$$

$$\therefore \quad x = 1 \pm \sqrt{7}$$

b $2x^2 + 3x - 6 = 0$ has
$a = 2$, $b = 3$, $c = -6$

$$\therefore \quad x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-6)}}{2(2)}$$

$$\therefore \quad x = \frac{-3 \pm \sqrt{9 + 48}}{4}$$

$$\therefore \quad x = \frac{-3 \pm \sqrt{57}}{4}$$

EXERCISE 3A.3

1 Use the quadratic formula to solve exactly for x:

a $x^2 - 4x - 3 = 0$
b $x^2 + 6x + 7 = 0$
c $x^2 + 1 = 4x$
d $x^2 + 4x = 1$
e $x^2 - 4x + 2 = 0$
f $2x^2 - 2x - 3 = 0$
g $3x^2 - 5x - 1 = 0$
h $-x^2 + 4x + 6 = 0$
i $-2x^2 + 7x - 2 = 0$

2 Rearrange the following equations so they are written in the form $ax^2 + bx + c = 0$, then use the quadratic formula to solve exactly for x.

a $(x + 2)(x - 1) = 2 - 3x$
b $(2x + 1)^2 = 3 - x$
c $(x - 2)^2 = 1 + x$
d $(3x + 1)^2 = -2x$
e $(x + 3)(2x + 1) = 9$
f $(2x + 3)(2x - 3) = x$
g $\dfrac{x - 1}{2 - x} = 2x + 1$
h $x - \dfrac{1}{x} = 1$
i $2x - \dfrac{1}{x} = 3$

B QUADRATIC INEQUALITIES

An **equation** is a mathematical statement that two expressions are equal.

Sometimes we have a statement that one expression is *greater than*, or else *greater than or equal to*, another. We call this an **inequality**.

$x^2 + 7x > 18$ is an example of a quadratic inequality.

While quadratic equations have 0, 1, or 2 solutions, quadratic inequalities may have 0, 1, or infinitely many solutions. We use interval notation to describe the set of solutions.

To solve quadratic inequalities we use these steps:

- Make the RHS zero by shifting all terms to the LHS.
- Fully factorise the LHS.
- Draw a sign diagram for the LHS.
- Determine the values required from the sign diagram.

Example 8

Self Tutor

Solve for x:

a $3x^2 + 5x \geqslant 2$

b $x^2 + 9 < 6x$

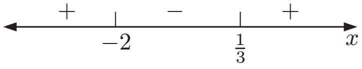
a

$$3x^2 + 5x \geqslant 2$$

$\therefore\ 3x^2 + 5x - 2 \geqslant 0$ {make RHS zero}

$\therefore\ (3x-1)(x+2) \geqslant 0$ {factorising LHS}

Sign diagram of LHS is



$\therefore\ x \leqslant -2$ or $x \geqslant \frac{1}{3}$.

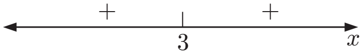
b

$$x^2 + 9 < 6x$$

$\therefore\ x^2 - 6x + 9 < 0$ {make RHS zero}

$\therefore\ (x-3)^2 < 0$ {factorising LHS}

Sign diagram of LHS is



So, the inequality is not true for any real x.

EXERCISE 3B

1 Solve for x:

a $(x-2)(x+3) \geqslant 0$

b $(x+1)(x-4) < 0$

c $(2x+1)(x-3) > 0$

d $x^2 - x \geqslant 0$

e $x^2 \geqslant 3x$

f $3x^2 + 2x < 0$

g $x^2 < 4$

h $2x^2 \geqslant 18$

i $x^2 + 4x + 4 > 0$

j $x^2 + 2x - 15 > 0$

k $x^2 - 11x + 28 \leqslant 0$

l $x(x+10) < -24$

m $x^2 - 30 \geqslant 13x$

n $2x^2 - x - 3 \geqslant 0$

o $4x^2 - 4x + 1 < 0$

p $6x^2 + 7x < 3$

q $3x^2 > 8(x+2)$

r $2x^2 - 4x + 2 < 0$

s $6x^2 + 1 \leqslant 5x$

t $(4x+1)(3x+2) \geqslant 16x - 4$

u $(2x+3)^2 < x + 6$

2 In $3x^2 + 12 \square 12x$, replace $\square$ with $>$, $\geqslant$, $<$, or $\leqslant$ so that the resulting inequality has:

a no solutions

b one solution

c infinitely many solutions.

C THE DISCRIMINANT OF A QUADRATIC

In the quadratic formula, the quantity $b^2 - 4ac$ under the square root sign is called the **discriminant**.

The symbol **delta** Δ is used to represent the discriminant, so $\Delta = b^2 - 4ac$.

The quadratic formula becomes $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$ where Δ replaces $b^2 - 4ac$.

- If $\Delta = 0$, $x = \dfrac{-b}{2a}$ is the **only solution** (a **repeated** or **double root**)
- If $\Delta > 0$, $\sqrt{\Delta}$ is a positive real number, so there are **two distinct real roots**
$$x = \frac{-b + \sqrt{\Delta}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{\Delta}}{2a}$$
- If $\Delta < 0$, $\sqrt{\Delta}$ is not a real number and so there are **no real roots**.
- If a, b, and c are rational and Δ is a **square** then the equation has two rational roots which can be found by factorisation.

Example 9

Self Tutor

Use the discriminant to determine the nature of the roots of:

a $2x^2 - 2x + 3 = 0$ **b** $3x^2 - 4x - 2 = 0$

a $\Delta = b^2 - 4ac$
$= (-2)^2 - 4(2)(3)$
$= -20$

Since $\Delta < 0$, there are no real roots.

b $\Delta = b^2 - 4ac$
$= (-4)^2 - 4(3)(-2)$
$= 40$

Since $\Delta > 0$, but 40 is not a square, there are 2 distinct irrational roots.

Example 10

Self Tutor

Consider $x^2 - 2x + m = 0$. Find the discriminant Δ, and hence find the values of m for which the equation has:

a a repeated root **b** 2 distinct real roots **c** no real roots.

$x^2 - 2x + m = 0$ has $a = 1$, $b = -2$, and $c = m$

$\therefore\ \Delta = b^2 - 4ac$
$= (-2)^2 - 4(1)(m)$
$= 4 - 4m$

a For a repeated root
$\Delta = 0$
$\therefore\ 4 - 4m = 0$
$\therefore\ 4 = 4m$
$\therefore\ m = 1$

b For 2 distinct real roots
$\Delta > 0$
$\therefore\ 4 - 4m > 0$
$\therefore\ -4m > -4$
$\therefore\ m < 1$

c For no real roots
$\Delta < 0$
$\therefore\ 4 - 4m < 0$
$\therefore\ -4m < -4$
$\therefore\ m > 1$

Example 11

Self Tutor

Consider the equation $kx^2 + (k+3)x = 1$. Find the discriminant Δ and draw its sign diagram. Hence, find the value of k for which the equation has:

a two distinct real roots

b two real roots

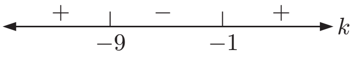
c a repeated root

d no real roots.

$kx^2 + (k+3)x - 1 = 0$ has $a = k$, $b = (k+3)$, and $c = -1$

$$\begin{aligned}\therefore\quad \Delta &= b^2 - 4ac\\ &= (k+3)^2 - 4(k)(-1)\\ &= k^2 + 6k + 9 + 4k\\ &= k^2 + 10k + 9\\ &= (k+9)(k+1)\end{aligned}$$

So, Δ has sign diagram:



a For two distinct real roots, $\Delta > 0$ $\therefore$ $k < -9$ or $k > -1$, $k \neq 0$.

b For two real roots, $\Delta \geqslant 0$ $\therefore$ $k \leqslant -9$ or $k \geqslant -1$, $k \neq 0$.

c For a repeated root, $\Delta = 0$ $\therefore$ $k = -9$ or $k = -1$.

d For no real roots, $\Delta < 0$ $\therefore$ $-9 < k < -1$.

Summary:

Factorisation of quadratic	*Roots of quadratic*	*Discriminant value*
two distinct linear factors	two real distinct roots	$\Delta > 0$
two identical linear factors	two identical real roots (repeated)	$\Delta = 0$
unable to factorise	no real roots	$\Delta < 0$

EXERCISE 3C

1 By using the discriminant only, state the nature of the solutions of:

a $x^2 + 7x - 3 = 0$ **b** $x^2 - 3x + 2 = 0$ **c** $3x^2 + 2x - 1 = 0$

d $5x^2 + 4x - 3 = 0$ **e** $x^2 + x + 5 = 0$ **f** $16x^2 - 8x + 1 = 0$

2 By using the discriminant only, determine which of the following quadratic equations have rational roots which can be found by factorisation.

a $6x^2 - 5x - 6 = 0$ **b** $2x^2 - 7x - 5 = 0$ **c** $3x^2 + 4x + 1 = 0$

d $6x^2 - 47x - 8 = 0$ **e** $4x^2 - 3x + 2 = 0$ **f** $8x^2 + 2x - 3 = 0$

3 For each of the following quadratic equations, determine the discriminant Δ in simplest form and draw its sign diagram. Hence find the value(s) of m for which the equation has:

i a repeated root **ii** two distinct real roots **iii** no real roots.

a $x^2 + 4x + m = 0$ **b** $mx^2 + 3x + 2 = 0$ **c** $mx^2 - 3x + 1 = 0$

4 For each of the following quadratic equations, find the discriminant Δ and hence draw its sign diagram. Find all values of k for which the equation has:

i two distinct real roots **ii** two real roots **iii** a repeated root **iv** no real roots.

a $2x^2 + kx - k = 0$

b $kx^2 - 2x + k = 0$

c $x^2 + (k+2)x + 4 = 0$

d $2x^2 + (k-2)x + 2 = 0$

e $x^2 + (3k-1)x + (2k+10) = 0$

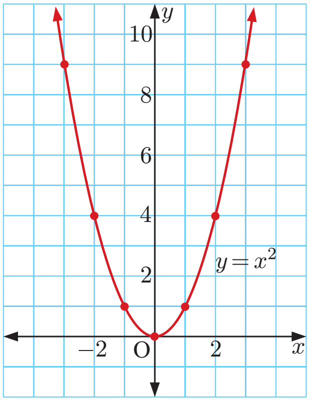
f $(k+1)x^2 + kx + k = 0$

D QUADRATIC FUNCTIONS

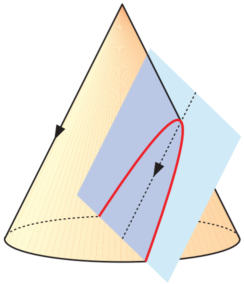
A **quadratic function** has the form $y = ax^2 + bx + c$ where $a \neq 0$.

The simplest quadratic function is $y = x^2$. Its graph can be drawn from a table of values.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9



The graph of a quadratic function is called a **parabola**.



The parabola is one of the **conic sections**, the others being circles, hyperbolae, and ellipses. They are called conic sections because they can be obtained by cutting a cone with a plane. A parabola is produced by cutting the cone with a plane parallel to its slant side.

There are many examples of parabolas in everyday life, including water fountains, suspension bridges, and radio telescopes.

TERMINOLOGY

The graph of a quadratic function $y = ax^2 + bx + c$, $a \neq 0$ is called a **parabola**.

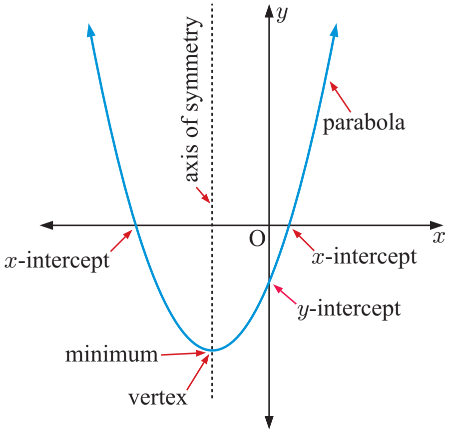
The point where the graph 'turns' is called the **vertex**.

If the graph opens upwards, the vertex is the **minimum** or **minimum turning point**, and the graph is **concave upwards**.

If the graph opens downwards, the vertex is the **maximum** or **maximum turning point**, and the graph is **concave downwards**.

The vertical line that passes through the vertex is called the **axis of symmetry**. Every parabola is symmetrical about its axis of symmetry.

The point where the graph crosses the y-axis is the **y-intercept**.



The points (if they exist) where the graph crosses the x-axis are called the **x-intercepts**. They correspond to the **roots** of the equation $y = 0$.

Discovery 1 — Graphing $y = a(x-p)(x-q)$

This Discovery is best done using a **graphing package** or **graphics calculator**.

GRAPHING PACKAGE

What to do:

1 **a** Use technology to help you to sketch:
$y = (x-1)(x-3)$, $y = 2(x-1)(x-3)$, $y = -(x-1)(x-3)$,
$y = -3(x-1)(x-3)$, and $y = -\frac{1}{2}(x-1)(x-3)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of a in $y = a(x-1)(x-3)$?

2 **a** Use technology to help you to sketch:
$y = 2(x-1)(x-4)$, $y = 2(x-3)(x-5)$, $y = 2(x+1)(x-2)$,
$y = 2x(x+5)$, and $y = 2(x+2)(x+4)$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of p and q in $y = 2(x-p)(x-q)$?

3 **a** Use technology to help you to sketch:
$y = 2(x-1)^2$, $y = 2(x-3)^2$, $y = 2(x+2)^2$, $y = 2x^2$

b Find the x-intercepts for each function in **a**.

c What is the geometrical significance of p in $y = 2(x-p)^2$?

4 Copy and complete:

- If a quadratic has the form $y = a(x-p)(x-q)$ then it ...... the x-axis at ......
- If a quadratic has the form $y = a(x-p)^2$ then it ...... the x-axis at ......

Discovery 2 Graphing $y = a(x-h)^2 + k$

This Discovery is also best done using technology.

GRAPHING PACKAGE

What to do:

1 **a** Use technology to help you to sketch:
$y = (x-3)^2 + 2$, $y = 2(x-3)^2 + 2$, $y = -2(x-3)^2 + 2$,
$y = -(x-3)^2 + 2$, and $y = -\frac{1}{3}(x-3)^2 + 2$

b Find the coordinates of the vertex for each function in **a**.

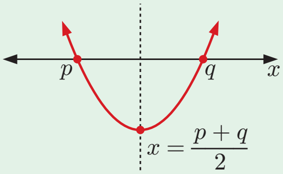
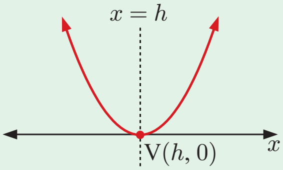
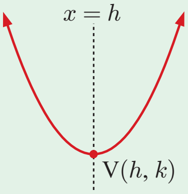
c What is the geometrical significance of a in $y = a(x-3)^2 + 2$?

2 **a** Use technology to help you to sketch:
$y = 2(x-1)^2 + 3$, $y = 2(x-2)^2 + 4$, $y = 2(x-3)^2 + 1$,
$y = 2(x+1)^2 + 4$, $y = 2(x+2)^2 - 5$, and $y = 2(x+3)^2 - 2$

b Find the coordinates of the vertex for each function in **a**.

c What is the geometrical significance of h and k in $y = 2(x-h)^2 + k$?

3 Copy and complete:
If a quadratic has the form $y = a(x-h)^2 + k$ then its vertex has coordinates ......
The graph of $y = a(x-h)^2 + k$ is a ...... of the graph of $y = ax^2$ with vector ......

Quadratic form, $a \neq 0$	*Graph*	*Facts*
• $y = a(x-p)(x-q)$ p, q are real		x-intercepts are p and q axis of symmetry is $x = \frac{p+q}{2}$ vertex is $\left(\frac{p+q}{2}, f\left(\frac{p+q}{2}\right)\right)$
• $y = a(x-h)^2$ h is real		touches x-axis at h axis of symmetry is $x = h$ vertex is $(h, 0)$
• $y = a(x-h)^2 + k$		axis of symmetry is $x = h$ vertex is (h, k)

You should have found that a, the coefficient of x^2, controls the width of the graph and whether it opens upwards or downwards.

For a quadratic function $y = ax^2 + bx + c$, $a \neq 0$:

- $a > 0$ produces the shape  called concave up.

  $a < 0$ produces the shape  called concave down.
- If $-1 < a < 1$, $a \neq 0$ the graph is wider than $y = x^2$.
  If $a < -1$ or $a > 1$ the graph is narrower than $y = x^2$.

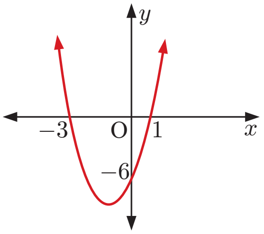
Example 12

Self Tutor

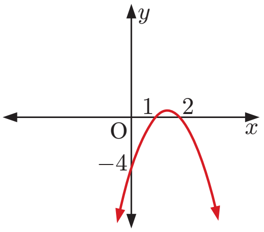
Using axes intercepts only, sketch the graphs of:

a $y = 2(x+3)(x-1)$ **b** $y = -2(x-1)(x-2)$ **c** $y = \frac{1}{2}(x+2)^2$

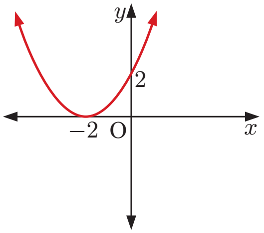
a $y = 2(x+3)(x-1)$
has x-intercepts -3, 1
When $x = 0$,
$$y = 2(3)(-1) = -6$$
$\therefore$ y-intercept is -6



b $y = -2(x-1)(x-2)$
has x-intercepts 1, 2
When $x = 0$,
$$y = -2(-1)(-2) = -4$$
$\therefore$ y-intercept is -4



c $y = \frac{1}{2}(x+2)^2$
touches x-axis at -2
When $x = 0$,
$$y = \tfrac{1}{2}(2)^2 = 2$$
$\therefore$ y-intercept is 2

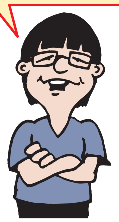


EXERCISE 3D.1

1 Using axes intercepts only, sketch the graphs of:

a $y = (x-4)(x+2)$
b $f(x) = -(x-4)(x+2)$
c $y = 2(x+3)(x+5)$
d $f(x) = -3(x+1)(x+5)$
e $f(x) = 2(x+3)^2$
f $y = -\frac{1}{4}(x+2)^2$

2 State the equation of the axis of symmetry for each graph in question **1**.



The axis of symmetry is midway between the x-intercepts.

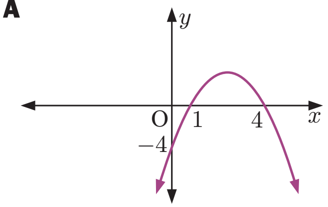
3 Match each quadratic function with its corresponding graph.

a $y = 2(x-1)(x-4)$ **b** $y = -(x+1)(x-4)$ **c** $y = (x-1)(x-4)$

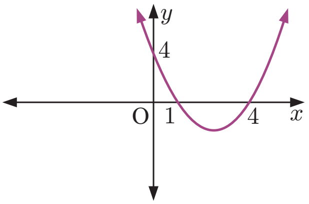
d $y = (x+1)(x-4)$ **e** $y = 2(x+4)(x-1)$ **f** $y = -3(x+4)(x-1)$

g $y = -(x-1)(x-4)$ **h** $y = -3(x-1)(x-4)$

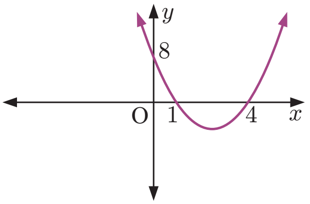
A



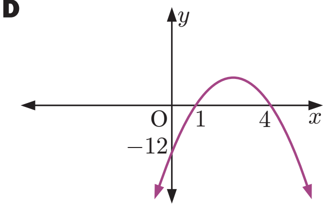
B



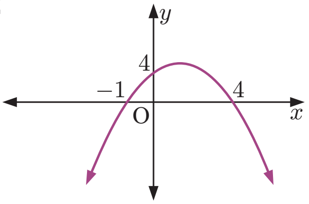
C



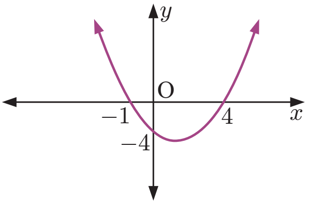
D



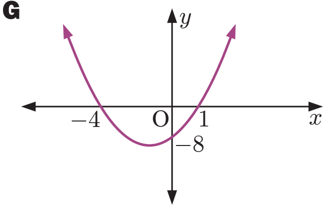
E



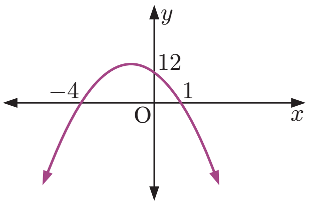
F



G



H



Example 13

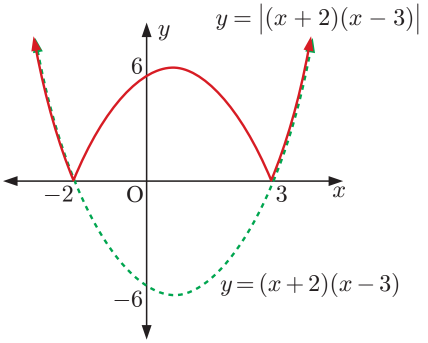
Self Tutor

Sketch the graph of $y = |(x+2)(x-3)|$.

We first sketch $y = (x+2)(x-3)$.

$y = (x+2)(x-3)$ has x-intercepts -2 and 3, and y-intercept $2(-3) = -6$.

The part of the graph that is below the x-axis is then reflected in the x-axis to produce the graph of $y = |(x+2)(x-3)|$.



4 Sketch the graph of:

a $y = |(x+4)(x-5)|$ **b** $f(x) = |-(x-1)(x-6)|$

c $y = |2(x-2)(x+2)|$ **d** $f(x) = \left|-3(x+3)^2\right|$

Example 14

Self Tutor

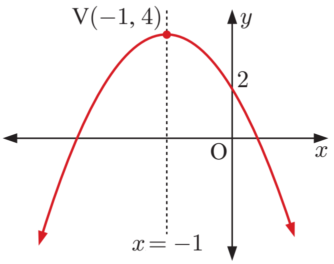
Use the vertex, axis of symmetry, and y-intercept to graph $y = -2(x+1)^2 + 4$.

The vertex is $(-1, 4)$.

The axis of symmetry is $x = -1$.

When $x = 0$, $y = -2(1)^2 + 4$
$= 2$

$a < 0$ so the shape is 



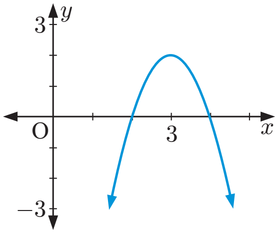
5 Use the vertex, axis of symmetry, and y-intercept to graph:

a $y = (x-1)^2 + 3$
b $f(x) = 2(x+2)^2 + 1$
c $y = -2(x-1)^2 - 3$
d $f(x) = \frac{1}{2}(x-3)^2 + 2$
e $y = -\frac{1}{3}(x-1)^2 + 4$
f $f(x) = -\frac{1}{10}(x+2)^2 - 3$

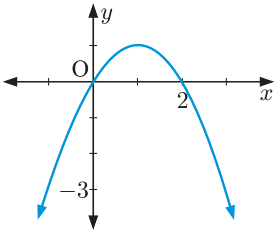
6 Match each quadratic function with its corresponding graph:

a $y = -(x+1)^2 + 3$
b $y = -2(x-3)^2 + 2$
c $y = x^2 + 2$
d $y = -(x-1)^2 + 1$
e $y = (x-2)^2 - 2$
f $y = \frac{1}{3}(x+3)^2 - 3$
g $y = -x^2$
h $y = -\frac{1}{2}(x-1)^2 + 1$
i $y = 2(x+2)^2 - 1$

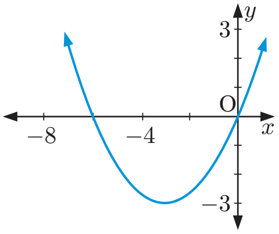
A



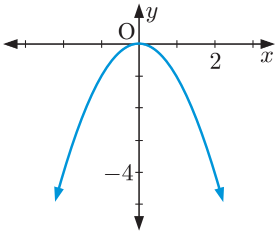
B



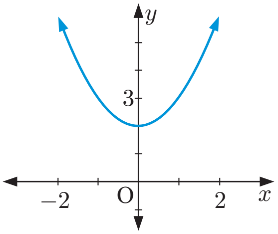
C



D



E



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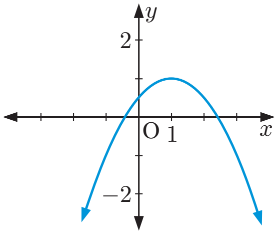
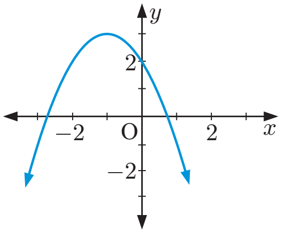
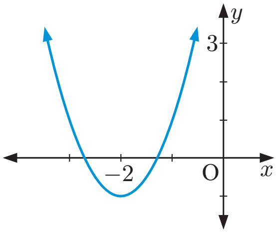


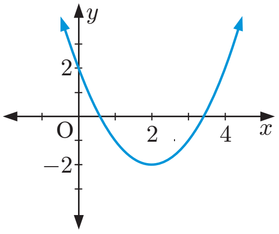
G



H



I



SKETCHING GRAPHS BY 'COMPLETING THE SQUARE'

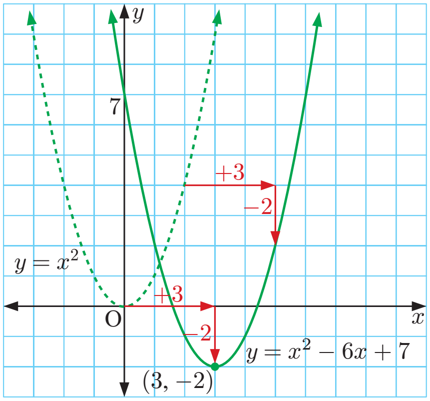
If we wish to graph a quadratic given in general form $y = ax^2 + bx + c$, one approach is to convert it to the form $y = a(x-h)^2 + k$ where we can read off the coordinates of the vertex (h, k). To do this, we '**complete the square**'.

Consider the simple case $y = x^2 - 6x + 7$, for which $a = 1$.

$$\begin{aligned} y &= x^2 - 6x + 7 \\ \therefore\ y &= \underbrace{x^2 - 6x + 3^2}\ \underbrace{+7 - 3^2} \\ \therefore\ y &= \quad (x-3)^2 \qquad\quad -2 \end{aligned}$$

So, the vertex is $(3, -2)$.

To obtain the graph of $y = x^2 - 6x + 7$ from the graph of $y = x^2$, we shift it 3 units to the right and 2 units down.



Example 15

Self Tutor

Write $y = x^2 + 4x + 3$ in the form $y = (x-h)^2 + k$ by 'completing the square'.

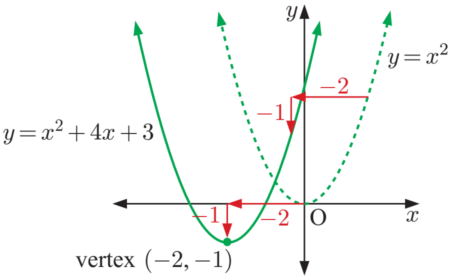
Hence sketch $y = x^2 + 4x + 3$, stating the coordinates of the vertex.

$$\begin{aligned} y &= x^2 + 4x + 3 \\ \therefore\ y &= x^2 + 4x + 2^2 + 3 - 2^2 \\ \therefore\ y &= (x+2)^2 - 1 \end{aligned}$$

shift 2 units left

shift 1 unit down

The vertex is $(-2, -1)$ and the y-intercept is 3.



Example 16

Self Tutor

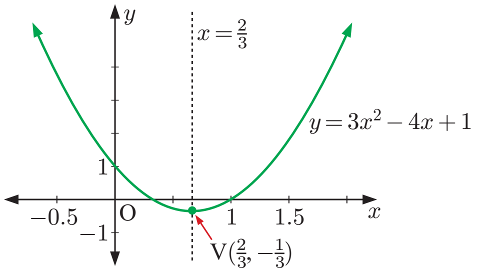
a Convert $y = 3x^2 - 4x + 1$ to the form $y = a(x-h)^2 + k$.

b Hence, write down the coordinates of its vertex and sketch the quadratic.

a $y = 3x^2 - 4x + 1$

$$\begin{aligned} &= 3[x^2 - \tfrac{4}{3}x + \tfrac{1}{3}] && \{\text{taking out a factor of 3}\} \\ &= 3[x^2 - 2(\tfrac{2}{3})x + (\tfrac{2}{3})^2 - (\tfrac{2}{3})^2 + \tfrac{1}{3}] && \{\text{completing the square}\} \\ &= 3[(x - \tfrac{2}{3})^2 - \tfrac{4}{9} + \tfrac{3}{9}] && \{\text{writing as a perfect square}\} \\ &= 3[(x - \tfrac{2}{3})^2 - \tfrac{1}{9}] \\ &= 3(x - \tfrac{2}{3})^2 - \tfrac{1}{3} \end{aligned}$$

b The vertex is $\left(\frac{2}{3}, -\frac{1}{3}\right)$ and the y-intercept is 1.



EXERCISE 3D.2

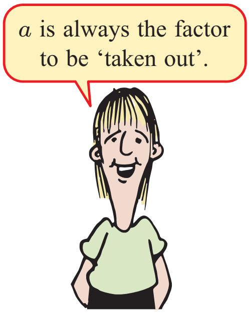
1 Write the following quadratics in the form $y = (x-h)^2 + k$ by 'completing the square'. Hence sketch each function, stating the coordinates of the vertex.

a $y = x^2 - 2x + 3$ **b** $y = x^2 + 4x - 2$ **c** $y = x^2 - 4x$
d $y = x^2 + 3x$ **e** $y = x^2 + 5x - 2$ **f** $y = x^2 - 3x + 2$
g $y = x^2 - 6x + 5$ **h** $y = x^2 + 8x - 2$ **i** $y = x^2 - 5x + 1$

2 For each of the following quadratics:

- **i** Write the quadratic in the form $y = a(x-h)^2 + k$.
- **ii** State the coordinates of the vertex.
- **iii** Find the y-intercept.
- **iv** Sketch the graph of the quadratic.

a $y = 2x^2 + 4x + 5$ **b** $y = 2x^2 - 8x + 3$
c $y = 2x^2 - 6x + 1$ **d** $y = 3x^2 - 6x + 5$
e $y = -x^2 + 4x + 2$ **f** $y = -2x^2 - 5x + 3$



QUADRATIC FUNCTIONS OF THE FORM $y = ax^2 + bx + c$

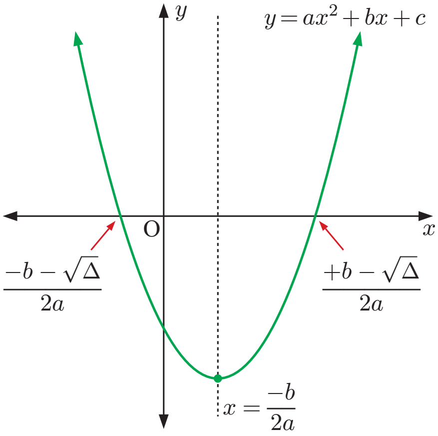
We now consider a method of graphing quadratics of the form $y = ax^2 + bx + c$ directly, without having to first convert them to a different form.

We know that the quadratic equation $ax^2 + bx + c = 0$ has solutions $\dfrac{-b-\sqrt{\Delta}}{2a}$ and $\dfrac{-b+\sqrt{\Delta}}{2a}$ where $\Delta = b^2 - 4ac$.

If $\Delta \geqslant 0$, these are the x-intercepts of the quadratic function $y = ax^2 + bx + c$.

The average of the values is $\dfrac{-b}{2a}$, so we conclude that:

- the axis of symmetry is $x = \dfrac{-b}{2a}$
- the vertex of the quadratic has x-coordinate $\dfrac{-b}{2a}$.



To graph a quadratic of the form $y = ax^2 + bx + c$, we:

- find the axis of symmetry $x = \frac{-b}{2a}$
- substitute to find the y-coordinate of the vertex
- state the y-intercept c
- find the x-intercepts by solving $ax^2 + bx + c = 0$, either by factorisation or using the quadratic formula.

Example 17

Self Tutor

Consider the quadratic $f(x) = 2x^2 + 8x - 10$.

a Find the axis of symmetry.
b Find the coordinates of the vertex.
c Find the axes intercepts.
d Hence, sketch the function.
e State the range of the function.

$f(x) = 2x^2 + 8x - 10$ has $a = 2$, $b = 8$, and $c = -10$.

$a > 0$, so the shape is 

a $\frac{-b}{2a} = \frac{-8}{2(2)} = -2$

The axis of symmetry is $x = -2$.

b $f(-2) = 2(-2)^2 + 8(-2) - 10$
$= -18$

The vertex is $(-2, -18)$.

c The y-intercept is -10.

When $y = 0$, $2x^2 + 8x - 10 = 0$

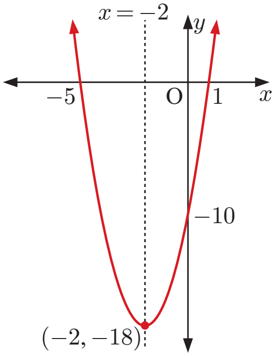
$\therefore\ 2(x^2 + 4x - 5) = 0$

$\therefore\ 2(x + 5)(x - 1) = 0$

$\therefore\ x = -5$ or 1

$\therefore$ the x-intercepts are -5 and 1.

d

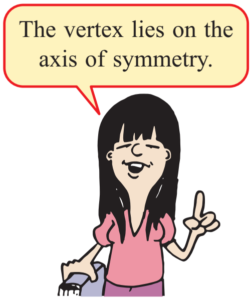


e The range is $\{y : y \geqslant -18\}$.

EXERCISE 3D.3

1 Locate the turning point or vertex for each of the following quadratic functions:

a $f(x) = x^2 - 4x + 2$
b $y = x^2 + 2x - 3$
c $y = 2x^2 + 4$
d $f(x) = -3x^2 + 1$
e $y = 2x^2 + 8x - 7$
f $f(x) = -x^2 - 4x - 9$
g $y = 2x^2 + 6x - 1$
h $f(x) = 2x^2 - 10x + 3$
i $y = -\frac{1}{2}x^2 + x - 5$



2 For each of the following quadratics:

i state the axis of symmetry
ii find the coordinates of the vertex
iii find the axes intercepts
iv sketch the quadratic
v state the range.

a $y = x^2 - 8x + 7$
b $y = -x^2 - 6x - 8$
c $f(x) = 6x - x^2$
d $y = -x^2 + 3x - 2$
e $y = 2x^2 + 4x - 24$
f $f(x) = -3x^2 + 4x - 1$
g $f(x) = 2x^2 - 5x + 2$
h $y = 4x^2 - 8x - 5$
i $y = -\frac{1}{4}x^2 + 2x - 3$

3 For each of the following quadratics:

i write the quadratic in factored form and hence find the roots
ii write the quadratic in completed square form and hence find the coordinates of the vertex
iii sketch the quadratic, showing the details you have found.

a $y = x^2 - 10x + 16$
b $y = x^2 + 10x + 9$
c $y = x^2 - 14x + 45$

4 Sketch the graph of:

a $y = \left|x^2 + 4x - 12\right|$
b $f(x) = \left|-x^2 - 3x + 10\right|$
c $y = \left|4x^2 - 12x + 5\right|$

Example 18

Self Tutor

Find the range of $y = x^2 - 6x - 2$ on the domain $-2 \leqslant x \leqslant 7$.

$y = x^2 - 6x - 2$ has $a = 1$, $b = -6$, and $c = -2$.

$a > 0$, so the shape is 

$$\frac{-b}{2a} = \frac{-(-6)}{2(1)} = 3$$

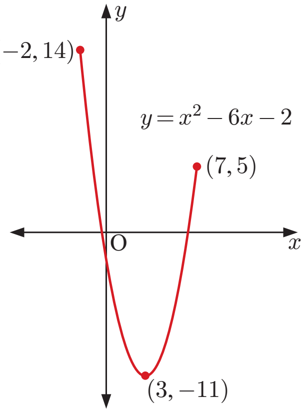
When $x = 3$, $y = 3^2 - 6(3) - 2$
$= -11$

$\therefore$ the vertex is $(3, -11)$.

When $x = -2$, $y = (-2)^2 - 6(-2) - 2$
$= 14$

When $x = 7$, $y = 7^2 - 6(7) - 2$
$= 5$

So, on the domain $\{x : -2 \leqslant x \leqslant 7\}$,
the range is $\{y : -11 \leqslant y \leqslant 14\}$.



To find the range of a function on a given domain, you must evaluate the function at the endpoints of the domain.

5 Find the range of:

a $f(x) = x^2 + 4x - 6$ on $-6 \leqslant x \leqslant 3$
b $y = -x^2 + 8x + 3$ on $0 \leqslant x \leqslant 7$
c $y = 2x^2 - 12x + 5$ on $-2 \leqslant x \leqslant 6$
d $f(x) = 7x - x^2$ on $-1 \leqslant x \leqslant 5$

Activity

Click on the icon to run a card game for quadratic functions.

CARD GAME

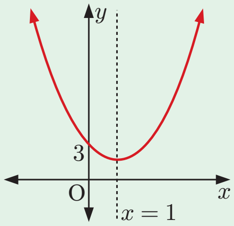
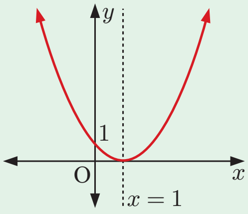
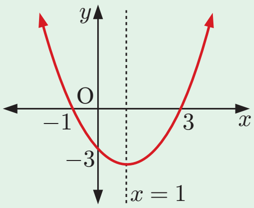
THE DISCRIMINANT AND THE QUADRATIC GRAPH

The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is $\Delta = b^2 - 4ac$.

We used Δ to determine the number of real roots of the equation. If they exist, these roots correspond to zeros of the quadratic $y = ax^2 + bx + c$. Δ therefore tells us about the relationship between a quadratic function and the x-axis.

The graphs of $y = x^2 - 2x + 3$, $y = x^2 - 2x + 1$, and $y = x^2 - 2x - 3$ all have the same axis of symmetry, $x = 1$.

Consider the following table:

$y = x^2 - 2x + 3$	$y = x^2 - 2x + 1$	$y = x^2 - 2x - 3$
		
$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(3)$ $= -8$	$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(1)$ $= 0$	$\Delta = b^2 - 4ac$ $= (-2)^2 - 4(1)(-3)$ $= 16$
$\Delta < 0$	$\Delta = 0$	$\Delta > 0$
does not cut the x-axis	touches the x-axis	cuts the x-axis twice

For a quadratic function $y = ax^2 + bx + c$, we consider the discriminant $\Delta = b^2 - 4ac$.

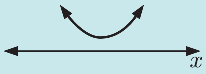
If $\Delta < 0$, the graph does not cut the x-axis.

If $\Delta = 0$, the graph *touches* the x-axis.

If $\Delta > 0$, the graph cuts the x-axis twice.

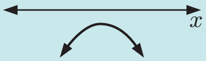
POSITIVE DEFINITE AND NEGATIVE DEFINITE QUADRATICS

Positive definite quadratics are quadratics which are positive for all values of x. So, $ax^2 + bx + c > 0$ for all $x \in \mathbb{R}$.



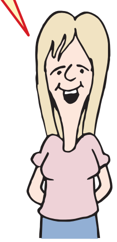
Test: A quadratic is **positive definite** if and only if $a > 0$ and $\Delta < 0$.

Negative definite quadratics are quadratics which are negative for all values of x. So, $ax^2 + bx + c < 0$ for all $x \in \mathbb{R}$.



Test: A quadratic is **negative definite** if and only if $a < 0$ and $\Delta < 0$.

The terms "positive definite" and "negative definite" are not needed for the syllabus.



Example 19

Self Tutor

Use the discriminant to determine the relationship between the graph of each function and the x-axis:

a $y = x^2 + 3x + 4$

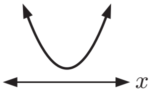
b $y = -2x^2 + 5x + 1$

a $a = 1, \quad b = 3, \quad c = 4$

$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= 9 - 4(1)(4) \\ &= -7 \end{aligned}$$

Since $\Delta < 0$, the graph does not cut the x-axis.

Since $a > 0$, the graph is concave up.



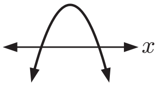
The graph is positive definite, which means that it lies entirely above the x-axis.

b $a = -2, \quad b = 5, \quad c = 1$

$$\begin{aligned} \therefore \quad \Delta &= b^2 - 4ac \\ &= 25 - 4(-2)(1) \\ &= 33 \end{aligned}$$

Since $\Delta > 0$, the graph cuts the x-axis twice.

Since $a < 0$, the graph is concave down.



EXERCISE 3D.4

1 Use the discriminant to determine the relationship between the graph and x-axis for:

a $y = x^2 + x - 2$

b $y = x^2 - 4x + 1$

c $f(x) = -x^2 - 3$

d $f(x) = x^2 + 7x - 2$

e $y = x^2 + 8x + 16$

f $f(x) = -2x^2 + 3x + 1$

g $y = 6x^2 + 5x - 4$

h $f(x) = -x^2 + x + 6$

i $y = 9x^2 + 6x + 1$

2 Consider the graph of $y = 2x^2 - 5x + 1$.

a Describe the shape of the graph.

b Use the discriminant to show that the graph cuts the x-axis twice.

c Find the x-intercepts, rounding your answers to 2 decimal places.

d State the y-intercept.

e Hence, sketch the function.

3 Consider the graph of $f(x) = -x^2 + 4x - 7$.

a Use the discriminant to show that the graph does not cut the x-axis.

b Is the graph positive definite or negative definite?

c Find the vertex and y-intercept.

d Hence, sketch the function.

4 Show that:

a $x^2 - 3x + 6 > 0$ for all x

b $4x - x^2 - 6 < 0$ for all x

c $2x^2 - 4x + 7$ is positive definite

d $-2x^2 + 3x - 4$ is negative definite.

5 Explain why $3x^2 + kx - 1$ is never positive definite for any value of k.

6 Under what conditions is $2x^2 + kx + 2$ positive definite?

E FINDING A QUADRATIC FROM ITS GRAPH

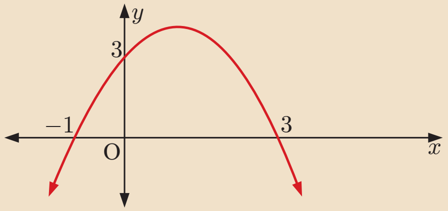
If we are given sufficient information on or about a graph, we can determine the quadratic function in whatever form is required.

Example 20

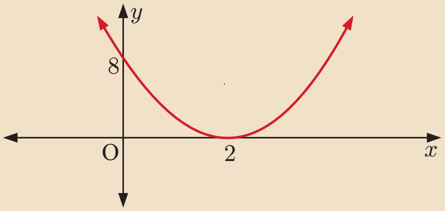
Self Tutor

Find the equation of the quadratic function with graph:

a



b



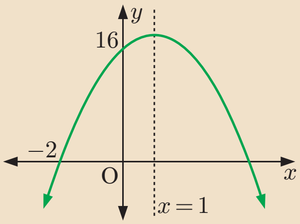
a Since the x-intercepts are -1 and 3,
$y = a(x+1)(x-3)$.
The graph is concave down, so $a < 0$.
When $x = 0$, $y = 3$
$\therefore\ 3 = a(1)(-3)$
$\therefore\ a = -1$
The quadratic function is
$y = -(x+1)(x-3)$.

b The graph touches the x-axis at $x = 2$,
so $y = a(x-2)^2$.
The graph is concave up, so $a > 0$.
When $x = 0$, $y = 8$
$\therefore\ 8 = a(-2)^2$
$\therefore\ a = 2$
The quadratic function is
$y = 2(x-2)^2$.

Example 21

Self Tutor

Find the equation of the quadratic function with graph:



The axis of symmetry $x = 1$ lies midway between the x-intercepts.

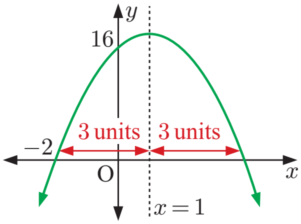
$\therefore$ the other x-intercept is 4.

$\therefore$ the quadratic has the form

$$y = a(x+2)(x-4) \quad \text{where} \quad a < 0$$

But when $x = 0$, $y = 16$
$\therefore\ 16 = a(2)(-4)$
$\therefore\ a = -2$

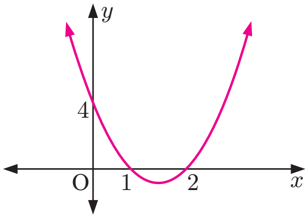
The quadratic is $y = -2(x+2)(x-4)$.



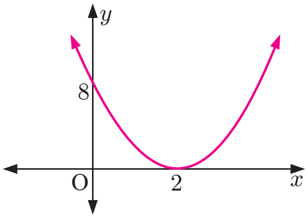
EXERCISE 3E

1 Find the equation of the quadratic with graph:

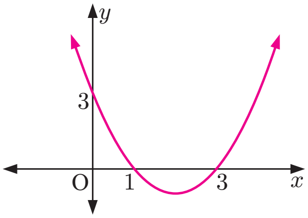
a



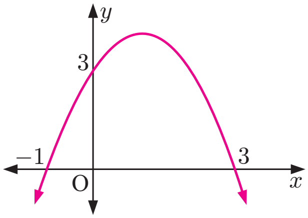
b



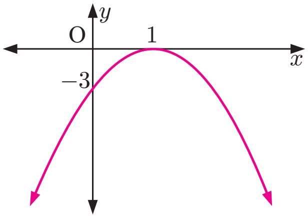
c



d



e



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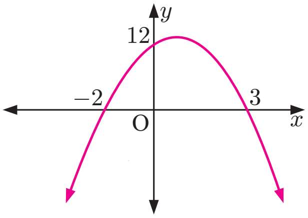
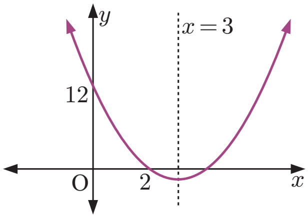
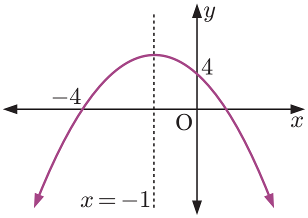


2 Find the quadratic with graph:

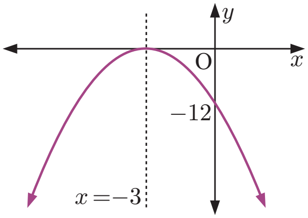
a



b



c



Example 22

Self Tutor

Find the equation of the quadratic whose graph cuts the x-axis at 4 and -3, and which passes through the point $(2, -20)$. Give your answer in the form $y = ax^2 + bx + c$.

Since the x-intercepts are 4 and -3, the quadratic has the form $y = a(x-4)(x+3)$ where $a \neq 0$.

When $x = 2$, $y = -20$

$$\therefore\ -20 = a(2-4)(2+3)$$
$$\therefore\ -20 = a(-2)(5)$$
$$\therefore\ a = 2$$

The quadratic is

$$\begin{aligned} y &= 2(x-4)(x+3) \\ &= 2(x^2 - x - 12) \\ &= 2x^2 - 2x - 24 \end{aligned}$$

3 Find, in the form $f(x) = ax^2 + bx + c$, the equation of the quadratic whose graph:

- **a** cuts the x-axis at 5 and 1, and passes through $(2, -9)$
- **b** cuts the x-axis at 2 and $-\frac{1}{2}$, and passes through $(3, -14)$
- **c** touches the x-axis at 3 and passes through $(-2, -25)$
- **d** touches the x-axis at -2 and passes through $(-1, 4)$

4 Find, in the form $f(x) = ax^2 + bx + c$, the equation of the quadratic whose graph:

a cuts the x-axis at 3, passes through $(5, 12)$, and has axis of symmetry $x = 2$

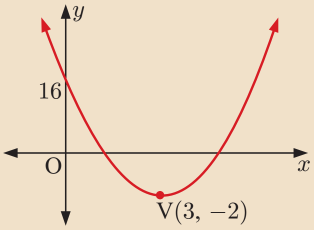
b cuts the x-axis at 5, passes through $(2, 5)$, and has axis of symmetry $x = 1$.

Example 23

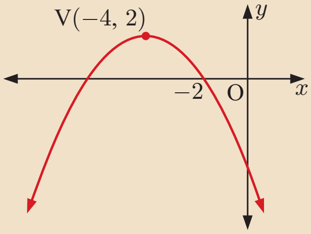
Self Tutor

Find the equation of each quadratic function given its graph:

a



b



a Since the vertex is $(3, -2)$, the quadratic has the form

$y = a(x-3)^2 - 2$ where $a > 0$.

When $x = 0$, $y = 16$

$$\therefore\ 16 = a(-3)^2 - 2$$
$$\therefore\ 16 = 9a - 2$$
$$\therefore\ 18 = 9a$$
$$\therefore\ a = 2$$

The quadratic is $y = 2(x-3)^2 - 2$.

b Since the vertex is $(-4, 2)$, the quadratic has the form

$y = a(x+4)^2 + 2$ where $a < 0$.

When $x = -2$, $y = 0$

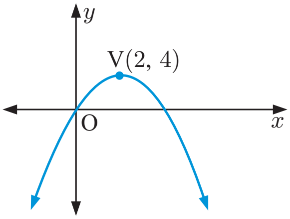
$$\therefore\ 0 = a(2)^2 + 2$$
$$\therefore\ 4a = -2$$
$$\therefore\ a = -\tfrac{1}{2}$$

The quadratic is

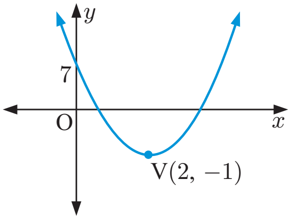
$y = -\frac{1}{2}(x+4)^2 + 2$.

5 If V is the vertex, find the equation of the quadratic function with graph:

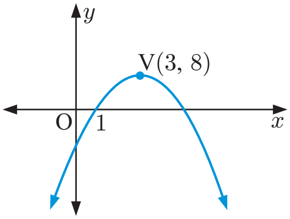
a



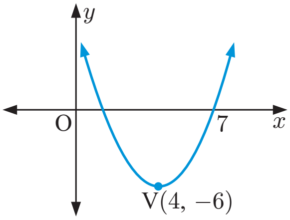
b



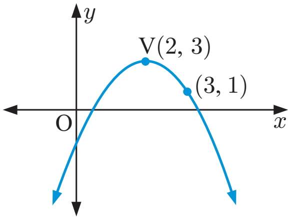
c



d



e



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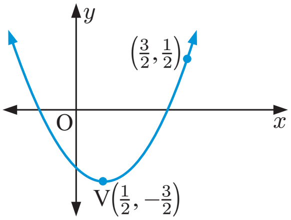


Discovery 3 Finding quadratic functions

For the quadratic function $y = 2x^2 + 3x + 7$ we can find a table of values for $x = 0, 1, 2, 3, 4, 5$.

x	0	1	2	3	4	5
y	7	12	21	34	51	72

We turn this table into a **difference table** by adding two further rows:

- the row Δ_1 gives the differences between successive y-values
- the row Δ_2 gives the differences between successive Δ_1-values.

x	0		1		2		3		4		5
y	7		12		21		34		51		72
Δ_1		5		9		$\boxed{13}$		17		$\boxed{21}$	
Δ_2			$\boxed{4}$		4		4		4		

$9 - 5$ $34 - 21$ $72 - 51$

What to do:

1 Construct difference tables for $x = 0, 1, 2, 3, 4, 5$ for each of the following quadratic functions:

a $y = x^2 + 4x + 3$ **b** $y = 3x^2 - 4x$

c $y = 5x - x^2$ **d** $y = 4x^2 - 5x + 2$

2 What do you notice about the Δ_2 row for each of the quadratic functions in **1**?

3 Consider the general quadratic $y = ax^2 + bx + c$, $a \neq 0$.

a Copy and complete the following difference table:

x	0		1		2		3		4		5
y	ⓒ		$a + b + c$		$4a + 2b + c$		......		......		......
Δ_1		○		......		......		......		......	
Δ_2			○		......		......		......		

b Comment on the Δ_2 row.

c What can the encircled numbers be used for?

4 Use your observations in **3** to determine, if possible, the quadratic functions with the following tables of values:

a

x	0	1	2	3	4
y	6	5	8	15	26

b

x	0	1	2	3	4
y	8	10	18	32	52

c

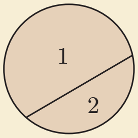
x	0	1	2	3	4
y	1	2	-1	-8	-19

d

x	0	1	2	3	4
y	5	3	-1	-7	-15

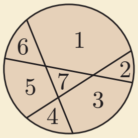
5 We wish to determine the **maximum** number of pieces into which a pizza can be cut using n cuts across it.

For example, for $n = 1$ we have



which has 2 pieces

for $n = 3$ we have



which has 7 pieces.

a Copy and complete:

Number of cuts, n	0	1	2	3	4	5
Maximum number of pieces, P_n						

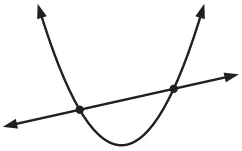
b Complete the Δ_1 and Δ_2 rows. Hence determine a quadratic formula for P_n.

c For a huge pizza with 12 cuts across it, find the maximum number of pieces which can result.

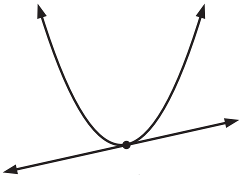
F WHERE FUNCTIONS MEET

Consider the graphs of a quadratic function and a linear function on the same set of axes.

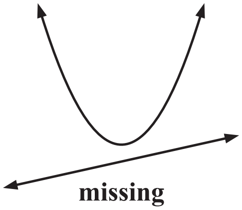
Notice that we could have:



cutting
(2 points of intersection)



touching
(1 point of intersection)



missing
(no points of intersection)

If the graphs meet, the coordinates of the points of intersection of the graphs can be found by *solving the two equations simultaneously.*

Example 24

Self Tutor

Find the coordinates of the points of intersection of the graphs with equations $y = x^2 - x - 18$ and $y = x - 3$.

$y = x^2 - x - 18$ meets $y = x - 3$ where

$$x^2 - x - 18 = x - 3$$
$$\therefore\ x^2 - 2x - 15 = 0 \qquad \{\text{RHS} = 0\}$$
$$\therefore\ (x-5)(x+3) = 0 \qquad \{\text{factorising}\}$$
$$\therefore\ x = 5 \text{ or } -3$$

Substituting into $y = x - 3$, when $x = 5$, $y = 2$ and when $x = -3$, $y = -6$.

$\therefore$ the graphs meet at $(5, 2)$ and $(-3, -6)$.

EXERCISE 3F

1 Find the coordinates of the point(s) of intersection of:

a $y = x^2 - 2x + 8$ and $y = x + 6$

b $f(x) = -x^2 + 3x + 9$ and $g(x) = 2x - 3$

c $y = x^2 - 4x + 3$ and $y = 2x - 6$

d $f(x) = -x^2 + 4x - 7$ and $g(x) = 5x - 4$

Example 25 Self Tutor

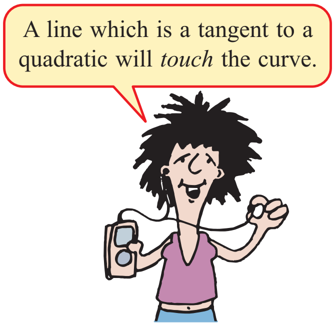
$y = 2x + k$ is a tangent to $y = 2x^2 - 3x + 4$. Find k.

$y = 2x + k$ meets $y = 2x^2 - 3x + 4$ where

$$2x^2 - 3x + 4 = 2x + k$$
$$\therefore \; 2x^2 - 5x + (4 - k) = 0$$

Since the graphs touch, this quadratic has $\Delta = 0$

$$\therefore \; (-5)^2 - 4(2)(4-k) = 0$$
$$\therefore \; 25 - 8(4-k) = 0$$
$$\therefore \; 25 - 32 + 8k = 0$$
$$\therefore \; 8k = 7$$
$$\therefore \; k = \tfrac{7}{8}$$



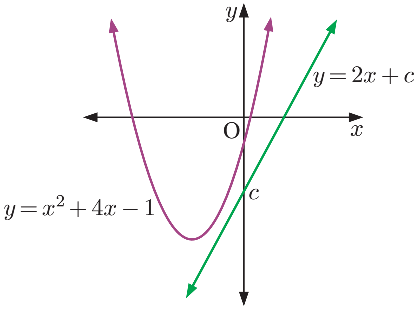
2 For which value of c is the line $y = 3x + c$ a tangent to the parabola with equation $y = x^2 - 5x + 7$?

3 Find the values of m for which the lines $y = mx - 2$ are tangents to the curve with equation $y = x^2 - 4x + 2$.

4 Find the gradients of the lines with y-intercept 1 that are tangents to the curve $f(x) = 3x^2 + 5x + 4$.

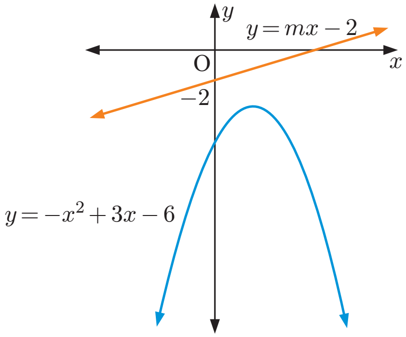
5
- **a** For what values of c do the lines $y = x + c$ never meet the parabola with equation $y = 2x^2 - 3x - 7$?
- **b** Choose one of the values of c found in part **a** above. Illustrate with a sketch that these graphs never meet.

6 Consider the curve $y = x^2 + 4x - 1$ and the line $y = 2x + c$. Find the value(s) of c for which the line:
- **a** meets the curve twice
- **b** is a tangent to the curve
- **c** does not meet the curve.



DEMO

7 Consider the curve $f(x) = -x^2 + 3x - 6$ and the line $g(x) = mx - 2$. Find the values of m for which the line:
- **a** meets the curve twice
- **b** is a tangent to the curve
- **c** does not meet the curve.



DEMO

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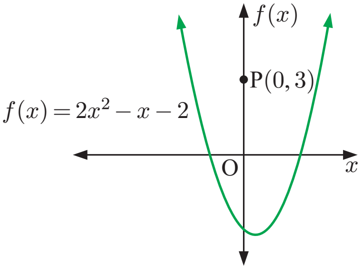


Show that any linear function passing through P(0, 3) will meet the curve $f(x) = 2x^2 - x - 2$ twice.

G PROBLEM SOLVING WITH QUADRATICS

Some real world problems can be solved using a quadratic equation. We are generally only interested in any **real solutions** which result.

Any answer we obtain must be checked to see if it is reasonable. For example:

- if we are finding a length then it must be positive and we reject any negative solutions
- if we are finding 'how many people are present' then clearly the answer must be a positive integer.

We employ the following general problem solving method:

Step 1: If the information is given in words, translate it into algebra using a variable such as x for the unknown. Write down the resulting equation. Be sure to define what the variable x represents, and include units if appropriate.

Step 2: Solve the equation by a suitable method.

Step 3: Examine the solutions carefully to see if they are acceptable.

Step 4: Give your answer in a sentence.

Example 26

Self Tutor

A rectangle has length 3 cm longer than its width. Its area is 42 cm². Find its width.

If the width is x cm then the length is $(x + 3)$ cm.

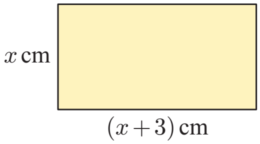
$$\therefore \quad x(x+3) = 42 \quad \{\text{equating areas}\}$$

$$\therefore \quad x^2 + 3x - 42 = 0$$

$$\therefore \quad x = \frac{-3 \pm \sqrt{3^2 - 4(1)(-42)}}{2}$$

$$\therefore \quad x = \frac{-3 \pm \sqrt{177}}{2}$$

$$\therefore \quad x \approx -8.15 \quad \text{or} \quad 5.15$$



We reject the negative solution as lengths are positive.

The width is about 5.15 cm.

EXERCISE 3G

1 Two integers differ by 12 and the sum of their squares is 74. Find the integers.

2 The sum of a number and its reciprocal is $\frac{26}{5}$. Find the number.

3 The sum of a natural number and its square is 210. Find the number.

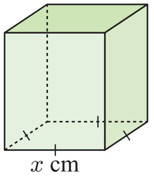
4 The product of two consecutive even numbers is 360. Find the numbers.

5 The number of diagonals of an n-sided polygon is given by the formula $D = \frac{n}{2}(n-3)$.

A polygon has 90 diagonals. How many sides does it have?

6 The length of a rectangle is 4 cm longer than its width. The rectangle has area 26 cm^2. Find its width.

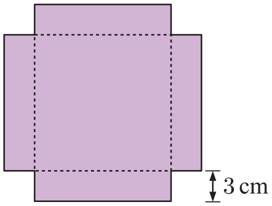
7 A rectangular box has a square base with sides of length x cm. Its height is 1 cm longer than its base side length. The total surface area of the box is 240 cm^2.



a Show that the total surface area is given by $A = 6x^2 + 4x\ \text{cm}^2$.

b Find the dimensions of the box.

8



An open box can hold 80 cm^3. It is made from a square piece of tinplate with 3 cm squares cut from each of its 4 corners. Find the dimensions of the original piece of tinplate.

Example 27

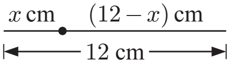
Self Tutor

Is it possible to bend a 12 cm length of wire to form the perpendicular sides of a right angled triangle with area 20 cm^2?

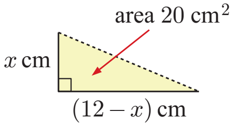
Suppose the wire is bent x cm from one end.

The area $A = \frac{1}{2}x(12-x)$

$$\therefore\ \tfrac{1}{2}x(12-x) = 20$$
$$\therefore\ x(12-x) = 40$$
$$\therefore\ 12x - x^2 - 40 = 0$$
$$\therefore\ x^2 - 12x + 40 = 0$$



becomes

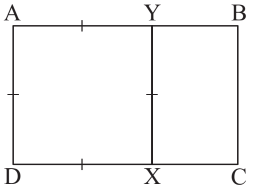


Now $\Delta = (-12)^2 - 4(1)(40)$
$= -16$ which is < 0

There are no real solutions, indicating this situation is **impossible**.

9 Is it possible to bend a 20 cm length of wire into the shape of a rectangle which has an area of 30 cm^2?

10 The rectangle ABCD is divided into a square and a smaller rectangle by [XY] which is parallel to its shorter sides. The smaller rectangle BCXY is *similar* to the original rectangle, so rectangle ABCD is a **golden rectangle**.



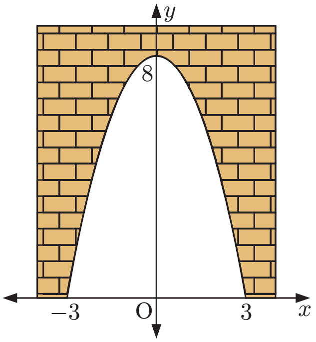
The ratio $\frac{\text{AB}}{\text{AD}}$ is called the **golden ratio**.

Show that the golden ratio is $\frac{1+\sqrt{5}}{2}$.

Hint: Let AB $= x$ units and AD $= 1$ unit.

11 A truck carrying a wide load needs to pass through the parabolic tunnel shown. The units are metres.
The truck is 5 m high and 4 m wide.

- **a** Find the quadratic function which describes the shape of the tunnel.
- **b** Determine whether the truck will fit.



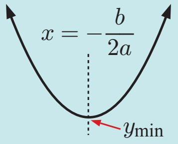
12 Answer the **Opening Problem** on page **64**.

H QUADRATIC OPTIMISATION

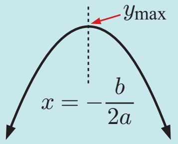
The process of finding the maximum or minimum value of a function is called **optimisation**.

For the quadratic function $y = ax^2 + bx + c$, we have already seen that the vertex has x-coordinate $-\frac{b}{2a}$.

- If $a > 0$, the **minimum** value of y occurs at $x = -\frac{b}{2a}$.



- If $a < 0$, the **maximum** value of y occurs at $x = -\frac{b}{2a}$.



Example 28

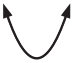
Self Tutor

Find the maximum or minimum value of the following quadratic functions, and the corresponding value of x:

a $y = x^2 + x - 3$

b $y = 3 + 3x - 2x^2$

a $y = x^2 + x - 3$ has
$a = 1$, $b = 1$, and $c = -3$.

Since $a > 0$, the shape is 

The minimum value occurs

when $x = \frac{-b}{2a} = -\frac{1}{2}$

and $y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 3 = -3\frac{1}{4}$

So, the minimum value of y is $-3\frac{1}{4}$, occurring when $x = -\frac{1}{2}$.

b $y = -2x^2 + 3x + 3$ has
$a = -2$, $b = 3$, and $c = 3$.

Since $a < 0$, the shape is 

The maximum value occurs

when $x = \frac{-b}{2a} = \frac{-3}{-4} = \frac{3}{4}$

and $y = -2(\frac{3}{4})^2 + 3(\frac{3}{4}) + 3 = 4\frac{1}{8}$

So, the maximum value of y is $4\frac{1}{8}$, occurring when $x = \frac{3}{4}$.

EXERCISE 3H

1 Find the maximum or minimum values of the following quadratic functions, and the corresponding values of x:

a $y = x^2 - 2x$
b $f(x) = 7 - 2x - x^2$
c $y = 8 + 2x - 3x^2$
d $f(x) = 2x^2 + x - 1$
e $y = 4x^2 - x + 5$
f $f(x) = 7x - 2x^2$

2 The profit in manufacturing x refrigerators per day, is given by the profit relation $P = -3x^2 + 240x - 800$ dollars.

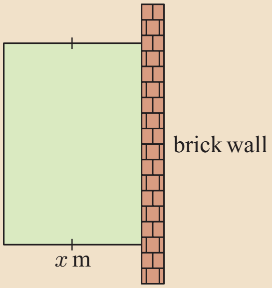
a How many refrigerators should be made each day to maximise the total profit?
b What is the maximum profit?

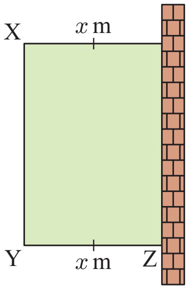
Example 29

Self Tutor

A gardener has 40 m of fencing to enclose a rectangular garden plot, where one side is an existing brick wall. Suppose the two new equal sides are x m long.

a Show that the area enclosed is given by $A = x(40 - 2x)$ m^2.
b Find the dimensions of the garden of maximum area.





a Side [XY] has length $(40 - 2x)$ m.
Now, area = length × width
$\therefore \quad A = x(40 - 2x)$ m^2

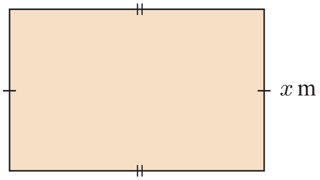
b $A = 0$ when $x = 0$ or 20.
The vertex of the function lies midway between these values, so $x = 10$.

Since $a < 0$, the shape is 

$\therefore$ the area is maximised when YZ = 10 m and XY = 20 m.

3 A rectangular plot is enclosed by 200 m of fencing and has an area of A square metres. Show that:

a $A = 100x - x^2$ where x m is the length of one of its sides
b the area is maximised if the rectangle is a square.



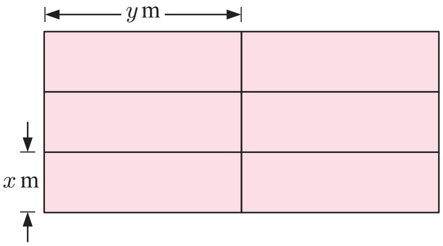
4 Three sides of a rectangular paddock are to be fenced, the fourth side being an existing straight water drain. If 1000 m of fencing is available, what dimensions should be used for the paddock so that it encloses the maximum possible area?

5 1800 m of fencing is available to fence six identical pens as shown in the diagram.

a Explain why $9x+8y=1800$.

b Show that the area of each pen is given by $A=-\frac{9}{8}x^2+225x$ m^2.

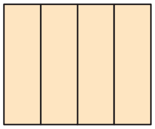
c If the area enclosed is to be maximised, what are the dimensions of each pen?



6 500 m of fencing is available to make 4 rectangular pens of identical shape. Find the dimensions that maximise the area of each pen if the plan is:

a

b



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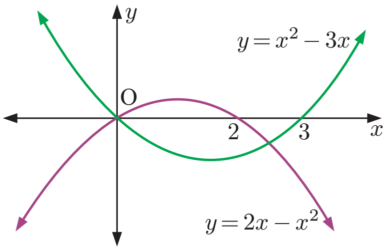


The graphs of $y=x^2-3x$ and $y=2x-x^2$ are illustrated.

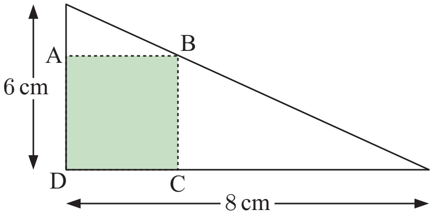
a Show that the graphs meet where $x=0$ and $x=2\frac{1}{2}$.

b Find the maximum vertical separation between the curves for $0\leqslant x\leqslant 2\frac{1}{2}$.

8 Infinitely many rectangles may be inscribed within the right angled triangle shown alongside. One of them is illustrated.

a Let AB $=x$ cm and BC $=y$ cm.
Use similar triangles to find y in terms of x.

b Find the dimensions of rectangle ABCD of maximum area.



Discovery 4 Sum and product of roots

What to do:

1 Suppose $ax^2+bx+c=0$ has roots p and q.

Prove that $p+q=\frac{-b}{a}$ and $pq=\frac{c}{a}$.

2 Suppose $2x^2-5x+1=0$ has roots p and q.
Without finding the values of p and q, find:

a $p+q$ **b** pq **c** p^2+q^2 **d** $\frac{1}{p}+\frac{1}{q}$

3 Find *all* quadratic equations with roots which are:

a one more than the roots of $2x^2-5x+1=0$

b the squares of the roots of $2x^2-5x+1=0$

c the reciprocals of the roots of $2x^2-5x+1=0$.

Review set 3A

1 Consider the quadratic function $f(x) = -2(x+2)(x-1)$.

- **a** State the x-intercepts.
- **b** State the equation of the axis of symmetry.
- **c** Find the y-intercept.
- **d** Find the coordinates of the vertex.
- **e** Sketch the function.
- **f** State the range of the function.

2 Solve the following equations, giving exact answers:

- **a** $3x^2 - 12x = 0$
- **b** $3x^2 - x - 10 = 0$
- **c** $x^2 - 11x = 60$

3 Solve using the quadratic formula:

- **a** $x^2 + 5x + 3 = 0$
- **b** $3x^2 + 11x - 2 = 0$

4 Solve for x:

- **a** $x^2 - 4x - 21 < 0$
- **b** $3x^2 - 2 \geqslant 5x$

5 Use the vertex, axis of symmetry, and y-intercept to graph:

- **a** $y = (x-2)^2 - 4$
- **b** $y = -\frac{1}{2}(x+4)^2 + 6$

6 Find, in the form $y = ax^2 + bx + c$, the equation of the quadratic whose graph:

- **a** touches the x-axis at 4 and passes through (2, 12)
- **b** has vertex (−4, 1) and passes through (1, 11).

7 Find the maximum or minimum value of the relation $f(x) = -2x^2 + 4x + 3$ and the value of x at which this occurs.

8 Find the points of intersection of $y = x^2 - 3x$ and $y = 3x^2 - 5x - 24$.

9 For what values of k does the graph of $y = -2x^2 + 5x + k$ not cut the x-axis?

10 Find the values of m for which $2x^2 - 3x + m = 0$ has:

- **a** a repeated root
- **b** two distinct real roots
- **c** no real roots.

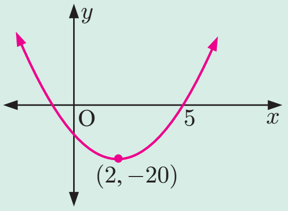
11 The sum of a number and its reciprocal is $2\frac{1}{30}$. Find the number.

12 Show that no line with a y-intercept of (0, 10) will ever be tangential to the curve with equation $y = 3x^2 + 7x - 2$.

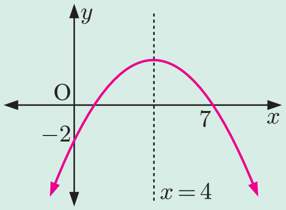
13
- **a** Write the quadratic $y = 2x^2 + 4x - 3$ in the form $y = a(x-h)^2 + k$.
- **b** Hence, sketch the graph of the quadratic.

14 Find the equation of the quadratic function with graph:

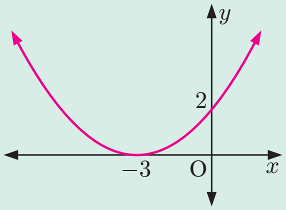
a



b



c



15 Find the range of $y = x^2 - 6x - 4$ on the domain $-1 \leqslant x \leqslant 8$.

16 When Annie hits a softball, the height of the ball above the ground after t seconds is given by $f(t) = -4.9t^2 + 19.6t + 1.4$ metres. Find the maximum height reached by the ball.

Review set 3B

1 Consider the quadratic function $y = \frac{1}{2}(x-2)^2 - 4$.

- **a** State the equation of the axis of symmetry.
- **b** Find the coordinates of the vertex.
- **c** Find the y-intercept.
- **d** Sketch the function.
- **e** State the range of the function.

2 Solve the following equations:

- **a** $x^2 - 5x - 3 = 0$
- **b** $2x^2 - 7x - 3 = 0$

3 Solve for x:

- **a** $x^2 + 5x \leqslant 14$
- **b** $2x^2 + 7x > 2(x+6)$

4 Consider the quadratic function $f(x) = -3x^2+8x+7$. Find the equation of the axis of symmetry, and the coordinates of the vertex.

5 Use the discriminant only to find the relationship between the graph and the x-axis for:

- **a** $y = 2x^2 + 3x - 7$
- **b** $y = -3x^2 - 7x + 4$

6 Determine whether each quadratic function is positive definite, negative definite, or neither:

- **a** $y = -2x^2 + 3x + 2$
- **b** $f(x) = 3x^2 + x + 11$

7 Find the equation of the quadratic function with vertex (2, 25) and y-intercept 1.

8 For what values of m does the line $y = mx - 10$ meet the curve $y = 3x^2 + 7x + 2$ twice?

9 Consider the quadratic function $y = 2x^2 + 4x - 1$.

- **a** State the axis of symmetry.
- **b** Find the coordinates of the vertex.
- **c** Find the axes intercepts.
- **d** Hence sketch the function.

10 Find the range of $y = -2x^2 + 6x + 1$ on the domain $-4 \leqslant x \leqslant 5$.

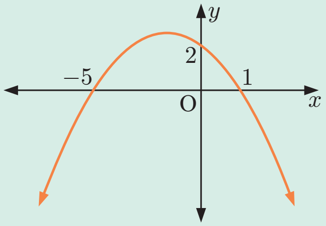
11 Find the values of k for which $kx^2 + kx - 2$ has:

- **a** a repeated root
- **b** two distinct real roots
- **c** no real roots.

12
- **a** For what values of c do the lines with equations $y = 3x + c$ intersect the parabola $y = x^2 + x - 5$ in two distinct points?
- **b** Choose one such value of c from part **a** and find the points of intersection in this case.

13 **a** Find the equation of the quadratic function illustrated.

b Find the vertex of the quadratic.

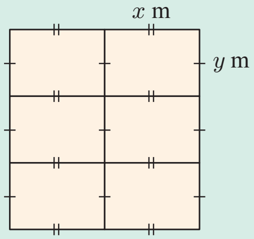


14 Find the maximum or minimum value of the quadratic, and the corresponding value of x:

a $y = 3x^2 + 4x + 7$

b $y = -2x^2 - 5x + 2$

15 600 m of fencing is used to construct 6 rectangular animal pens as shown.



a Show that the area A of each pen is $A = x\left(\frac{600-8x}{9}\right)$ m^2.

b Find the dimensions of each pen so that it has the maximum possible area.

c What is the area of each pen in this case?

16 Sketch the graph of $f(x) = \left|x^2 + x - 20\right|$.

4

Surds, indices, and exponentials

Contents:

Opening problem

The interior of a freezer has temperature $-10°C$. When a packet of peas is placed in the freezer, its temperature after t minutes is given by $T(t) = -10 + 32 \times 2^{-0.2t}$ °C.

Things to think about:

- **a** What was the temperature of the packet of peas:
  - **i** when it was first placed in the freezer
  - **ii** after 5 minutes
  - **iii** after 10 minutes
  - **iv** after 15 minutes?
- **b** What does the graph of temperature over time look like?
- **c** According to this model, will the temperature of the packet of peas ever reach $-10°C$? Explain your answer.

We often deal with numbers that are repeatedly multiplied together. Mathematicians use **indices**, also called **powers** or **exponents**, to construct such expressions.

Indices have many applications in the areas of finance, engineering, physics, electronics, biology, and computer science.

A SURDS

A **radical** is any number which is written with the **radical sign** $\sqrt{}$.

A **surd** is a real, irrational radical such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, or $\sqrt{6}$. Surds are present in solutions to some quadratic equations. $\sqrt{4}$ is a radical, but is not a surd as it simplifies to 2.

$\sqrt{a}$ is the non-negative number such that $\sqrt{a} \times \sqrt{a} = a$.

Important properties of surds are:

- $\sqrt{a}$ is never negative, so $\sqrt{a} \geqslant 0$.
- $\sqrt{a}$ is only real if $a \geqslant 0$.
- $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ for $a \geqslant 0$ and $b \geqslant 0$.
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ for $a \geqslant 0$ and $b > 0$.

Example 1

Self Tutor

Write as a single surd: **a** $\sqrt{2} \times \sqrt{3}$ **b** $\frac{\sqrt{18}}{\sqrt{6}}$

a
$$\begin{aligned} &\sqrt{2} \times \sqrt{3} \\ &= \sqrt{2 \times 3} \\ &= \sqrt{6} \end{aligned}$$

b
$$\begin{aligned} &\frac{\sqrt{18}}{\sqrt{6}} \\ &= \sqrt{\frac{18}{6}} \\ &= \sqrt{3} \end{aligned}$$

EXERCISE 4A.1

1 Write as a single surd or rational number:

a $\sqrt{11}\times\sqrt{11}$
b $\sqrt{3}\times\sqrt{5}$
c $(\sqrt{3})^2$
d $\sqrt{5}\times\sqrt{6}$
e $2\sqrt{2}\times\sqrt{2}$
f $3\sqrt{2}\times2\sqrt{2}$
g $3\sqrt{7}\times2\sqrt{7}$
h $(3\sqrt{5})^2$
i $\dfrac{\sqrt{12}}{\sqrt{2}}$
j $\dfrac{\sqrt{18}}{\sqrt{3}}$
k $\dfrac{\sqrt{20}}{\sqrt{5}}$
l $\dfrac{\sqrt{6}\times\sqrt{10}}{\sqrt{12}}$

Example 2

Self Tutor

Write $\sqrt{18}$ in the form $a\sqrt{b}$ where a and b are integers and a is as large as possible.

$$\begin{aligned}&\sqrt{18}\\ =\;&\sqrt{9\times2} && \{9 \text{ is the largest perfect square factor of } 18\}\\ =\;&\sqrt{9}\times\sqrt{2}\\ =\;&3\sqrt{2}\end{aligned}$$

2 Write in the form $a\sqrt{b}$ where a and b are integers and a is as large as possible:

a $\sqrt{8}$
b $\sqrt{12}$
c $\sqrt{20}$
d $\sqrt{32}$
e $\sqrt{27}$
f $\sqrt{45}$
g $\sqrt{48}$
h $\sqrt{54}$
i $\sqrt{50}$
j $\sqrt{80}$
k $\sqrt{96}$
l $\sqrt{108}$

OPERATING WITH SURDS

The rules for adding, subtracting, and multiplying by surds are the same as those for ordinary algebra.

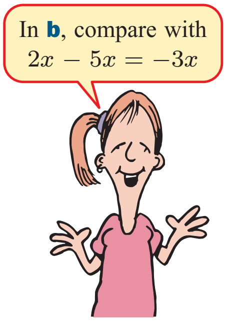
Example 3

Self Tutor

Simplify: **a** $3\sqrt{3}+5\sqrt{3}$ **b** $2\sqrt{2}-5\sqrt{2}$

a $$\begin{aligned}&3\sqrt{3}+5\sqrt{3}\\ =\;&8\sqrt{3}\end{aligned}$$

b $$\begin{aligned}&2\sqrt{2}-5\sqrt{2}\\ =\;&-3\sqrt{2}\end{aligned}$$



Example 4

Self Tutor

Simplify:

a $\sqrt{5}(6-\sqrt{5})$ **b** $(6+\sqrt{3})(1+2\sqrt{3})$

a $$\begin{aligned}&\sqrt{5}(6-\sqrt{5})\\ =\;&\sqrt{5}\times6+\sqrt{5}\times-\sqrt{5}\\ =\;&6\sqrt{5}-5\end{aligned}$$

b $$\begin{aligned}&(6+\sqrt{3})(1+2\sqrt{3})\\ =\;&6+6(2\sqrt{3})+\sqrt{3}(1)+\sqrt{3}(2\sqrt{3})\\ =\;&6+12\sqrt{3}+\sqrt{3}+6\\ =\;&12+13\sqrt{3}\end{aligned}$$

EXERCISE 4A.2

1 Simplify:

a $2\sqrt{2}+3\sqrt{2}$
b $2\sqrt{2}-3\sqrt{2}$
c $5\sqrt{5}-3\sqrt{5}$
d $5\sqrt{5}+3\sqrt{5}$
e $3\sqrt{5}-5\sqrt{5}$
f $7\sqrt{3}+2\sqrt{3}$
g $9\sqrt{6}-12\sqrt{6}$
h $\sqrt{2}+\sqrt{2}+\sqrt{2}$

2 Simplify:

a $\sqrt{2}(3-\sqrt{2})$
b $\sqrt{5}(\sqrt{5}+1)$
c $\sqrt{10}(3+2\sqrt{10})$
d $\sqrt{7}(3\sqrt{7}-4)$
e $-\sqrt{3}(5+\sqrt{3})$
f $2\sqrt{6}(\sqrt{6}-7)$
g $-\sqrt{8}(\sqrt{8}-5)$
h $-3\sqrt{2}(4-6\sqrt{2})$

3 Simplify:

a $(5+\sqrt{2})(4+\sqrt{2})$
b $(7+2\sqrt{3})(4+\sqrt{3})$
c $(9-\sqrt{7})(4+2\sqrt{7})$
d $(\sqrt{3}+1)(2-3\sqrt{3})$
e $(\sqrt{8}-6)(2\sqrt{8}-3)$
f $(2\sqrt{5}-7)(1-4\sqrt{5})$

Example 5

Self Tutor

Simplify:

a $(5-\sqrt{2})^2$
b $(7+2\sqrt{5})(7-2\sqrt{5})$

a
$$\begin{aligned} &(5-\sqrt{2})^2 \\ &= 5^2+2(5)(-\sqrt{2})+(\sqrt{2})^2 \\ &= 25-10\sqrt{2}+2 \\ &= 27-10\sqrt{2} \end{aligned}$$

b
$$\begin{aligned} &(7+2\sqrt{5})(7-2\sqrt{5}) \\ &= 7^2-(2\sqrt{5})^2 \\ &= 49-(4\times 5) \\ &= 29 \end{aligned}$$

4 Simplify:

a $(3+\sqrt{2})^2$
b $(6-\sqrt{3})^2$
c $(\sqrt{5}+1)^2$
d $(\sqrt{8}-3)^2$
e $(4+2\sqrt{3})^2$
f $(3\sqrt{5}+1)^2$
g $(7-2\sqrt{10})^2$
h $(5\sqrt{6}-4)^2$

5 Simplify:

a $(3+\sqrt{7})(3-\sqrt{7})$
b $(\sqrt{2}+5)(\sqrt{2}-5)$
c $(4-\sqrt{3})(4+\sqrt{3})$
d $(2\sqrt{2}+1)(2\sqrt{2}-1)$
e $(4+3\sqrt{8})(4-3\sqrt{8})$
f $(9\sqrt{3}-5)(9\sqrt{3}+5)$

DIVISION BY SURDS

Numbers like $\frac{6}{\sqrt{2}}$ and $\frac{9}{5+\sqrt{2}}$ involve dividing by a surd.

It is customary to 'simplify' these numbers by rewriting them without the surd in the denominator.

For any fraction of the form $\frac{b}{\sqrt{a}}$, we can remove the surd from the denominator by multiplying by $\frac{\sqrt{a}}{\sqrt{a}}$.

Since $\frac{\sqrt{a}}{\sqrt{a}}=1$, this does not change the value of the fraction.

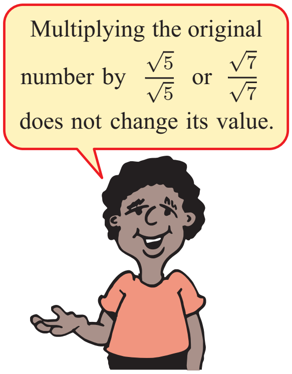
Example 6 Self Tutor

Write with an integer denominator:

a $\frac{6}{\sqrt{5}}$ **b** $\frac{35}{\sqrt{7}}$

a
$$\frac{6}{\sqrt{5}}$$
$$= \frac{6}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{6\sqrt{5}}{5}$$

b
$$\frac{35}{\sqrt{7}}$$
$$= \frac{35}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}}$$
$$= \frac{{}^{5}\cancel{35}\sqrt{7}}{{}_{1}\cancel{7}}$$
$$= 5\sqrt{7}$$



For any fraction of the form $\frac{c}{a+\sqrt{b}}$, we can remove the surd from the denominator by multiplying by $\frac{a-\sqrt{b}}{a-\sqrt{b}}$.

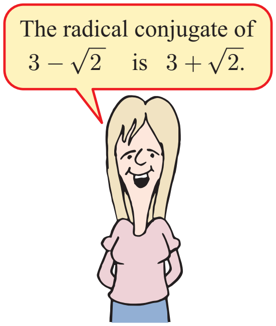
Expressions such as $a+\sqrt{b}$ and $a-\sqrt{b}$ are known as **radical conjugates**. They are identical except for the sign in the middle.

The product of radical conjugates is rational, since we have the difference between two squares. Multiplying by $\frac{a-\sqrt{b}}{a-\sqrt{b}}$ therefore produces a rational denominator, so it is sometimes called **rationalising the denominator**.

Example 7 Self Tutor

Write $\frac{5}{3-\sqrt{2}}$ with an integer denominator.

$$\frac{5}{3-\sqrt{2}} = \left(\frac{5}{3-\sqrt{2}}\right)\left(\frac{3+\sqrt{2}}{3+\sqrt{2}}\right)$$
$$= \frac{5(3+\sqrt{2})}{3^2-(\sqrt{2})^2}$$
$$= \frac{15+5\sqrt{2}}{7}$$



EXERCISE 4A.3

1 Write with integer denominator:

a $\frac{1}{\sqrt{3}}$ **b** $\frac{3}{\sqrt{3}}$ **c** $\frac{9}{\sqrt{3}}$ **d** $\frac{11}{\sqrt{3}}$ **e** $\frac{\sqrt{2}}{3\sqrt{3}}$

f $\frac{2}{\sqrt{2}}$ **g** $\frac{6}{\sqrt{2}}$ **h** $\frac{12}{\sqrt{2}}$ **i** $\frac{\sqrt{3}}{\sqrt{2}}$ **j** $\frac{1}{4\sqrt{2}}$

2 Write with integer denominator:

a $\frac{5}{\sqrt{5}}$
b $\frac{15}{\sqrt{5}}$
c $\frac{-3}{\sqrt{5}}$
d $\frac{200}{\sqrt{5}}$
e $\frac{1}{3\sqrt{5}}$
f $\frac{7}{\sqrt{7}}$
g $\frac{21}{\sqrt{7}}$
h $\frac{2}{\sqrt{11}}$
i $\frac{26}{\sqrt{13}}$
j $\frac{1}{(\sqrt{3})^3}$

3 Rationalise the denominator:

a $\frac{1}{3+\sqrt{2}}$
b $\frac{2}{3-\sqrt{2}}$
c $\frac{1}{2+\sqrt{5}}$
d $\frac{\sqrt{2}}{2-\sqrt{2}}$
e $\frac{10}{\sqrt{6}-1}$
f $\frac{\sqrt{3}}{\sqrt{7}+2}$
g $\frac{1+\sqrt{2}}{1-\sqrt{2}}$
h $\frac{\sqrt{3}}{4-\sqrt{3}}$
i $\frac{-2\sqrt{2}}{1-\sqrt{2}}$
j $\frac{1+\sqrt{5}}{2-\sqrt{5}}$
k $\frac{\sqrt{3}+2}{\sqrt{3}-1}$
l $\frac{\sqrt{10}-7}{\sqrt{10}+4}$

Example 8 — Self Tutor

Write $\frac{1}{5+\sqrt{2}}$ in the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Q}$.

$$\begin{aligned}\frac{1}{5+\sqrt{2}} &= \left(\frac{1}{5+\sqrt{2}}\right) \times \left(\frac{5-\sqrt{2}}{5-\sqrt{2}}\right)\\ &= \frac{5-\sqrt{2}}{25-2}\\ &= \frac{5-\sqrt{2}}{23}\\ &= \tfrac{5}{23} - \tfrac{1}{23}\sqrt{2}\end{aligned}$$

4 Write in the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Q}$:

a $\frac{3}{\sqrt{2}-3}$
b $\frac{4}{2+\sqrt{2}}$
c $\frac{\sqrt{2}}{\sqrt{2}-5}$
d $\frac{-2\sqrt{2}}{\sqrt{2}+1}$

5 Write in the form $a+b\sqrt{3}$ where $a, b \in \mathbb{Q}$:

a $\frac{4}{1-\sqrt{3}}$
b $\frac{6}{\sqrt{3}+2}$
c $\frac{\sqrt{3}}{2-\sqrt{3}}$
d $\frac{1+2\sqrt{3}}{3+\sqrt{3}}$

6 **a** Suppose a, b, and c are integers, $c > 0$. Show that $(a+b\sqrt{c})(a-b\sqrt{c})$ is also an integer.

b Write with an integer denominator:

i $\frac{1}{1+2\sqrt{3}}$
ii $\frac{\sqrt{2}}{3\sqrt{2}-5}$
iii $\frac{\sqrt{2}-1}{3-2\sqrt{2}}$

7 **a** Suppose a and b are positive integers. Show that $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$ is also an integer.

b Write with an integer denominator:

i $\frac{1}{\sqrt{2}+\sqrt{3}}$
ii $\frac{\sqrt{3}}{\sqrt{3}-\sqrt{5}}$
iii $\frac{\sqrt{11}-\sqrt{14}}{\sqrt{11}+\sqrt{14}}$

8 Solve the equation $2x - 3\sqrt{3} = 1 - x\sqrt{3}$. Give your solution in the form $x = a + b\sqrt{3}$, where a and b are integers.

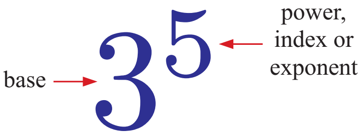
9 Find the positive solution of the equation $(9+\sqrt{5})x^2 + (5-2\sqrt{5})x - 5 = 0$. Give your answer in the form $a+b\sqrt{5}$, where $a, b \in \mathbb{Q}$.

B INDICES

If n is a positive integer, then a^n is the product of n factors of a.

$$a^n = \underbrace{a \times a \times a \times a \times \;\ldots\ldots\; \times a}_{n \text{ factors}}$$

We say that a is the **base**, and n is the **index** or **exponent**.



NEGATIVE BASES

$(-1)^1 = -1$

$(-1)^2 = -1 \times -1 = 1$

$(-1)^3 = -1 \times -1 \times -1 = -1$

$(-1)^4 = -1 \times -1 \times -1 \times -1 = 1$

$(-2)^1 = -2$

$(-2)^2 = -2 \times -2 = 4$

$(-2)^3 = -2 \times -2 \times -2 = -8$

$(-2)^4 = -2 \times -2 \times -2 \times -2 = 16$

From the patterns above we can see that:

A **negative** base raised to an **odd** index is **negative**.
A **negative** base raised to an **even** index is **positive**.

EXERCISE 4B

1 List the first six powers of:

a 2 **b** 3 **c** 4

2 Copy and complete the values of these common powers:

a $5^1 = ...$, $5^2 = ...$, $5^3 = ...$, $5^4 = ...$

b $6^1 = ...$, $6^2 = ...$, $6^3 = ...$, $6^4 = ...$

c $7^1 = ...$, $7^2 = ...$, $7^3 = ...$, $7^4 = ...$

3 Simplify, then use a calculator to check your answer:

a $(-1)^5$ **b** $(-1)^6$ **c** $(-1)^{14}$ **d** $(-1)^{19}$ **e** $(-1)^8$ **f** -1^8

g $-(-1)^8$ **h** $(-2)^5$ **i** -2^5 **j** $-(-2)^6$ **k** $(-5)^4$ **l** $-(-5)^4$

4 Use your calculator to find the value of:

a 4^7 **b** 7^4 **c** -5^5 **d** $(-5)^5$ **e** 8^6 **f** $(-8)^6$

g -8^6 **h** 2.13^9 **i** -2.13^9 **j** $(-2.13)^9$

5 Use your calculator to find the values of:

a 9^{-1} **b** $\frac{1}{9^1}$ **c** 6^{-2} **d** $\frac{1}{6^2}$ **e** 3^{-4} **f** $\frac{1}{3^4}$

g 17^0 **h** $(0.366)^0$

What do you notice?

6 Consider $3^1, 3^2, 3^3, 3^4, 3^5$ .... Look for a pattern and hence find the last digit of 3^{101}.

7 What is the last digit of 7^{217}?

Historical note

Nicomachus discovered an interesting number pattern involving cubes and sums of odd numbers.

Nicomachus was born in Roman Syria (now Jerash, Jordan) around 100 AD. He wrote in Greek, and was a Pythagorean, which means he followed the teaching of **Pythagoras**.

$$\begin{aligned} 1 &= 1^3 \\ 3+5 = 8 &= 2^3 \\ 7+9+11 = 27 &= 3^3 \\ &\vdots \end{aligned}$$

C INDEX LAWS

The **index laws** for $m, n \in \mathbb{Z}$ are:

$a^m \times a^n = a^{m+n}$	To **multiply** numbers with the **same base**, keep the base and **add** the indices.
$\dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$	To **divide** numbers with the same base, keep the base and **subtract** the indices.
$(a^m)^n = a^{m \times n}$	When **raising** a **power** to a **power**, keep the base and **multiply** the indices.
$(ab)^n = a^n b^n$	The power of a product is the product of the powers.
$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad b \neq 0$	The power of a quotient is the quotient of the powers.
$a^0 = 1, \quad a \neq 0$	Any non-zero number raised to the power of zero is 1.
$a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$ and in particular $a^{-1} = \dfrac{1}{a}, \quad a \neq 0.$	

Example 9

Self Tutor

Simplify using the index laws:

a $3^5 \times 3^4$

b $\dfrac{5^3}{5^5}$

c $(m^4)^3$

a
$$\begin{aligned} &3^5 \times 3^4 \\ &= 3^{5+4} \\ &= 3^9 \end{aligned}$$

b
$$\begin{aligned} &\frac{5^3}{5^5} \\ &= 5^{3-5} \\ &= 5^{-2} \\ &= \tfrac{1}{25} \end{aligned}$$

c
$$\begin{aligned} &(m^4)^3 \\ &= m^{4\times 3} \\ &= m^{12} \end{aligned}$$

EXERCISE 4C

1 Simplify using the index laws:

a $5^4 \times 5^7$ **b** $d^2 \times d^6$ **c** $\dfrac{k^8}{k^3}$ **d** $\dfrac{7^5}{7^6}$ **e** $(x^2)^5$ **f** $(3^4)^4$

g $\dfrac{p^3}{p^7}$ **h** $n^3 \times n^9$ **i** $(5^t)^3$ **j** $7^x \times 7^2$ **k** $\dfrac{10^3}{10^q}$ **l** $(c^4)^m$

Example 10

Self Tutor

Write as powers of 2:

a 16 **b** $\frac{1}{16}$ **c** 1 **d** 4×2^n **e** $\frac{2^m}{8}$

a
$$\begin{aligned}&16\\&= 2\times 2\times 2\times 2\\&= 2^4\end{aligned}$$

b
$$\begin{aligned}&\tfrac{1}{16}\\&= \frac{1}{2^4}\\&= 2^{-4}\end{aligned}$$

c
$$\begin{aligned}&1\\&= 2^0\end{aligned}$$

d
$$\begin{aligned}&4\times 2^n\\&= 2^2\times 2^n\\&= 2^{2+n}\end{aligned}$$

e
$$\begin{aligned}&\frac{2^m}{8}\\&= \frac{2^m}{2^3}\\&= 2^{m-3}\end{aligned}$$

2 Write as powers of 2:

a 4 **b** $\frac{1}{4}$ **c** 8 **d** $\frac{1}{8}$ **e** 32 **f** $\frac{1}{32}$

g 2 **h** $\frac{1}{2}$ **i** 64 **j** $\frac{1}{64}$ **k** 128 **l** $\frac{1}{128}$

3 Write as powers of 3:

a 9 **b** $\frac{1}{9}$ **c** 27 **d** $\frac{1}{27}$ **e** 3 **f** $\frac{1}{3}$

g 81 **h** $\frac{1}{81}$ **i** 1 **j** 243 **k** $\frac{1}{243}$

4 Write as a single power of 2:

a 2×2^a **b** 4×2^b **c** 8×2^t **d** $(2^{x+1})^2$ **e** $(2^{1-n})^{-1}$

f $\frac{2^c}{4}$ **g** $\frac{2^m}{2^{-m}}$ **h** $\frac{4}{2^{1-n}}$ **i** $\frac{2^{x+1}}{2^x}$ **j** $\frac{4^x}{2^{1-x}}$

5 Write as a single power of 3:

a 9×3^p **b** 27^a **c** 3×9^n **d** 27×3^d **e** 9×27^t

f $\frac{3^y}{3}$ **g** $\frac{3}{3^y}$ **h** $\frac{9}{27^t}$ **i** $\frac{9^a}{3^{1-a}}$ **j** $\frac{9^{n+1}}{3^{2n-1}}$

Example 11

Self Tutor

Write in simplest form, without brackets:

a $\left(-3a^2\right)^4$ **b** $\left(-\frac{2a^2}{b}\right)^3$

a
$$\begin{aligned}&\left(-3a^2\right)^4\\&= (-3)^4\times(a^2)^4\\&= 81\times a^{2\times 4}\\&= 81a^8\end{aligned}$$

b
$$\begin{aligned}&\left(-\frac{2a^2}{b}\right)^3\\&= \frac{(-2)^3\times(a^2)^3}{b^3}\\&= \frac{-8a^6}{b^3}\end{aligned}$$

6 Write without brackets:

a $(2a)^2$ **b** $(3b)^3$ **c** $(ab)^4$ **d** $(pq)^3$ **e** $\left(\frac{m}{n}\right)^2$

f $\left(\frac{a}{3}\right)^3$ **g** $\left(\frac{b}{c}\right)^4$ **h** $\left(\frac{2a}{b}\right)^0$ **i** $\left(\frac{m}{3n}\right)^4$ **j** $\left(\frac{xy}{2}\right)^3$

7 Write the following in simplest form, without brackets:

a $(-2a)^2$ **b** $(-6b^2)^2$ **c** $(-2a)^3$ **d** $(-3m^2n^2)^3$

e $(-2ab^4)^4$ **f** $\left(\frac{-2a^2}{b^2}\right)^3$ **g** $\left(\frac{-4a^3}{b}\right)^2$ **h** $\left(\frac{-3p^2}{q^3}\right)^2$

i $\frac{(2x^2y)^2}{x}$ **j** $\frac{(4a^2b)^3}{2ab^2}$ **k** $\frac{(-5a^6b^3)^2}{5b^8}$ **l** $\frac{(-2x^7y^4)^3}{4x^3y^{15}}$

Example 12 Self Tutor

Write without negative exponents: $\frac{a^{-3}b^2}{c^{-1}}$

$$a^{-3} = \frac{1}{a^3} \quad \text{and} \quad \frac{1}{c^{-1}} = c^1$$

$$\therefore \quad \frac{a^{-3}b^2}{c^{-1}} = \frac{b^2c}{a^3}$$

8 Write without negative exponents:

a ab^{-2} **b** $(ab)^{-2}$ **c** $(2ab^{-1})^2$ **d** $(3a^{-2}b)^2$ **e** $\frac{a^2b^{-1}}{c^2}$

f $\frac{a^2b^{-1}}{c^{-2}}$ **g** $\frac{1}{a^{-3}}$ **h** $\frac{a^{-2}}{b^{-3}}$ **i** $\frac{2a^{-1}}{d^2}$ **j** $\frac{12a}{m^{-3}}$

Example 13 Self Tutor

Write $\frac{1}{2^{1-n}}$ in non-fractional form.

$$\begin{aligned}\frac{1}{2^{1-n}} &= 2^{-(1-n)}\\ &= 2^{-1+n}\\ &= 2^{n-1}\end{aligned}$$

9 Write in non-fractional form:

a $\frac{1}{a^n}$ **b** $\frac{1}{b^{-n}}$ **c** $\frac{1}{3^{2-n}}$ **d** $\frac{a^n}{b^{-m}}$ **e** $\frac{a^{-n}}{a^{2+n}}$

10 Simplify, giving your answers in simplest rational form:

a $\left(\frac{5}{3}\right)^0$ **b** $\left(\frac{7}{4}\right)^{-1}$ **c** $\left(\frac{1}{6}\right)^{-1}$ **d** $\frac{3^3}{3^0}$

e $\left(\frac{4}{3}\right)^{-2}$ **f** $2^1 + 2^{-1}$ **g** $\left(1\frac{2}{3}\right)^{-3}$ **h** $5^2 + 5^1 + 5^{-1}$

11 Write as powers of 2, 3 and/or 5:

a $\frac{1}{9}$ **b** $\frac{1}{16}$ **c** $\frac{1}{125}$ **d** $\frac{3}{5}$

e $\frac{4}{27}$ **f** $\dfrac{2^c}{8 \times 9}$ **g** $\dfrac{9^k}{10}$ **h** $\dfrac{6^p}{75}$

12 Read about Nicomachus' pattern on page **108** and find the series of odd numbers for:

a 5^3 **b** 7^3 **c** 12^3

D RATIONAL INDICES

The index laws used previously can also be applied to **rational indices**, or indices which are written as a fraction.

The notation a^n is defined to mean "a multiplied together n times". Since we cannot multiply a together "half a time", the notation $a^{\frac{1}{2}}$ is an extension of the meaning of this notation. The goal is to extend the meaning of a^n so that the fundamental law

$$a^n a^m = a^{n+m}$$

remains true. If we assume that $a > 0$ then this law holds for rational indices.

Since $x^3 = -8$ has $x = -2$ as a solution, we would like to write

$$x = x^{\frac{3}{3}} = (x^3)^{\frac{1}{3}} = (-8)^{\frac{1}{3}} = (-2^3)^{\frac{1}{3}} = (-2)^{\frac{3}{3}} = -2.$$

Under some circumstances it is therefore possible to extend the meaning of a^n when n is rational and $a \leqslant 0$. However, this is not generally so easy, and so for this course we restrict ourselves to cases where $a > 0$.

For $a > 0$, notice that $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$ {index laws}
and $\sqrt{a} \times \sqrt{a} = a$ also.

So, $$\boldsymbol{a^{\frac{1}{2}} = \sqrt{a}}$$ {by direct comparison}

Likewise $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^1 = a$
and $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$

suggests $$\boldsymbol{a^{\frac{1}{3}} = \sqrt[3]{a}}$$

In general, $$\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}}$$ where $\sqrt[n]{a}$ reads 'the nth root of a', for $n \in \mathbb{Z}^+$.

We can now determine that

$$\begin{aligned} &\sqrt[n]{a^m} \\ &= (a^m)^{\frac{1}{n}} \\ &= a^{\frac{m}{n}} \end{aligned}$$

$$\therefore \quad \boldsymbol{a^{\frac{m}{n}} = \sqrt[n]{a^m}} \quad \text{for } a > 0,\ n \in \mathbb{Z}^+,\ m \in \mathbb{Z}$$

Example 14

Self Tutor

Write as a single power of 2:

a $\sqrt[3]{2}$ **b** $\frac{1}{\sqrt{2}}$ **c** $\sqrt[5]{4}$

a $\sqrt[3]{2}$

$= 2^{\frac{1}{3}}$

b $\frac{1}{\sqrt{2}}$

$= \frac{1}{2^{\frac{1}{2}}}$

$= 2^{-\frac{1}{2}}$

c $\sqrt[5]{4}$

$= (2^2)^{\frac{1}{5}}$

$= 2^{2\times\frac{1}{5}}$

$= 2^{\frac{2}{5}}$

EXERCISE 4D

1 Write as a single power of 2:

a $\sqrt[5]{2}$ **b** $\frac{1}{\sqrt[5]{2}}$ **c** $2\sqrt{2}$ **d** $4\sqrt{2}$ **e** $\frac{1}{\sqrt[3]{2}}$

f $2 \times \sqrt[3]{2}$ **g** $\frac{4}{\sqrt{2}}$ **h** $(\sqrt{2})^3$ **i** $\frac{1}{\sqrt[3]{16}}$ **j** $\frac{1}{\sqrt{8}}$

2 Write as a single power of 3:

a $\sqrt[3]{3}$ **b** $\frac{1}{\sqrt[3]{3}}$ **c** $\sqrt[4]{3}$ **d** $3\sqrt{3}$ **e** $\frac{1}{9\sqrt{3}}$

3 Write the following in the form a^x where a is a prime number and x is rational:

a $\sqrt[3]{7}$ **b** $\sqrt[4]{27}$ **c** $\sqrt[5]{16}$ **d** $\sqrt[3]{32}$ **e** $\sqrt[7]{49}$

f $\frac{1}{\sqrt[3]{7}}$ **g** $\frac{1}{\sqrt[4]{27}}$ **h** $\frac{1}{\sqrt[5]{16}}$ **i** $\frac{1}{\sqrt[3]{32}}$ **j** $\frac{1}{\sqrt[7]{49}}$

4 Use your calculator to evaluate:

a $3^{\frac{3}{4}}$ **b** $2^{\frac{7}{8}}$ **c** $2^{-\frac{1}{3}}$ **d** $4^{-\frac{3}{5}}$ **e** $\sqrt[4]{8}$

f $\sqrt[5]{27}$ **g** $\frac{1}{\sqrt[3]{7}}$

Example 15

Self Tutor

Without using a calculator, write in simplest rational form:

a $8^{\frac{4}{3}}$ **b** $27^{-\frac{2}{3}}$

a $8^{\frac{4}{3}}$

$= (2^3)^{\frac{4}{3}}$

$= 2^{3\times\frac{4}{3}}$ $\{(a^m)^n = a^{mn}\}$

$= 2^4$

$= 16$

b $27^{-\frac{2}{3}}$

$= (3^3)^{-\frac{2}{3}}$

$= 3^{3\times-\frac{2}{3}}$

$= 3^{-2}$

$= \frac{1}{9}$

5 Without using a calculator, write in simplest rational form:

a $4^{\frac{3}{2}}$ **b** $8^{\frac{5}{3}}$ **c** $16^{\frac{3}{4}}$ **d** $25^{\frac{3}{2}}$ **e** $32^{\frac{2}{5}}$

f $4^{-\frac{1}{2}}$ **g** $9^{-\frac{3}{2}}$ **h** $8^{-\frac{4}{3}}$ **i** $27^{-\frac{4}{3}}$ **j** $125^{-\frac{2}{3}}$

E ALGEBRAIC EXPANSION AND FACTORISATION

EXPANSION

We can use the usual expansion laws to simplify expressions containing indices:

$$\begin{aligned} \boldsymbol{a(b+c)} &= \boldsymbol{ab+ac} \\ \boldsymbol{(a+b)(c+d)} &= \boldsymbol{ac+ad+bc+bd} \\ \boldsymbol{(a+b)(a-b)} &= \boldsymbol{a^2-b^2} \\ \boldsymbol{(a+b)^2} &= \boldsymbol{a^2+2ab+b^2} \\ \boldsymbol{(a-b)^2} &= \boldsymbol{a^2-2ab+b^2} \end{aligned}$$

Example 16

Self Tutor

Expand and simplify: $x^{-\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}-3x^{-\frac{1}{2}})$

$x^{-\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}-3x^{-\frac{1}{2}})$

$=x^{-\frac{1}{2}}\times x^{\frac{3}{2}}+x^{-\frac{1}{2}}\times 2x^{\frac{1}{2}}-x^{-\frac{1}{2}}\times 3x^{-\frac{1}{2}}$ {each term is × by $x^{-\frac{1}{2}}$}

$=x^1+2x^0-3x^{-1}$ {adding exponents}

$=x+2-\dfrac{3}{x}$

Example 17

Self Tutor

Expand and simplify:

a $(2^x+3)(2^x+1)$ **b** $(7^x+7^{-x})^2$

a $(2^x+3)(2^x+1)$

$=2^x\times 2^x+2^x+3\times 2^x+3$

$=2^{2x}+4\times 2^x+3$

$=4^x+2^{2+x}+3$

b $(7^x+7^{-x})^2$

$=(7^x)^2+2\times 7^x\times 7^{-x}+(7^{-x})^2$

$=7^{2x}+2\times 7^0+7^{-2x}$

$=7^{2x}+2+7^{-2x}$

EXERCISE 4E.1

1 Expand and simplify:

a $x^2(x^3+2x^2+1)$ **b** $2^x(2^x+1)$ **c** $x^{\frac{1}{2}}(x^{\frac{1}{2}}+x^{-\frac{1}{2}})$

d $7^x(7^x+2)$ **e** $3^x(2-3^{-x})$ **f** $x^{\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}})$

g $2^{-x}(2^x+5)$ **h** $5^{-x}(5^{2x}+5^x)$ **i** $x^{-\frac{1}{2}}(x^2+x+x^{\frac{1}{2}})$

2 Expand and simplify:

a $(2^x - 1)(2^x + 3)$ **b** $(3^x + 2)(3^x + 5)$ **c** $(5^x - 2)(5^x - 4)$

d $(2^x + 3)^2$ **e** $(3^x - 1)^2$ **f** $(4^x + 7)^2$

3 Expand and simplify:

a $(x^{\frac{1}{2}} + 2)(x^{\frac{1}{2}} - 2)$ **b** $(2^x + 3)(2^x - 3)$ **c** $(x^{\frac{1}{2}} + x^{-\frac{1}{2}})(x^{\frac{1}{2}} - x^{-\frac{1}{2}})$

d $(x + \frac{2}{x})^2$ **e** $(7^x - 7^{-x})^2$ **f** $(5 - 2^{-x})^2$

g $(x^{\frac{2}{3}} + x^{\frac{1}{3}})^2$ **h** $(x^{\frac{3}{2}} - x^{\frac{1}{2}})^2$ **i** $(2x^{\frac{1}{2}} - x^{-\frac{1}{2}})^2$

FACTORISATION AND SIMPLIFICATION

Example 18

Self Tutor

Factorise: **a** $2^{n+3} + 2^n$ **b** $2^{n+3} + 8$ **c** $2^{3n} + 2^{2n}$

a
$$\begin{aligned} &2^{n+3} + 2^n \\ &= 2^n 2^3 + 2^n \\ &= 2^n(2^3 + 1) \\ &= 2^n \times 9 \end{aligned}$$

b
$$\begin{aligned} &2^{n+3} + 8 \\ &= 2^n 2^3 + 8 \\ &= 8(2^n) + 8 \\ &= 8(2^n + 1) \end{aligned}$$

c
$$\begin{aligned} &2^{3n} + 2^{2n} \\ &= 2^{2n} 2^n + 2^{2n} \\ &= 2^{2n}(2^n + 1) \end{aligned}$$

Example 19

Self Tutor

Factorise: **a** $4^x - 9$ **b** $9^x + 4(3^x) + 4$

a
$$\begin{aligned} &4^x - 9 \\ &= (2^x)^2 - 3^2 && \{\text{compare } a^2 - b^2 = (a+b)(a-b)\} \\ &= (2^x + 3)(2^x - 3) \end{aligned}$$

b
$$\begin{aligned} &9^x + 4(3^x) + 4 \\ &= (3^x)^2 + 4(3^x) + 4 && \{\text{compare } a^2 + 4a + 4\} \\ &= (3^x + 2)^2 && \{\text{as } a^2 + 4a + 4 = (a+2)^2\} \end{aligned}$$

EXERCISE 4E.2

1 Factorise:

a $5^{2x} + 5^x$ **b** $3^{n+2} + 3^n$ **c** $7^n + 7^{3n}$

d $5^{n+1} - 5$ **e** $6^{n+2} - 6$ **f** $4^{n+2} - 16$

2 Factorise:

a $9^x - 4$ **b** $4^x - 25$ **c** $16 - 9^x$

d $25 - 4^x$ **e** $9^x - 4^x$ **f** $4^x + 6(2^x) + 9$

g $9^x + 10(3^x) + 25$ **h** $4^x - 14(2^x) + 49$ **i** $25^x - 4(5^x) + 4$

3 Factorise:

a $4^x + 9(2^x) + 18$ **b** $4^x - 2^x - 20$ **c** $9^x + 9(3^x) + 14$

d $9^x + 4(3^x) - 5$ **e** $25^x + 5^x - 2$ **f** $49^x - 7^{x+1} + 12$

Example 20

Self Tutor

Simplify:

a $\dfrac{6^n}{3^n}$ **b** $\dfrac{4^n}{6^n}$

a $\dfrac{6^n}{3^n}$ **or** $\dfrac{6^n}{3^n}$

$= \dfrac{2^n\cancel{3^n}}{{}_1\cancel{3^n}}$ $= \left(\frac{6}{3}\right)^n$

$= 2^n$ $= 2^n$

b $\dfrac{4^n}{6^n}$ **or** $\dfrac{4^n}{6^n}$

$= \dfrac{\cancel{2^n}2^n}{\cancel{2^n}3^n}$ $= \left(\frac{4}{6}\right)^n$

$= \dfrac{2^n}{3^n}$ $= \left(\frac{2}{3}\right)^n$

4 Simplify:

a $\dfrac{12^n}{6^n}$ **b** $\dfrac{20^a}{2^a}$ **c** $\dfrac{6^b}{2^b}$ **d** $\dfrac{4^n}{20^n}$

e $\dfrac{35^x}{7^x}$ **f** $\dfrac{6^a}{8^a}$ **g** $\dfrac{5^{n+1}}{5^n}$ **h** $\dfrac{5^{n+1}}{5}$

Example 21

Self Tutor

Simplify:

a $\dfrac{3^n + 6^n}{3^n}$ **b** $\dfrac{2^{m+2} - 2^m}{2^m}$ **c** $\dfrac{2^{m+3} + 2^m}{9}$

a $\dfrac{3^n + 6^n}{3^n}$

$= \dfrac{3^n + 2^n3^n}{3^n}$

$= \dfrac{\cancel{3^n}(1 + 2^n)}{{}_1\cancel{3^n}}$

$= 1 + 2^n$

b $\dfrac{2^{m+2} - 2^m}{2^m}$

$= \dfrac{2^m2^2 - 2^m}{2^m}$

$= \dfrac{\cancel{2^m}(4 - 1)}{{}_1\cancel{2^m}}$

$= 3$

c $\dfrac{2^{m+3} + 2^m}{9}$

$= \dfrac{2^m2^3 + 2^m}{9}$

$= \dfrac{2^m\cancel{(8 + 1)}}{{}_1\cancel{9}}$

$= 2^m$

5 Simplify:

a $\dfrac{6^m + 2^m}{2^m}$ **b** $\dfrac{2^n + 12^n}{2^n}$ **c** $\dfrac{8^n + 4^n}{2^n}$

d $\dfrac{12^x - 3^x}{3^x}$ **e** $\dfrac{6^n + 12^n}{1 + 2^n}$ **f** $\dfrac{5^{n+1} - 5^n}{4}$

g $\dfrac{5^{n+1} - 5^n}{5^n}$ **h** $\dfrac{4^n - 2^n}{2^n}$ **i** $\dfrac{2^n - 2^{n-1}}{2^n}$

6 Simplify:

a $2^n(n + 1) + 2^n(n - 1)$ **b** $3^n\left(\dfrac{n - 1}{6}\right) - 3^n\left(\dfrac{n + 1}{6}\right)$

F EXPONENTIAL EQUATIONS

An **exponential equation** is an equation in which the unknown occurs as part of the index or exponent.

For example: $2^x = 8$ and $30 \times 3^x = 7$ are both exponential equations.

There are a number of methods we can use to solve exponential equations. These include graphing, using technology, and by using **logarithms**, which we will study in **Chapter 5**. However, in some cases we can solve algebraically.

If the base numbers are the same, we can **equate indices**.
If $\boldsymbol{a^x = a^k}$ then $\boldsymbol{x = k}$.

For example, if $2^x = 8$ then $2^x = 2^3$. Thus $x = 3$, and this is the only solution.

Example 22

Self Tutor

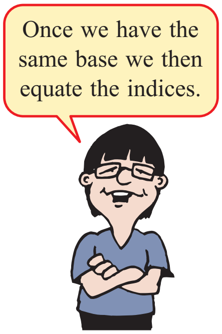
Solve for x:

a $2^x = 16$

b $3^{x+2} = \frac{1}{27}$

a
$$2^x = 16$$
$$\therefore\ 2^x = 2^4$$
$$\therefore\ x = 4$$

b
$$3^{x+2} = \tfrac{1}{27}$$
$$\therefore\ 3^{x+2} = 3^{-3}$$
$$\therefore\ x + 2 = -3$$
$$\therefore\ x = -5$$



Example 23

Self Tutor

Solve for x:

a $4^x = 8$

b $9^{x-2} = \frac{1}{3}$

a
$$4^x = 8$$
$$\therefore\ (2^2)^x = 2^3$$
$$\therefore\ 2^{2x} = 2^3$$
$$\therefore\ 2x = 3$$
$$\therefore\ x = \tfrac{3}{2}$$

b
$$9^{x-2} = \tfrac{1}{3}$$
$$\therefore\ (3^2)^{x-2} = 3^{-1}$$
$$\therefore\ 3^{2(x-2)} = 3^{-1}$$
$$\therefore\ 2(x-2) = -1$$
$$\therefore\ 2x - 4 = -1$$
$$\therefore\ 2x = 3$$
$$\therefore\ x = \tfrac{3}{2}$$

EXERCISE 4F

1 Solve for x:

a $2^x = 8$ **b** $5^x = 25$ **c** $3^x = 81$ **d** $7^x = 1$

e $3^x = \frac{1}{3}$ **f** $2^x = \sqrt{2}$ **g** $5^x = \frac{1}{125}$ **h** $4^{x+1} = 64$

i $2^{x-2} = \frac{1}{32}$ **j** $3^{x+1} = \frac{1}{27}$ **k** $7^{x+1} = 343$ **l** $5^{1-2x} = \frac{1}{5}$

2 Solve for x:

a $8^x = 32$ **b** $4^x = \frac{1}{8}$ **c** $9^x = \frac{1}{27}$ **d** $25^x = \frac{1}{5}$

e $27^x = \frac{1}{9}$ **f** $16^x = \frac{1}{32}$ **g** $4^{x+2} = 128$ **h** $25^{1-x} = \frac{1}{125}$

i $4^{4x-1} = \frac{1}{2}$ **j** $9^{x-3} = 27$ **k** $(\frac{1}{2})^{x+1} = 8$ **l** $(\frac{1}{3})^{x+2} = 9$

m $81^x = 27^{-x}$ **n** $(\frac{1}{4})^{1-x} = 32$ **o** $(\frac{1}{7})^x = 49$ **p** $(\frac{1}{3})^{x+1} = 243$

3 Solve for x, if possible:

a $4^{2x+1} = 8^{1-x}$ **b** $9^{2-x} = (\frac{1}{3})^{2x+1}$ **c** $2^x \times 8^{1-x} = \frac{1}{4}$

4 Solve for x:

a $\dfrac{3^{2x+1}}{3^x} = 9^x$ **b** $\dfrac{25^x}{5^{x+4}} = 25^{1-x}$ **c** $\dfrac{4^x}{2^{x+2}} = \dfrac{2^{x+1}}{8^x}$

d $\dfrac{5^{2x-5}}{125^x} = \dfrac{25^{1-2x}}{5^{x+2}}$ **e** $\dfrac{4^x}{8^{2-x}} = 2^x \times 4^{x-1}$ **f** $\dfrac{9^{2x}}{27^{2-x}} = \dfrac{81^{3x+1}}{3^{1-2x}}$

5 Solve for x:

a $3 \times 2^x = 24$ **b** $7 \times 2^x = 28$ **c** $3 \times 2^{x+1} = 24$

d $12 \times 3^{-x} = \frac{4}{3}$ **e** $4 \times (\frac{1}{3})^x = 36$ **f** $5 \times (\frac{1}{2})^x = 20$

Example 24

Self Tutor

Solve for x: $4^x + 2^x - 20 = 0$

$$
\begin{aligned}
& 4^x + 2^x - 20 = 0 \\
\therefore \quad & (2^x)^2 + 2^x - 20 = 0 && \{\text{compare } a^2 + a - 20 = 0\} \\
\therefore \quad & (2^x - 4)(2^x + 5) = 0 && \{a^2 + a - 20 = (a-4)(a+5)\} \\
\therefore \quad & 2^x = 4 \text{ or } 2^x = -5 \\
\therefore \quad & 2^x = 2^2 && \{2^x \text{ cannot be negative}\} \\
\therefore \quad & x = 2
\end{aligned}
$$

6 Solve for x:

a $4^x - 6(2^x) + 8 = 0$ **b** $4^x - 2^x - 2 = 0$ **c** $9^x - 12(3^x) + 27 = 0$

d $9^x = 3^x + 6$ **e** $25^x - 23(5^x) - 50 = 0$ **f** $49^x + 1 = 2(7^x)$

G EXPONENTIAL FUNCTIONS

We have already seen how to evaluate b^n when $n \in \mathbb{Q}$, or in other words when n is a rational number.

But what about b^n when $n \in \mathbb{R}$, so n is real but not necessarily rational?

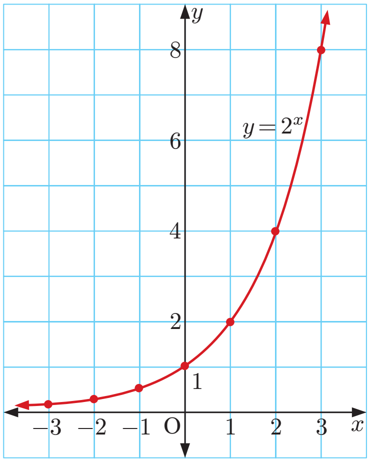
To answer this question, we can look at graphs of exponential functions.

The most simple general **exponential function** has the form $y = b^x$ where $b > 0$, $b \neq 1$.

For example, $y = 2^x$ is an exponential function.

We construct a table of values from which we graph the function:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



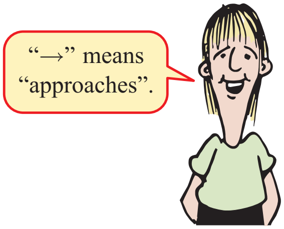
When $x = -10$, $y = 2^{-10} \approx 0.001$.
When $x = -50$, $y = 2^{-50} \approx 8.88 \times 10^{-16}$.

As x becomes large and negative, the graph of $y = 2^x$ approaches the x-axis from above but never touches it, since 2^x becomes very small but never zero.

So, as $x \to -\infty$, $y \to 0$ from above.

We say that $y = 2^x$ is '**asymptotic** to the x-axis' or '$y = 0$ is a **horizontal asymptote**'.

We now have a well-defined meaning for b^n where $b, n \in \mathbb{R}$ because simple exponential functions have smooth increasing or decreasing graphs.



Discovery 1 — Graphs of exponential functions

In this Discovery we examine the graphs of various families of exponential functions.

Click on the icon to run the **dynamic graphing package**, or else you could use your **graphics calculator**.

DYNAMIC GRAPHING PACKAGE

What to do:

1 Explore the family of curves of the form $y = b^x$ where $b > 0$.
For example, consider $y = 2^x$, $y = 3^x$, $y = 10^x$, and $y = (1.3)^x$.

- **a** What effect does changing b have on the shape of the graph?
- **b** What is the y-intercept of each graph?
- **c** What is the horizontal asymptote of each graph?

2 Explore the family of curves of the form $y = 2^x + d$ where d is a constant.
For example, consider $y = 2^x$, $y = 2^x + 1$, and $y = 2^x - 2$.

- **a** What effect does changing d have on the position of the graph?
- **b** What effect does changing d have on the shape of the graph?
- **c** What is the horizontal asymptote of each graph?
- **d** What is the horizontal asymptote of $y = 2^x + d$?
- **e** To graph $y = 2^x + d$ from $y = 2^x$ what transformation is used?

3 Explore the family of curves of the form $y = 2^{x-c}$.
For example, consider $y = 2^x$, $y = 2^{x-1}$, $y = 2^{x+2}$, and $y = 2^{x-3}$.

- **a** What effect does changing c have on the position of the graph?
- **b** What effect does changing c have on the shape of the graph?
- **c** What is the horizontal asymptote of each graph?
- **d** To graph $y = 2^{x-c}$ from $y = 2^x$ what transformation is used?

4 Explore the relationship between $y = b^x$ and $y = b^{-x}$ where $b > 0$.
For example, consider $y = 2^x$ and $y = 2^{-x}$.

- **a** What is the y-intercept of each graph?
- **b** What is the horizontal asymptote of each graph?
- **c** What transformation moves $y = 2^x$ to $y = 2^{-x}$?

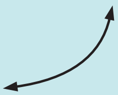
5 Explore the family of curves of the form $y = a \times 2^x$ where a is a constant.

- **a** Consider functions where $a > 0$, such as $y = 2^x$, $y = 3 \times 2^x$, and $y = \frac{1}{2} \times 2^x$.
  Comment on the effect on the graph.
- **b** Consider functions where $a < 0$, such as $y = -2^x$, $y = -3 \times 2^x$, and $y = -\frac{1}{2} \times 2^x$.
  Comment on the effect on the graph.
- **c** What is the horizontal asymptote of each graph? Explain your answer.

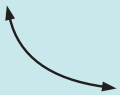
From **Discovery 1** you should have found that:

For the general exponential function $\boldsymbol{y = a \times b^{x-c} + d}$ where $b > 0$, $b \neq 1$, $a \neq 0$:

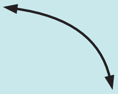
- b controls how steeply the graph increases or decreases
- c controls horizontal translation
- d controls vertical translation
- the equation of the horizontal asymptote is $y = d$
- if $a > 0$, $b > 1$
  the function is
  increasing



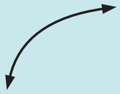
- if $a > 0$, $0 < b < 1$
  the function is
  decreasing



- if $a < 0$, $b > 1$
  the function is
  decreasing

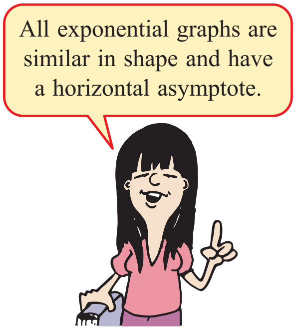


- if $a < 0$, $0 < b < 1$
  the function is
  increasing.



We can sketch reasonably accurate graphs of exponential functions using:

- the horizontal asymptote
- the y-intercept
- two other points, for example, when $x = 2$, $x = -2$



Example 25

Self Tutor

Sketch the graph of $y = 2^{-x} - 3$.
Hence state the domain and range of $f(x) = 2^{-x} - 3$.

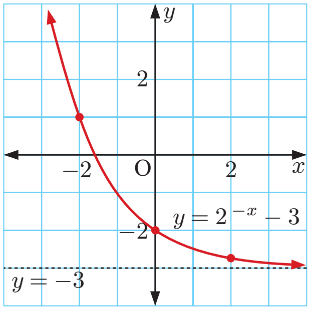
For $y = 2^{-x} - 3$,
the horizontal asymptote is $y = -3$.

When $x = 0$, $y = 2^0 - 3$
$= 1 - 3$
$= -2$

$\therefore$ the y-intercept is -2.

When $x = 2$, $y = 2^{-2} - 3$
$= \frac{1}{4} - 3$
$= -2\frac{3}{4}$

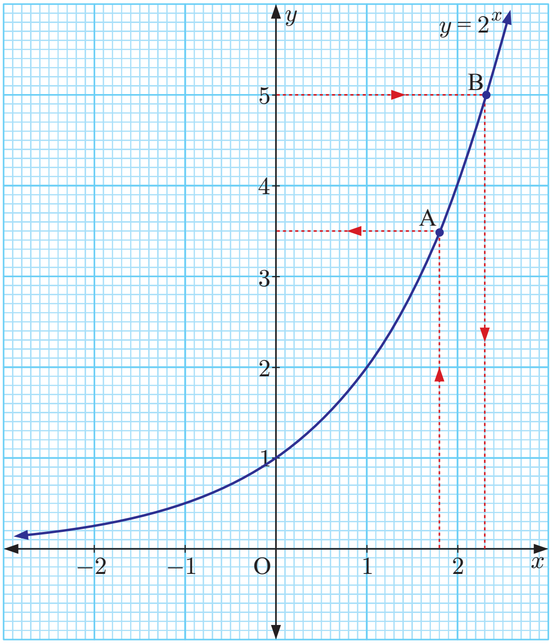
When $x = -2$, $y = 2^2 - 3 = 1$



The domain is $\{x : x \in \mathbb{R}\}$. The range is $\{y : y > -3\}$.

Consider the graph of $y = 2^x$ alongside. We can use the graph to estimate:

- the value of 2^x for a given value of x, for example $2^{1.8} \approx 3.5$ {point A}
- the solutions of the exponential equation $2^x = b$, for example if $2^x = 5$ then $x \approx 2.3$ {point B}.



EXERCISE 4G

1 Use the graph of $y = 2^x$ to estimate the value of:

a $2^{\frac{1}{2}}$ or $\sqrt{2}$ **b** $2^{0.8}$ **c** $2^{1.5}$ **d** $2^{-\sqrt{2}}$

2 Use the graph of $y = 2^x$ to estimate the solution to:

a $2^x = 3$ **b** $2^x = 0.6$ **c** $2^x = 4.3$ **d** $2^x = 0.3$

3 Use the graph of $y = 2^x$ to explain why $2^x = 0$ has no solutions.

4 Suppose $f(x) = 2 \times 3^x$. Find:

a $f(0)$ **b** $f(3)$ **c** $f(-2)$

5 Suppose $g(x) = 5^x + 2$.

a Find $g(0)$ and $g(-1)$. **b** Find a such that $g(a) = 27$.

6 Draw freehand sketches of the following pairs of graphs using your observations from the previous **Discovery**:

GRAPHING PACKAGE

a $y = 2^x$ and $y = 2^x - 2$ **b** $y = 2^x$ and $y = 2^{-x}$

c $y = 2^x$ and $y = 2^{x-2}$ **d** $y = 2^x$ and $y = 2(2^x)$

7 Draw freehand sketches of the following pairs of graphs:

a $y = 3^x$ and $y = 3^{-x}$ **b** $y = 3^x$ and $y = 3^x + 1$

c $y = 3^x$ and $y = -3^x$ **d** $y = 3^x$ and $y = 3^{x-1}$

8 For each of the functions below:

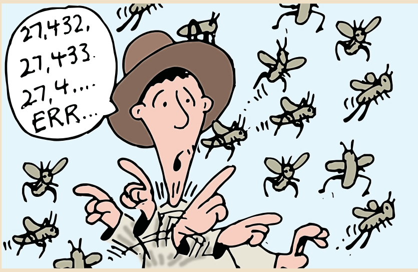
- **i** Sketch the graph of the function.
- **ii** State the domain and range.
- **iii** Use your calculator to find the value of y when $x = \sqrt{2}$.
- **iv** Discuss the behaviour of y as $x \to \pm\infty$.
- **v** Determine the horizontal asymptotes.

a $y = 2^x + 1$ **b** $f(x) = 2 - 2^x$ **c** $y = 2^{-x} + 3$ **d** $f(x) = 3 - 2^{-x}$

Example 26

Self Tutor

An entomologist monitoring a grasshopper plague notices that the area affected by the grasshoppers is given by $A(n) = 1000 \times 2^{0.2n}$ hectares, where n is the number of weeks after the initial observation.



a Find the original affected area.

b Find the affected area after:

i 5 weeks **ii** 10 weeks **iii** 12 weeks.

c Draw the graph of A against n.

d How long will it take for the affected area to reach 8000 hectares?

a $$\begin{aligned} A(0) &= 1000 \times 2^0 \\ &= 1000 \times 1 \\ &= 1000 \end{aligned}$$ $\therefore$ the original affected area was 1000 ha.

b **i** $A(5) = 1000 \times 2^1$

$= 2000$

The affected area is 2000 ha.

ii $A(10) = 1000 \times 2^2$

$= 4000$

The affected area is 4000 ha.

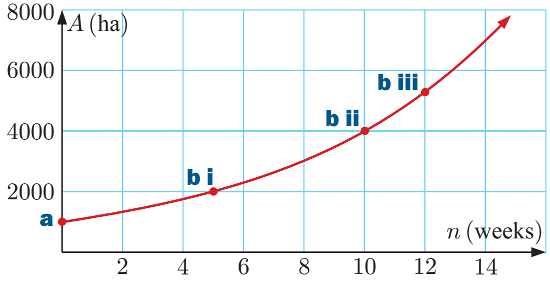
iii $A(12) = 1000 \times 2^{0.2\times 12}$

$= 1000 \times 2^{2.4}$

≈ 5280

The affected area is about 5280 ha.

c



d We need to find n such that

$$A(n) = 8000$$
$$\therefore\ 1000 \times 2^{0.2n} = 8000$$
$$\therefore\ 2^{0.2n} = 8$$
$$\therefore\ 2^{0.2n} = 2^3$$
$$\therefore\ 0.2n = 3$$
$$\therefore\ n = 15$$

So, it will take 15 weeks.

9 A breeding program to ensure the survival of pygmy possums is established with an initial population of 50 (25 pairs). From a previous program, the expected population P in n years' time is given by $P(n) = P_0 \times 2^{0.5n}$.

© Matt West, Healesville Sanctuary

a What is the value of P_0?

b What is the expected population after:

i 2 years **ii** 6 years **iii** 10 years?

c Sketch the graph of P against n using **a** and **b** only.

d How long will it take for the population to reach 800?

GRAPHING PACKAGE

10 The weight W of bacteria in a culture t hours after establishment is given by $W(t) = 100 \times 3^{0.1t}$ grams.

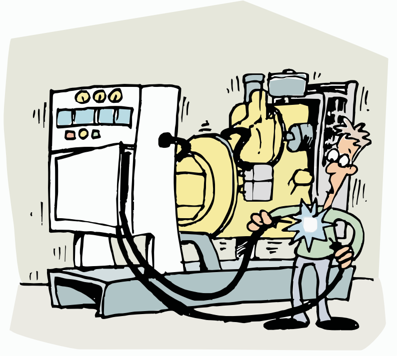
a Find the initial weight.

b Find the weight after: **i** 5 hours **ii** 10 hours **iii** 24 hours.

c Sketch the graph of W against t using the results of **a** and **b** only.

d How long will it take for the weight to reach 900 g?

11 The current flowing in an electrical circuit t seconds after it is switched off is given by $I(t) = 32 \times 4^{-t}$ amps.



a What current was flowing at the instant when it was switched off?

b What current was still flowing after:

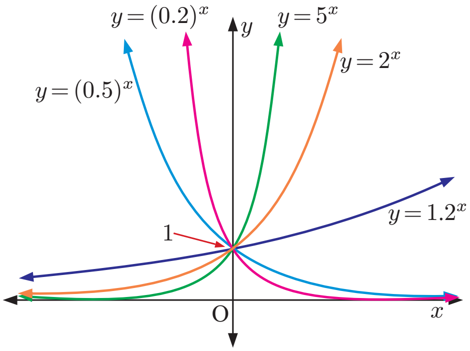
i 1 second **ii** 2 seconds?

c Plot I against t.

d How long will it take for the current to reach $\frac{1}{2}$ amp?

12 Answer the **Opening Problem** on page **102**.

H THE NATURAL EXPONENTIAL e^x



We have seen that the simplest exponential functions are of the form $f(x) = b^x$ where $b > 0$, $b \neq 1$.

Graphs of some of these functions are shown alongside.

We can see that for all positive values of the base b, the graph is always positive.

Hence $\boxed{b^x > 0 \text{ for all } b > 0.}$

There are an infinite number of possible choices for the base number.

However, where exponential data is examined in science, engineering, and finance, the base $e \approx 2.7183$ is commonly used.

e is a special number in mathematics. It is irrational like π, and just as π is the ratio of a circle's circumference to its diameter, e also has a physical meaning. We explore this meaning in the following **Discovery**.

Discovery 2 Continuous compound interest

A formula for calculating the amount to which an investment grows is $u_n = u_0(1+i)^n$ where:

u_n is the final amount, u_0 is the initial amount,

i is the interest rate per compounding period,

n is the number of periods or number of times the interest is compounded.

We will investigate the final value of an investment for various values of n, and allow n to get extremely large.

What to do:

1 Suppose \$1000 is invested for one year at a fixed rate of 6% per annum. Use your calculator to find the final amount or *maturing value* if the interest is paid:

- **a** annually $(n = 1,\ i = 6\% = 0.06)$
- **b** quarterly $(n = 4,\ i = \frac{6\%}{4} = 0.015)$
- **c** monthly
- **d** daily
- **e** by the second
- **f** by the millisecond.

2 Comment on your answers from **1**.

3 If r is the percentage rate per year, t is the number of years, and N is the number of interest payments per year, then $i = \frac{r}{N}$ and $n = Nt$.

The growth formula becomes $u_n = u_0\left(1 + \frac{r}{N}\right)^{Nt}$.

If we let $a = \frac{N}{r}$, show that $u_n = u_0\left[\left(1 + \frac{1}{a}\right)^a\right]^{rt}$.

4 For continuous compound growth, the number of interest payments per year N gets very large.

- **a** Explain why a gets very large as N gets very large.
- **b** Copy and complete the table, giving your answers as accurately as technology permits.

a	$\left(1+\frac{1}{a}\right)^a$
10	
100	
1000	
10 000	
100 000	
1 000 000	
10 000 000	

5 You should have found that for very large values of a,

$$\left(1+\frac{1}{a}\right)^a \approx 2.718\,281\,828\,459....$$

Use the $\boxed{e^x}$ key of your calculator to find the value of e^1. What do you notice?

6 For continuous growth, $u_n = u_0 e^{rt}$ where u_0 is the initial amount
r is the annual percentage rate
t is the number of years

Use this formula to find the final value if \$1000 is invested for 4 years at a fixed rate of 6% per annum, where the interest is calculated continuously.

From **Discovery 2** we observe that:

If interest is paid *continuously* or *instantaneously* then the formula for calculating a compounding amount $u_n = u_0(1+i)^n$ can be replaced by $u_n = u_0 e^{rt}$, where r is the percentage rate per annum and t is the number of years.

Historical note

The natural exponential e was first described in 1683 by Swiss mathematician **Jacob Bernoulli**. He discovered the number while studying compound interest, just as we did in **Discovery 2**.

The natural exponential was first called e by Swiss mathematician and physicist **Leonhard Euler** in a letter to the German mathematician **Christian Goldbach** in 1731. The number was then published with this notation in 1736.

In 1748 Euler evaluated e correct to 18 decimal places.

One may think that e was chosen because it was the first letter of Euler's name or for the word exponential, but it is likely that it was just the next vowel available since he had already used a in his work.

Leonhard Euler

EXERCISE 4H

GRAPHING PACKAGE

1 Sketch, on the same set of axes, the graphs of $y = 2^x$, $y = e^x$, and $y = 3^x$. Comment on any observations.

2 Sketch, on the same set of axes, the graphs of $y = e^x$ and $y = e^{-x}$. What is the geometric connection between these two graphs?

3 For the general exponential function $y = ae^{kx}$, what is the y-intercept?

4 Consider $y = 2e^x$.

a Explain why y can never be < 0. **b** Find y if: **i** $x = -20$ **ii** $x = 20$.

5 Find, to 3 significant figures, the value of:

a e^2 **b** e^3 **c** $e^{0.7}$ **d** $\sqrt{e}$ **e** e^{-1}

6 Write the following as powers of e:

a $\sqrt{e}$ **b** $\frac{1}{\sqrt{e}}$ **c** $\frac{1}{e^2}$ **d** $e\sqrt{e}$

7 On the same set of axes, sketch and clearly label the graphs of:

$f : x \mapsto e^x, \quad g : x \mapsto e^{x-2}, \quad h : x \mapsto e^x + 3$

State the domain and range of each function.

8 On the same set of axes, sketch and clearly label the graphs of:

$f : x \mapsto e^x, \quad g : x \mapsto -e^x, \quad h : x \mapsto 10 - e^x$

State the domain and range of each function.

9 Expand and simplify:

a $(e^x + 1)^2$ **b** $(1 + e^x)(1 - e^x)$ **c** $e^x(e^{-x} - 3)$

10 Solve for x:

a $e^x = \sqrt{e}$ **b** $e^{\frac{1}{2}x} = \frac{1}{e^2}$

11 Suppose $f : x \mapsto e^x$ and $g : x \mapsto 3x + 2$.

a Find $fg(x)$ and $gf(x)$. **b** Solve $fg(x) = \frac{1}{e}$.

12 Consider the function $f(x) = e^x$.

a On the same set of axes, sketch $y = f(x)$, $y = x$, and $y = f^{-1}(x)$.

b State the domain and range of f^{-1}.

Activity

Click on the icon to run a card game for exponential functions.

CARD GAME

Review set 4A

1 Simplify:

a $5\sqrt{3}(4 - \sqrt{3})$ **b** $(6 - 5\sqrt{2})^2$

2 Write with integer denominator:

a $\frac{2}{\sqrt{3}}$ **b** $\frac{\sqrt{7}}{\sqrt{5}}$ **c** $\frac{1}{4\sqrt{7}}$

3 Simplify using the laws of exponents:

a $a^4b^5 \times a^2b^2$ **b** $6xy^5 \div 9x^2y^5$ **c** $\frac{5(x^2y)^2}{(5x^2)^2}$

4 Let $f(x) = 3^x$.

a Write down the value of: **i** $f(4)$ **ii** $f(-1)$

b Find the value of k such that $f(x+2) = k\,f(x)$, $k \in \mathbb{Z}$.

5 Write without brackets or negative exponents:

a $x^{-2} \times x^{-3}$ **b** $2(ab)^{-2}$ **c** $2ab^{-2}$

6 Write as a single power of 3:

a $\dfrac{27}{9^a}$ **b** $(\sqrt{3})^{1-x} \times 9^{1-2x}$

7 Evaluate:

a $8^{\frac{2}{3}}$ **b** $27^{-\frac{2}{3}}$

8 Write without negative exponents:

a mn^{-2} **b** $(mn)^{-3}$ **c** $\dfrac{m^2n^{-1}}{p^{-2}}$ **d** $(4m^{-1}n)^2$

9 Expand and simplify:

a $(3 - e^x)^2$ **b** $(\sqrt{x}+2)(\sqrt{x}-2)$ **c** $2^{-x}(2^{2x} + 2^x)$

10 Find the positive solution of the equation $(8+\sqrt{13})x^2 + (2-\sqrt{13})x - 1 = 0$.
Give your solution in the form $x = a + b\sqrt{13}$, where $a, b \in \mathbb{Q}$.

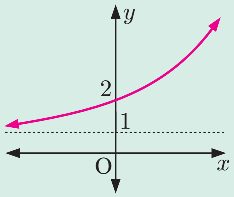
11 Solve for x:

a $2^{x-3} = \frac{1}{32}$ **b** $9^x = 27^{2-2x}$ **c** $e^{2x} = \dfrac{1}{\sqrt{e}}$

12 Match each equation to its corresponding graph:

a $y = -e^x$ **b** $y = 3 \times 2^x$ **c** $y = e^x + 1$ **d** $y = 3^{-x}$ **e** $y = -e^{-x}$

A



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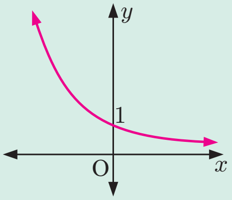

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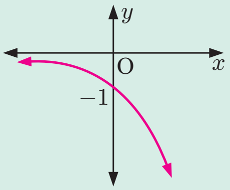
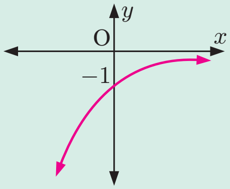
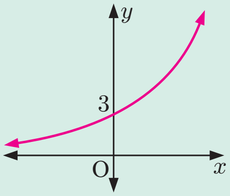


D



E



13 If $f(x) = 3 \times 2^x$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-2)$

14 Consider the function $f : x \mapsto e^{-x} - 3$.

a State the range of the function. **b** Find the value of $f(0)$.

c Solve $f(x) = \dfrac{\sqrt{e} - 3e}{e}$.

15 The temperature of a dish t minutes after it is removed from the microwave, is given by $T = 80 \times 2^{-0.1t}$ °C.

a Find the initial temperature of the dish.

b Find the temperature after: **i** 10 minutes **ii** 20 minutes.

c Draw the graph of T against t for $t \geqslant 0$.

d Find the time taken for the temperature of the dish to fall to 10°C.

Review set 4B

1 Simplify:

a $(7+2\sqrt{3})(5-3\sqrt{3})$ **b** $(6+2\sqrt{2})(6-2\sqrt{2})$

2 Rationalise the denominator:

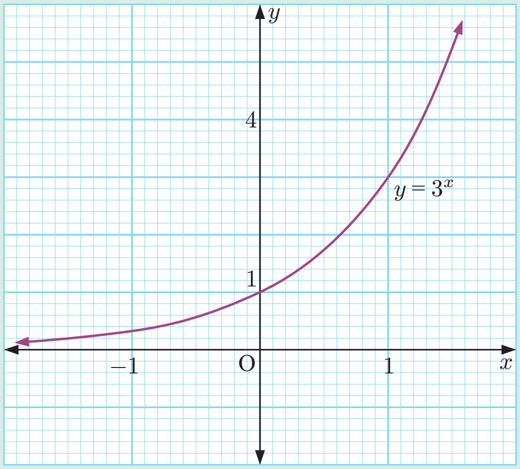
a $\dfrac{1}{5-\sqrt{3}}$ **b** $\dfrac{\sqrt{11}}{\sqrt{7}-2}$ **c** $\dfrac{8+\sqrt{2}}{3-\sqrt{2}}$ **d** $\dfrac{4+5\sqrt{5}}{6-3\sqrt{5}}$

3 Given the graph of $y = 3^x$ shown, estimate solutions to the exponential equations:

a $3^x = 5$

b $3^x = \frac{1}{2}$

c $6 \times 3^x = 20$



4 Write each of the following in the form $a \pm b\sqrt{2}$ where $a, b \in \mathbb{Z}^+$:

a $(\sqrt{2}-1)^2$ **b** $\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$ **c** $\dfrac{1}{(\sqrt{2}+1)^2}$ **d** $\dfrac{1}{3+2\sqrt{2}}$

5 Simplify using the laws of exponents:

a $(a^7)^3$ **b** $pq^2 \times p^3q^4$ **c** $\dfrac{8ab^5}{2a^4b^4}$

6 Write the following as a power of 2:

a 2×2^{-4} **b** $16 \div 2^{-3}$ **c** 8^4

7 Write the following without brackets:

a $(2m^3)^2$ **b** $\left(\dfrac{-a^3}{b}\right)^3$ **c** $\dfrac{(3x^2y)^2}{3x}$ **d** $\dfrac{(2a^{\frac{1}{2}}b^{\frac{1}{5}})^4}{a}$

8 Simplify $\dfrac{2^{x+1}}{2^{1-x}}$.

9 Write as powers of 5 in simplest form:

a 1 **b** $5\sqrt{5}$ **c** $\dfrac{1}{\sqrt[4]{5}}$ **d** 25^{a+3}

10 Expand and simplify:

a $e^x(e^{-x}+e^x)$ **b** $(2^x+5)^2$ **c** $(x^{\frac{1}{2}}-7)(x^{\frac{1}{2}}+7)$

11 Solve for x:

a $6\times 2^x = 192$ **b** $4\times(\frac{1}{3})^x = 324$

12 Solve for x without using a calculator:

a $4^{x+1}=\left(\frac{1}{8}\right)^x$ **b** $\dfrac{25^x}{5^{x-3}}=\dfrac{5^x}{125^{x-2}}$ **c** $\dfrac{3^{x+2}}{9^{3-x}}=\dfrac{27^{1-2x}}{3^{2x}}$

13 Suppose $f(x)=2^{-x}+1$.

a Find $f(\frac{1}{2})$. **b** Find a such that $f(a)=3$.

14 On the same set of axes draw the graphs of $y=2^x$ and $y=2^x-4$. Include on your graph the y-intercept and the equation of the horizontal asymptote of each function.

15 Consider $y=3^x-5$.

a Find y when $x=0,\ \pm1,\ \pm2$. **b** Discuss y as $x\to\pm\infty$.

c Sketch the graph of $y=3^x-5$. **d** State the range of the function.

16 Consider $f: x\mapsto e^{2x-1}$ and $g: x\mapsto e^{\sqrt{2}x}$.

a State the range of f.

b Find the exact value of $g(\sqrt{2})$.

c Solve $f(x)=g(x)$, giving your answer in the form $x=a+b\sqrt{2}$ where $a,\ b\in\mathbb{Q}$.

5

Logarithms

Contents:

Opening problem

In a plentiful springtime, a population of 1000 mice will double every week.

The population after t weeks is given by the exponential function $P(t) = 1000 \times 2^t$ mice.

Things to think about:

- **a** What does the graph of the population over time look like?
- **b** How long will it take for the population to reach 20 000 mice?
- **c** Can we write a function for t in terms of P, which determines the time at which the population P is reached?
- **d** What does the graph of this function look like?

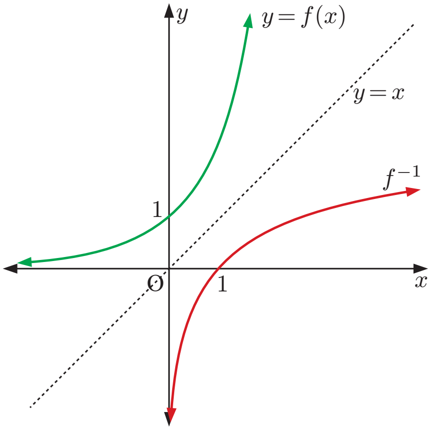
A LOGARITHMS IN BASE 10

Consider the exponential function $f : x \mapsto 10^x$ or $f(x) = 10^x$.

The graph of $y = f(x)$ is shown alongside, along with its inverse function f^{-1}.

Since f is defined by $y = 10^x$,
f^{-1} is defined by $x = 10^y$.
{interchanging x and y}

y is the exponent to which the base 10 is raised in order to get x.



We write this as $y = \log_{10} x$ or $\lg x$ and say that y is the **logarithm in base 10, of x**.

Logarithms are thus defined to be the inverse of exponential functions:

If $f(x) = 10^x$ then $f^{-1}(x) = \log_{10} x$ or $\lg x$.

WORKING WITH LOGARITHMS

Many positive numbers can be easily written in the form 10^x.

For example:

$$\begin{aligned} 10\,000 &= 10^4 \\ 1000 &= 10^3 \\ 100 &= 10^2 \\ 10 &= 10^1 \\ 1 &= 10^0 \\ 0.1 &= 10^{-1} \\ 0.01 &= 10^{-2} \\ 0.001 &= 10^{-3} \end{aligned}$$

Numbers like $\sqrt{10}$, $10\sqrt{10}$ and $\frac{1}{\sqrt[5]{10}}$ can also be written in the form 10^x as follows:

$$\sqrt{10} = 10^{\frac{1}{2}} \qquad 10\sqrt{10} = 10^1 \times 10^{0.5} \qquad \frac{1}{\sqrt[5]{10}} = 10^{-\frac{1}{5}}$$
$$= 10^{0.5} \qquad\qquad = 10^{1.5} \qquad\qquad = 10^{-0.2}$$

In fact, all positive numbers can be written in the form 10^x.

The **logarithm in base 10** of a positive number is the power that 10 must be raised to in order to obtain the number.

For example:

- Since $1000 = 10^3$, we write $\log_{10}(1000) = 3$
  or $\lg(1000) = 3$.
- Since $0.01 = 10^{-2}$, we write $\log_{10}(0.01) = -2$
  or $\lg(0.01) = -2$.

We hence conclude that:

$\boldsymbol{\lg 10^x = x}$ for any $x \in \mathbb{R}$.

$\boldsymbol{a = 10^{\lg a}}$ for any $a > 0$.

lg a means $\log_{10} a$.
a must be positive since $10^x > 0$ for all $x \in \mathbb{R}$.

Example 1 Self Tutor

Without using a calculator, find:

a $\log 100$ **b** $\log(\sqrt[4]{10})$

a $\log 100 = \log 10^2 = 2$

b $\log(\sqrt[4]{10}) = \log(10^{\frac{1}{4}}) = \frac{1}{4}$

EXERCISE 5A

1 Without using a calculator, find:

a $\lg 10\,000$ **b** $\lg 0.001$ **c** $\lg 10$ **d** $\lg 1$

e $\lg \sqrt{10}$ **f** $\lg(\sqrt[3]{10})$ **g** $\lg\left(\frac{1}{\sqrt[4]{10}}\right)$ **h** $\lg\left(10\sqrt{10}\right)$

i $\lg \sqrt[3]{100}$ **j** $\lg\left(\frac{100}{\sqrt{10}}\right)$ **k** $\lg\left(10 \times \sqrt[3]{10}\right)$ **l** $\lg\left(1000\sqrt{10}\right)$

Check your answers using your calculator.

2 Simplify:

a $\lg 10^n$ **b** $\lg\left(10^a \times 100\right)$ **c** $\lg\left(\frac{10}{10^m}\right)$ **d** $\lg\left(\frac{10^a}{10^b}\right)$

Example 2

Self Tutor

Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

a 8 **b** 800 **c** 0.08

a $8 = 10^{\lg 8} \approx 10^{0.9031}$

b $800 = 10^{\lg 800} \approx 10^{2.9031}$

c $0.08 = 10^{\lg 0.08} \approx 10^{-1.0969}$

3 **a** Use your calculator to find $\lg 41$, giving your answer correct to 4 decimal places.
b Hence, write 41 as a power of 10.

4 Use your calculator to write the following in the form 10^x where x is correct to 4 decimal places:

a 6 **b** 60 **c** 6000 **d** 0.6 **e** 0.006
f 15 **g** 1500 **h** 1.5 **i** 0.15 **j** 0.000 15

5 Explain why you cannot find the logarithm of a negative number.

Example 3

Self Tutor

a Use your calculator to find: **i** $\lg 2$ **ii** $\lg 20$
b Explain why $\lg 20 = \lg 2 + 1$.

a **i** $\lg 2 \approx 0.3010$
ii $\lg 20 \approx 1.3010$

b
$$\begin{aligned}\lg 20 &= \lg(2 \times 10)\\ &\approx \lg(10^{0.3010} \times 10^1)\\ &\approx \lg 10^{1.3010} \quad \{\text{adding exponents}\}\\ &\approx 1.3010\\ &\approx \lg 2 + 1\end{aligned}$$

6 **a** Use your calculator to find: **i** $\lg 3$ **ii** $\lg 300$
b Explain why $\lg 300 = \lg 3 + 2$.

7 **a** Use your calculator to find: **i** $\lg 5$ **ii** $\lg 0.05$
b Explain why $\lg 0.05 = \lg 5 - 2$.

Example 4

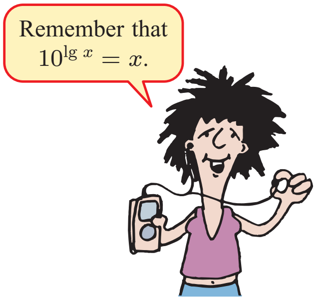
Self Tutor

Find x if:

a $\lg x = 3$ **b** $\lg x \approx -0.271$

a
$$\begin{aligned}\lg x &= 3\\ \therefore\ 10^{\lg x} &= 10^3\\ \therefore\ x &= 1000\end{aligned}$$

b
$$\begin{aligned}\lg x &\approx -0.271\\ \therefore\ 10^{\lg x} &\approx 10^{-0.271}\\ \therefore\ x &\approx 0.536\end{aligned}$$



8 Find x if:

a $\lg x = 2$	**b** $\lg x = 1$	**c** $\lg x = 0$
d $\lg x = -1$	**e** $\lg x = \frac{1}{2}$	**f** $\lg x = -\frac{1}{2}$
g $\lg x = 4$	**h** $\lg x = -5$	**i** $\lg x \approx 0.8351$
j $\lg x \approx 2.1457$	**k** $\lg x \approx -1.378$	**l** $\lg x \approx -3.1997$

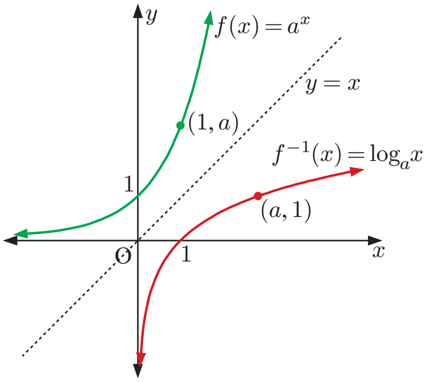
B LOGARITHMS IN BASE a

In the previous section we defined logarithms in base 10 as the inverse of the exponential function $f(x) = 10^x$.

If $f(x) = 10^x$ then $f^{-1}(x) = \log_{10} x$.

We can use the same principle to define logarithms in other bases:

If $f(x) = a^x$ then $f^{-1}(x) = \log_a x$.



If $b = a^x$, $a \neq 1$, $a > 0$, we say that x is the **logarithm in base a, of b**, and that $b = a^x \Leftrightarrow x = \log_a b$, $b > 0$.

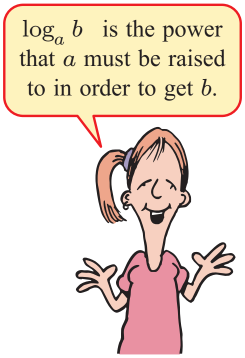
$b = a^x \Leftrightarrow x = \log_a b$ is read as "$b = a^x$ if and only if $x = \log_a b$".

It is a short way of writing:
"if $b = a^x$ then $x = \log_a b$, and if $x = \log_a b$ then $b = a^x$".

$b = a^x$ and $x = \log_a b$ are *equivalent* or *interchangeable* statements.

For example:

- $8 = 2^3$ means that $3 = \log_2 8$ and vice versa.
- $\log_5 25 = 2$ means that $25 = 5^2$ and vice versa.



If $y = a^x$ then $x = \log_a y$, and so

If $x = a^y$ then $y = \log_a x$, and so

$\boldsymbol{x = \log_a a^x}$.

$\boldsymbol{x = a^{\log_a x}}$ provided $x > 0$.

Example 5

Self Tutor

a Write an equivalent exponential equation for $\log_{10} 1000 = 3$.

b Write an equivalent logarithmic equation for $3^4 = 81$.

a From $\log_{10} 1000 = 3$ we deduce that $10^3 = 1000$.

b From $3^4 = 81$ we deduce that $\log_3 81 = 4$.

EXERCISE 5B

1 Write an equivalent exponential equation for:

a $\log_{10} 100 = 2$ **b** $\log_{10} 10\,000 = 4$ **c** $\log_{10}(0.1) = -1$

d $\log_{10} \sqrt{10} = \frac{1}{2}$ **e** $\log_2 8 = 3$ **f** $\log_3 9 = 2$

g $\log_2(\frac{1}{4}) = -2$ **h** $\log_3 \sqrt{27} = 1.5$ **i** $\log_5 \left(\frac{1}{\sqrt{5}}\right) = -\frac{1}{2}$

2 Write an equivalent logarithmic equation for:

a $2^2 = 4$ **b** $4^3 = 64$ **c** $5^2 = 25$

d $7^2 = 49$ **e** $2^6 = 64$ **f** $2^{-3} = \frac{1}{8}$

g $10^{-2} = 0.01$ **h** $2^{-1} = \frac{1}{2}$ **i** $3^{-3} = \frac{1}{27}$

Example 6

Self Tutor

Find:

a $\log_2 16$ **b** $\log_5 0.2$ **c** $\log_{10} \sqrt[5]{100}$ **d** $\log_2 \left(\frac{1}{\sqrt{2}}\right)$

a
$$\begin{aligned}&\log_2 16\\ &= \log_2 2^4\\ &= 4\end{aligned}$$

b
$$\begin{aligned}&\log_5 0.2\\ &= \log_5(\tfrac{1}{5})\\ &= \log_5 5^{-1}\\ &= -1\end{aligned}$$

c
$$\begin{aligned}&\log_{10} \sqrt[5]{100}\\ &= \log_{10} \left(10^2\right)^{\frac{1}{5}}\\ &= \log_{10} 10^{\frac{2}{5}}\\ &= \tfrac{2}{5}\end{aligned}$$

d
$$\begin{aligned}&\log_2 \left(\frac{1}{\sqrt{2}}\right)\\ &= \log_2 2^{-\frac{1}{2}}\\ &= -\tfrac{1}{2}\end{aligned}$$

3 Find:

a $\log_{10} 100\,000$ **b** $\log_{10}(0.01)$ **c** $\log_3 \sqrt{3}$ **d** $\log_2 8$

e $\log_2 64$ **f** $\log_2 128$ **g** $\log_5 25$ **h** $\log_5 125$

i $\log_2(0.125)$ **j** $\log_9 3$ **k** $\log_4 16$ **l** $\log_{36} 6$

m $\log_3 243$ **n** $\log_2 \sqrt[3]{2}$ **o** $\log_a a^n$ **p** $\log_8 2$

q $\log_t \left(\frac{1}{t}\right)$ **r** $\log_6 6\sqrt{6}$ **s** $\log_4 1$ **t** $\log_9 9$

4 Use your calculator to find:

a $\log_{10} 152$ **b** $\log_{10} 25$ **c** $\log_{10} 74$ **d** $\log_{10} 0.8$

5 Solve for x:

a $\log_2 x = 3$ **b** $\log_4 x = \frac{1}{2}$ **c** $\log_x 81 = 4$ **d** $\log_2(x - 6) = 3$

6 Simplify:

a $\log_4 16$ **b** $\log_2 4$ **c** $\log_3 \left(\frac{1}{3}\right)$ **d** $\log_{10} \sqrt[4]{1000}$

e $\log_7 \left(\frac{1}{\sqrt{7}}\right)$ **f** $\log_5(25\sqrt{5})$ **g** $\log_3 \left(\frac{1}{\sqrt{27}}\right)$ **h** $\log_4 \left(\frac{1}{2\sqrt{2}}\right)$

i $\log_x x^2$ **j** $\log_x \sqrt{x}$ **k** $\log_m m^3$ **l** $\log_x(x\sqrt{x})$

m $\log_n \left(\frac{1}{n}\right)$ **n** $\log_a \left(\frac{1}{a^2}\right)$ **o** $\log_a \left(\frac{1}{\sqrt{a}}\right)$ **p** $\log_m \sqrt{m^5}$

Discussion

We have seen that $\sqrt{2}$ cannot be written in the form $\frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$. We therefore say $\sqrt{2}$ is irrational.

More generally, $\sqrt{a}$ is only rational if a is a perfect square.

What about logarithms? The following is a proof that $\log_2 3$ is irrational.

Proof: If $\log_2 3$ is rational, then $\log_2 3 = \frac{p}{q}$ where $p, q \in \mathbb{Z}$, $q \neq 0$

$$\therefore \quad 3 = 2^{\frac{p}{q}}$$
$$\therefore \quad 3^q = 2^p$$

The left hand side is always odd, and the right hand side is always even, so the statement is impossible.

Hence $\log_2 3$ must be irrational.

Under what circumstances will $\log_a b$ be rational?

C LAWS OF LOGARITHMS

Discovery The laws of logarithms

What to do:

1 Use your calculator to find:

a $\lg 2 + \lg 3$ **b** $\lg 3 + \lg 7$ **c** $\lg 4 + \lg 20$

d $\lg 6$ **e** $\lg 21$ **f** $\lg 80$

From your answers, suggest a possible simplification for $\lg a + \lg b$.

2 Use your calculator to find:

a $\lg 6 - \lg 2$ **b** $\lg 12 - \lg 3$ **c** $\lg 3 - \lg 5$

d $\lg 3$ **e** $\lg 4$ **f** $\lg(0.6)$

From your answers, suggest a possible simplification for $\lg a - \lg b$.

3 Use your calculator to find:

a $3 \lg 2$ **b** $2 \lg 5$ **c** $-4 \lg 3$

d $\lg(2^3)$ **e** $\lg(5^2)$ **f** $\lg(3^{-4})$

From your answers, suggest a possible simplification for $n \lg a$.

From the **Discovery**, you should have found the three important **laws of logarithms**:

If A and B are both positive then:

- $\lg A + \lg B = \lg(AB)$
- $\lg A - \lg B = \lg\left(\frac{A}{B}\right)$
- $n \lg A = \lg(A^n)$

More generally, in any base c where $c \neq 1$, $c > 0$, we have these **laws of logarithms**:

If A and B are both positive then:

- $\boldsymbol{\log_c A + \log_c B = \log_c(AB)}$
- $\boldsymbol{\log_c A - \log_c B = \log_c\left(\frac{A}{B}\right)}$
- $\boldsymbol{n \log_c A = \log_c(A^n)}$

Proof:

- $$\begin{aligned}&\log_c(AB)\\ &= \log_c(c^{\log_c A} \times c^{\log_c B})\\ &= \log_c(c^{\log_c A + \log_c B})\\ &= \log_c A + \log_c B\end{aligned}$$
- $$\begin{aligned}&\log_c\left(\frac{A}{B}\right)\\ &= \log_c\left(\frac{c^{\log_c A}}{c^{\log_c B}}\right)\\ &= \log_c(c^{\log_c A - \log_c B})\\ &= \log_c A - \log_c B\end{aligned}$$
- $$\begin{aligned}&\log_c(A^n)\\ &= \log_c((c^{\log_c A})^n)\\ &= \log_c(c^{n\log_c A})\\ &= n\log_c A\end{aligned}$$

Example 7

Self Tutor

Use the laws of logarithms to write the following as a single logarithm or as an integer:

a $\lg 5 + \lg 3$ **b** $\log_3 24 - \log_3 8$ **c** $\log_2 5 - 1$

a $$\begin{aligned}&\lg 5 + \lg 3\\ &= \lg(5 \times 3)\\ &= \lg 15\end{aligned}$$

b $$\begin{aligned}&\log_3 24 - \log_3 8\\ &= \log_3\left(\tfrac{24}{8}\right)\\ &= \log_3 3\\ &= 1\end{aligned}$$

c $$\begin{aligned}&\log_2 5 - 1\\ &= \log_2 5 - \log_2 2^1\\ &= \log_2\left(\tfrac{5}{2}\right)\end{aligned}$$

Example 8

Self Tutor

Simplify by writing as a single logarithm or as a rational number:

a $2\lg 7 - 3\lg 2$ **b** $2\lg 3 + 3$ **c** $\dfrac{\lg 8}{\lg 4}$

a $$\begin{aligned}&2\lg 7 - 3\lg 2\\ &= \lg(7^2) - \lg(2^3)\\ &= \lg 49 - \lg 8\\ &= \lg\left(\tfrac{49}{8}\right)\end{aligned}$$

b $$\begin{aligned}&2\lg 3 + 3\\ &= \lg(3^2) + \lg(10^3)\\ &= \lg 9 + \lg 1000\\ &= \lg(9000)\end{aligned}$$

c $$\begin{aligned}&\frac{\lg 8}{\lg 4}\\ &= \frac{\lg 2^3}{\lg 2^2}\\ &= \frac{3\lg 2}{2\lg 2}\\ &= \tfrac{3}{2}\end{aligned}$$

EXERCISE 5C

1 Write as a single logarithm or as an integer:

a $\lg 8+\lg 2$
b $\lg 4+\lg 5$
c $\lg 40-\lg 5$
d $\lg p-\lg m$
e $\log_4 8-\log_4 2$
f $\lg 5+\lg(0.4)$
g $\lg 2+\lg 3+\lg 4$
h $1+\log_2 3$
i $\lg 4-1$
j $\lg 5+\lg 4-\lg 2$
k $2+\lg 2$
l $t+\lg w$
m $\log_m 40-2$
n $\log_3 6-\log_3 2-\log_3 3$
o $\lg 50-4$
p $3-\log_5 50$
q $\log_5 100-\log_5 4$
r $\lg\left(\frac{4}{3}\right)+\lg 3+\lg 7$

2 Write as a single logarithm or integer:

a $5\lg 2+\lg 3$
b $2\lg 3+3\lg 2$
c $3\lg 4-\lg 8$
d $2\log_3 5-3\log_3 2$
e $\frac{1}{2}\log_6 4+\log_6 3$
f $\frac{1}{3}\lg\left(\frac{1}{8}\right)$
g $3-\lg 2-2\lg 5$
h $1-3\lg 2+\lg 20$
i $2-\frac{1}{2}\log_n 4-\log_n 5$

3 Simplify without using a calculator:

a $\dfrac{\lg 4}{\lg 2}$
b $\dfrac{\log_5 27}{\log_5 9}$
c $\dfrac{\lg 8}{\lg 2}$
d $\dfrac{\lg 3}{\lg 9}$
e $\dfrac{\log_3 25}{\log_3(0.2)}$
f $\dfrac{\log_4 8}{\log_4(0.25)}$

Check your answers using a calculator.

Example 9

Self Tutor

Show that:

a $\lg\left(\frac{1}{9}\right)=-2\lg 3$
b $\lg 500=3-\lg 2$

a
$$\begin{aligned}&\lg\left(\tfrac{1}{9}\right)\\&=\lg(3^{-2})\\&=-2\lg 3\end{aligned}$$

b
$$\begin{aligned}&\lg 500\\&=\lg\left(\tfrac{1000}{2}\right)\\&=\lg 1000-\lg 2\\&=\lg 10^3-\lg 2\\&=3-\lg 2\end{aligned}$$

4 Show that:

a $\lg 9=2\lg 3$
b $\lg\sqrt{2}=\frac{1}{2}\lg 2$
c $\lg\left(\frac{1}{8}\right)=-3\lg 2$
d $\lg\left(\frac{1}{5}\right)=-\lg 5$
e $\lg 5=1-\lg 2$
f $\lg 5000=4-\lg 2$
g $\log_6 4+\log_6 9=2$
h $\log_{15} 3-\log_{15} 45=-1$
i $2\log_{12} 2+\frac{1}{2}\log_{12} 9=1$

5 Find the exact value of:

a $3\lg 2+2\lg 5-\frac{1}{2}\lg 4$
b $2\log_2 3-\log_2 6-\frac{1}{2}\log_2 9$
c $5\log_6 2+2\log_6 3-\frac{1}{2}\log_6 16-\log_6 12$

6 If $x = \log_2 P$, $y = \log_2 Q$, and $z = \log_2 R$, write in terms of x, y, and z:

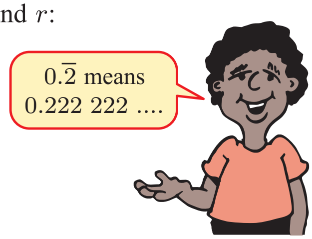
a $\log_2(PR)$ **b** $\log_2(RQ^2)$ **c** $\log_2\left(\dfrac{PR}{Q}\right)$

d $\log_2(P^2\sqrt{Q})$ **e** $\log_2\left(\dfrac{Q^3}{\sqrt{R}}\right)$ **f** $\log_2\left(\dfrac{R^2\sqrt{Q}}{P^3}\right)$

7 If $p = \log_b 2$, $q = \log_b 3$, and $r = \log_b 5$, write in terms of p, q, and r:

a $\log_b 6$ **b** $\log_b 45$ **c** $\log_b 108$

d $\log_b\left(\frac{5\sqrt{3}}{2}\right)$ **e** $\log_b\left(\frac{5}{32}\right)$ **f** $\log_b(0.\overline{2})$



8 If $\log_t M = 1.29$ and $\log_t N^2 = 1.72$, find:

a $\log_t N$ **b** $\log_t(MN)$ **c** $\log_t\left(\dfrac{N^2}{\sqrt{M}}\right)$

9 Suppose $\log_b P = 5$ and $\log_b(P^3Q^2) = 21$. Find $\log_b Q$.

10 Suppose that $\log_t(AB^3) = 15$ and $\log_t\left(\dfrac{A^2}{B}\right) = 9$.

a Write *two* equations connecting $\log_t A$ and $\log_t B$.
b Find the values of $\log_t A$ and $\log_t B$.
c Find $\log_t\left(B^5\sqrt{A}\right)$.
d Write B in terms of t.

D LOGARITHMIC EQUATIONS

We can use the laws of logarithms to write equations in a different form. This can be particularly useful if an unknown appears as an exponent.

For the logarithmic function, for every value of y, there is only one corresponding value of x. We can therefore take the logarithm of both sides of an equation without changing the solution. However, we can only do this if both sides are positive.

Example 10

Self Tutor

Write these as logarithmic equations (in base 10):

a $y = 5 \times 3^x$ **b** $P = \dfrac{20}{\sqrt{n}}$

a
$$\begin{aligned} y &= 5 \times 3^x \\ \therefore\ \lg y &= \lg(5 \times 3^x) \\ \therefore\ \lg y &= \lg 5 + \lg 3^x \\ \therefore\ \lg y &= \lg 5 + x \lg 3 \end{aligned}$$

b
$$\begin{aligned} P &= \frac{20}{\sqrt{n}} \\ \therefore\ \lg P &= \lg\left(\frac{20}{n^{\frac{1}{2}}}\right) \\ \therefore\ \lg P &= \lg 20 - \lg n^{\frac{1}{2}} \\ \therefore\ \lg P &= \lg 20 - \tfrac{1}{2}\lg n \end{aligned}$$

Example 11

Self Tutor

Write the following equations without logarithms:

a $\lg y = x \lg 4 + \lg 3$ **b** $\log_2 M = 3\log_2 a - 5$

a

$$\begin{aligned} \lg y &= x \lg 4 + \lg 3 \\ \therefore\ \lg y &= \lg 4^x + \lg 3 \\ \therefore\ \lg y &= \lg(3 \times 4^x) \\ \therefore\ y &= 3 \times 4^x \end{aligned}$$

b

$$\begin{aligned} \log_2 M &= 3\log_2 a - 5 \\ \therefore\ \log_2 M &= \log_2 a^3 - \log_2 2^5 \\ \therefore\ \log_2 M &= \log_2\left(\frac{a^3}{32}\right) \\ \therefore\ M &= \frac{a^3}{32} \end{aligned}$$

EXERCISE 5D.1

1 Write the following as logarithmic equations in base 10, assuming all terms are positive:

- **a** $y = 2^x$
- **b** $y = x^3$
- **c** $M = d^4$
- **d** $T = 5^x$
- **e** $y = \sqrt{x}$
- **f** $y = 7 \times 3^x$
- **g** $S = \frac{9}{t}$
- **h** $M = 100 \times 7^x$
- **i** $T = 5\sqrt{d}$
- **j** $F = \frac{1000}{\sqrt{n}}$
- **k** $S = 200 \times 2^t$
- **l** $y = \sqrt{\frac{15}{x}}$

2 Write the following equations without logarithms:

- **a** $\lg y = x \lg 7$
- **b** $\lg D = \lg x + \lg 2$
- **c** $\log_a F = \log_a 5 - \log_a t$
- **d** $\lg y = x \lg 2 + \lg 6$
- **e** $\lg P = \frac{1}{2} \lg x$
- **f** $\lg N = -\frac{1}{3} \lg p$
- **g** $\lg P = 3 \lg x + 1$
- **h** $\lg y = x - \lg 2$
- **i** $\lg y = 2 \lg x - 1$
- **j** $\log_2 T = 5 \log_2 k + 1$
- **k** $\log_3 P = 4 \log_3 n - 2$
- **l** $\log_2 y = 4x + 3$

3 Suppose $\lg y = 3 \lg x - \lg 2$.

- **a** Write y in terms of x, without using logarithms.
- **b** Find y when: **i** $x = 2$ **ii** $x = 4$

4 Suppose $\lg y = \frac{1}{3}x + 2$.

- **a** Write y in the form $y = a(10^{bx})$ where $a, b \in \mathbb{Q}$.
- **b** Find y when: **i** $x = 0$ **ii** $x = 3$

5 Copy and complete:

- **a** If there is a *power* relationship between y and x, for example $y = 5x^3$, then there is a *linear* relationship between $\lg y$ and ......
- **b** If there is an *exponential* relationship between y and x, for example $y = 4 \times 2^x$, then there is a *linear* relationship between ...... and ......

SOLVING LOGARITHMIC EQUATIONS

Logarithmic equations can often be solved using the laws of logarithms. However, we must always check that our solutions satisfy the original equation, remembering that $\lg x$ is only defined for $x > 0$.

Example 12

Self Tutor

Solve for x:

a $\lg(x-6) + \lg 3 = 2\lg 6$

b $\lg x + \lg(x+5) = \lg 14$

a
$$\begin{aligned} \lg(x-6) + \lg 3 &= 2\lg 6 \\ \therefore\ \lg(x-6) &= \lg 6^2 - \lg 3 \\ \therefore\ \lg(x-6) &= \lg\left(\tfrac{36}{3}\right) \\ \therefore\ x-6 &= 12 \\ \therefore\ x &= 18 \end{aligned}$$

Check: $x-6>0$, so $x>6$ ✓

b
$$\begin{aligned} \lg x + \lg(x+5) &= \lg 14 \\ \therefore\ \lg(x(x+5)) &= \lg 14 \\ \therefore\ x(x+5) &= 14 \\ \therefore\ x^2+5x-14 &= 0 \\ \therefore\ (x+7)(x-2) &= 0 \\ \therefore\ x &= -7 \text{ or } 2 \end{aligned}$$

But $x>0$ and $x+5>0$

$\therefore$ $x = 2$ is the only valid solution.

EXERCISE 5D.2

1 Solve for x:

a $\lg(x-4) = \lg 3 + \lg 7$

b $\lg(x+5) - \lg 8 = 2\lg 3$

c $\lg(2x) = 1 + \frac{1}{2}\lg 16$

d $\log_2 x = 3\log_2 5 - 6$

e $\lg x - \lg(x-4) = \lg 5$

f $\log_5(x-2) - \log_5(x+2) = \log_5 3$

g $\log_3 x - 2 = \log_3(x-1)$

h $\lg(x+2) - 1 = \lg(x-3) - \lg 12$

2 Solve for x:

a $\lg x + \lg(x+1) = \lg 30$

b $\log_5(x+9) + \log_5(x+2) = \log_5(20x)$

c $\log_7 x = \log_7 8 - \log_7(6-x)$

d $\log_6(x+4) + \log_6(x-1) = 1$

e $\lg x + \lg(2x+8) = 1$

f $\lg(x+2) + \lg(x+7) = \lg(2x+2)$

g $2\log_2 x - \log_2(8-3x) = 1$

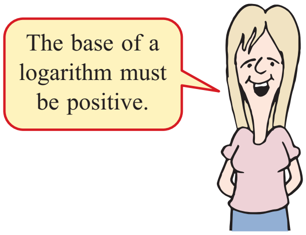
h $\log_2 x + \log_2(2x-7) = 2$

Example 13

Self Tutor

Solve for x: $\log_x 3 + \log_x 12 = 2$

$$\begin{aligned} \log_x 3 + \log_x 12 &= 2 \\ \therefore\ \log_x(3\times 12) &= \log_x(x^2) \\ \therefore\ 36 &= x^2 \\ \therefore\ x &= 6 \qquad \{\text{since } x>0\} \end{aligned}$$



3 Solve for x:

a $\log_x 32 - \log_x 4 = 1$

b $\log_x 45 = 2 + \log_x 5$

c $\log_x 54 = 3 - \log_x 4$

d $2\log_x 2 - 3 = \log_x\left(\frac{1}{16}\right)$

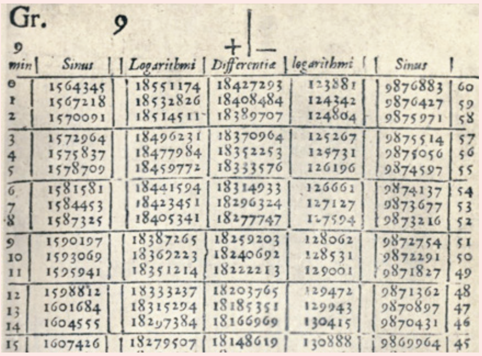
Historical note — The invention of logarithm

It is easy to take modern technology, such as the electronic calculator, for granted. Until electronic computers became affordable in the 1980s, a "calculator" was a *profession*, literally someone who would spend their time performing calculations by hand. They used mechanical calculators and technology such as logarithms. They often worked in banks, but sometimes for astronomers and other scientists.

The logarithm was invented by **John Napier** (1550 - 1617) and first published in 1614 in a Latin book which translates as a *Description of the Wonderful Canon of Logarithms.* John Napier was the 8th Lord of Merchiston, which is now part of Edinburgh, Scotland. Napier wrote a number of other books on many subjects including religion and mathematics. One of his other inventions was a device for performing long multiplication which is now called "Napier's Bones". Other calculators, such as slide rules, used logarithms as part of their design. He also popularised the use of the decimal point in mathematical notation.

John Napier

In Napier's time, mathematicians did not use the same notation a^b for indices, nor did they make use of the general concept of a function as described in this course. It was therefore impossible for Napier to explain logarithms as we have done. Instead, Napier's definition was based on the continuous movement of two points.



Gr. 9

9

+ | —

min	Sinus	Logarithmi	Differentiæ	logarithmi	Sinus	
0	1564345	18551174	18427293	123881	9876883	60
1	1567218	18532826	18408484	124342	9876427	59
2	1570091	18514511	18389707	124804	9875971	58
3	1572964	18496231	18370964	125267	9875514	57
4	1575837	18477984	18352253	125731	9875056	56
5	1578709	18459772	18333576	126196	9874597	55
6	1581581	18441594	18314933	126661	9874137	54
7	1584453	18423451	18296324	127127	9873677	53
8	1587325	18405341	18277747	127594	9873216	52
9	1590197	18387265	18259203	128062	9872754	51
10	1593069	18369223	18240692	128531	9872291	50
11	1595941	18351214	18222213	129001	9871827	49
12	1598812	18333237	18203765	129472	9871362	48
13	1601684	18315294	18185351	129943	9870897	47
14	1604555	18297384	18166969	130415	9870431	46
15	1607426	18279507	18148619	130888	9869964	45

To enable people to actually use logarithms, he calculated tables of numbers by hand to seven places of decimals. This took him many years of work. To find the logarithm of a particular number, you would look it up in the table. Although this seems awkward to us, it is much quicker to use tables than calculate multiplication, division, and square roots by hand.

Logarithms were an extremely important development and they had an immediate effect on the seventeenth century scientific community. **Johannes Kepler** used Napier's tables to assist with his calculations. This helped him develop his laws of planetary motion. Without logarithms these calculations would have taken many years. Kepler published a letter congratulating and acknowledging Napier. Kepler's laws gave **Sir Isaac Newton** important evidence to support his theory of universal gravitation. 200 years later, **Laplace** said that logarithms "by shortening the labours, doubled the life of the astronomer".

Johannes Kepler

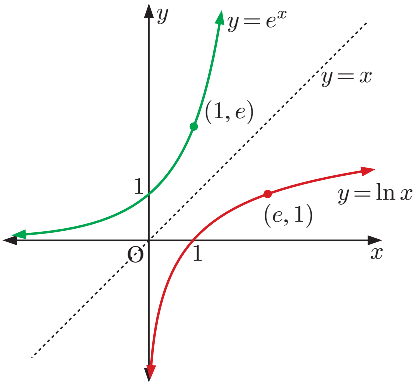
E NATURAL LOGARITHMS

In **Chapter 4** we came across the **natural exponential** $e \approx 2.718\,28$.

Given the exponential function $f(x) = e^x$, the inverse function $f^{-1} = \log_e x$ is the logarithm in base e.

We use $\ln x$ to represent $\log_e x$, and call $\ln x$ the **natural logarithm** of x.

$y = \ln x$ is the reflection of $y = e^x$ in the mirror line $y = x$.



Notice that:
- $\ln 1 = \ln e^0 = 0$
- $\ln e = \ln e^1 = 1$
- $\ln e^2 = 2$
- $\ln \sqrt{e} = \ln e^{\frac{1}{2}} = \frac{1}{2}$
- $\ln\left(\frac{1}{e}\right) = \ln e^{-1} = -1$

$\boldsymbol{\ln e^x = x}$ and $\boldsymbol{e^{\ln x} = x}$.

Since $a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}$, $\boldsymbol{a^x = e^{x \ln a}}, \quad a > 0.$

EXERCISE 5E.1

1 Without using a calculator find:

a $\ln e^2$ **b** $\ln e^3$ **c** $\ln \sqrt{e}$ **d** $\ln 1$

e $\ln\left(\frac{1}{e}\right)$ **f** $\ln \sqrt[3]{e}$ **g** $\ln\left(\frac{1}{e^2}\right)$ **h** $\ln\left(\frac{1}{\sqrt{e}}\right)$

Check your answers using a calculator.

2 Simplify:

a $e^{\ln 3}$ **b** $e^{2\ln 3}$ **c** $e^{-\ln 5}$ **d** $e^{-2\ln 2}$

3 Explain why $\ln(-2)$ and $\ln 0$ cannot be found.

4 Simplify:

a $\ln e^a$ **b** $\ln(e \times e^a)$ **c** $\ln\left(e^a \times e^b\right)$ **d** $\ln(e^a)^b$ **e** $\ln\left(\frac{e^a}{e^b}\right)$

Example 14 Self Tutor

Use your calculator to write the following in the form e^k where k is correct to 4 decimal places:

a 50 **b** 0.005

a 50
$= e^{\ln 50}$ {using $x = e^{\ln x}$}
$\approx e^{3.9120}$

b 0.005
$= e^{\ln 0.005}$
$\approx e^{-5.2983}$

5 Use your calculator to write the following in the form e^k where k is correct to 4 decimal places:

a 6 **b** 60 **c** 6000 **d** 0.6 **e** 0.006
f 15 **g** 1500 **h** 1.5 **i** 0.15 **j** 0.000 15

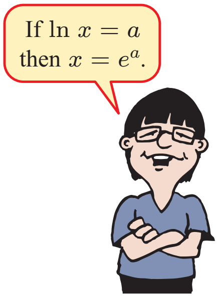
Example 15 Self Tutor

Find x if:

a $\ln x = 2.17$ **b** $\ln x = -0.384$

a $\ln x = 2.17$
$\therefore \quad x = e^{2.17}$
$\therefore \quad x \approx 8.76$

b $\ln x = -0.384$
$\therefore \quad x = e^{-0.384}$
$\therefore \quad x \approx 0.681$



6 Find x if:

a $\ln x = 3$ **b** $\ln x = 1$ **c** $\ln x = 0$ **d** $\ln x = -1$
e $\ln x = -5$ **f** $\ln x \approx 0.835$ **g** $\ln x \approx 2.145$ **h** $\ln x \approx -3.2971$

LAWS OF NATURAL LOGARITHMS

The laws for natural logarithms are the laws for logarithms written in base e:

For positive A and B:

- $\ln A + \ln B = \ln(AB)$
- $\ln A - \ln B = \ln\left(\frac{A}{B}\right)$
- $n \ln A = \ln(A^n)$

Example 16 Self Tutor

Use the laws of logarithms to write the following as a single logarithm:

a $\ln 5 + \ln 3$ **b** $\ln 24 - \ln 8$ **c** $\ln 5 - 1$

a $\ln 5 + \ln 3$
$= \ln(5 \times 3)$
$= \ln 15$

b $\ln 24 - \ln 8$
$= \ln\left(\frac{24}{8}\right)$
$= \ln 3$

c $\ln 5 - 1$
$= \ln 5 - \ln\ e^1$
$= \ln\left(\frac{5}{e}\right)$

Example 17 Self Tutor

Use the laws of logarithms to simplify:

a $2\ln 7 - 3\ln 2$ **b** $2\ln 3 + 3$

a $2\ln 7 - 3\ln 2$
$= \ln(7^2) - \ln(2^3)$
$= \ln 49 - \ln 8$
$= \ln\left(\frac{49}{8}\right)$

b $2\ln 3 + 3$
$= \ln(3^2) + \ln e^3$
$= \ln 9 + \ln e^3$
$= \ln(9e^3)$

EXERCISE 5E.2

1 Write as a single logarithm or integer:

a $\ln 15 + \ln 3$
b $\ln 15 - \ln 3$
c $\ln 20 - \ln 5$
d $\ln 4 + \ln 6$
e $\ln 5 + \ln(0.2)$
f $\ln 2 + \ln 3 + \ln 5$
g $1 + \ln 4$
h $\ln 6 - 1$
i $\ln 5 + \ln 8 - \ln 2$
j $2 + \ln 4$
k $\ln 20 - 2$
l $\ln 12 - \ln 4 - \ln 3$

2 Write in the form $\ln a,\ \ a \in \mathbb{Q}$:

a $5 \ln 3 + \ln 4$
b $3 \ln 2 + 2 \ln 5$
c $3 \ln 2 - \ln 8$
d $3 \ln 4 - 2 \ln 2$
e $\frac{1}{3} \ln 8 + \ln 3$
f $\frac{1}{3} \ln(\frac{1}{27})$
g $-\ln 2$
h $-\ln(\frac{1}{2})$
i $-2 \ln(\frac{1}{4})$

Example 18

Self Tutor

Show that:

a $\ln\left(\frac{1}{9}\right) = -2 \ln 3$
b $\ln\left(\frac{e}{4}\right) = 1 - 2 \ln 2$

a
$$\begin{aligned} &\ln\left(\tfrac{1}{9}\right) \\ &= \ln(3^{-2}) \\ &= -2 \ln 3 \end{aligned}$$

b
$$\begin{aligned} \ln\left(\frac{e}{4}\right) &= \ln e - \ln 4 \\ &= \ln e^1 - \ln 2^2 \\ &= 1 - 2 \ln 2 \end{aligned}$$

3 Show that:

a $\ln 27 = 3 \ln 3$
b $\ln \sqrt{3} = \frac{1}{2} \ln 3$
c $\ln(\frac{1}{16}) = -4 \ln 2$
d $\ln(\frac{1}{6}) = -\ln 6$
e $\ln\left(\frac{1}{\sqrt{2}}\right) = -\frac{1}{2} \ln 2$
f $\ln\left(\frac{e}{5}\right) = 1 - \ln 5$

4 Show that:

a $\ln \sqrt[3]{5} = \frac{1}{3} \ln 5$
b $\ln(\frac{1}{32}) = -5 \ln 2$
c $\ln\left(\frac{1}{\sqrt[5]{2}}\right) = -\frac{1}{5} \ln 2$
d $\ln\left(\frac{e^2}{8}\right) = 2 - 3 \ln 2$

Example 19

Self Tutor

Write the following equations without logarithms:

a $\ln A = 2 \ln c + 3$
b $\ln M = 3a - \ln 2$

a
$$\begin{aligned} \ln A &= 2 \ln c + 3 \\ \therefore\ \ln A &= \ln c^2 + \ln e^3 \\ \therefore\ \ln A &= \ln(c^2 e^3) \\ \therefore\ A &= c^2 e^3 \end{aligned}$$

b
$$\begin{aligned} \ln M &= 3a - \ln 2 \\ \therefore\ \ln M &= \ln e^{3a} - \ln 2 \\ \therefore\ \ln M &= \ln\left(\frac{e^{3a}}{2}\right) \\ \therefore\ M &= \tfrac{1}{2} e^{3a} \end{aligned}$$

5 Write the following equations without logarithms, assuming all terms are positive:

a $\ln D = \ln x + 1$ **b** $\ln F = -\ln p + 2$ **c** $\ln P = 2x + \ln 5$

d $\ln M = 2\ln y + 3$ **e** $\ln B = 3t - \ln 4$ **f** $\ln N = -\frac{1}{3}\ln g$

g $\ln Q \approx 3\ln x + 2.159$ **h** $\ln D \approx 0.4\ln n - 0.6582$ **i** $\ln T \approx -x + 1.578$

F SOLVING EXPONENTIAL EQUATIONS USING LOGARITHMS

In **Chapter 4** we found solutions to simple exponential equations where we could make equal bases and then equate exponents. However, it is not always easy to make the bases the same. In these situations we use **logarithms** to find the solution.

Example 20

Self Tutor

Solve for x, giving your answers correct to 3 significant figures:

a $2^x = 7$ **b** $5^{3x-1} = 90$

a
$$\begin{aligned} 2^x &= 7 \\ \therefore\ \lg 2^x &= \lg 7 \\ \therefore\ x\lg 2 &= \lg 7 \quad \{\lg(a^n) = n\lg a\} \\ \therefore\ x &= \frac{\lg 7}{\lg 2} \\ \therefore\ x &\approx 2.81 \end{aligned}$$

b
$$\begin{aligned} 5^{3x-1} &= 90 \\ \therefore\ \lg 5^{3x-1} &= \lg 90 \\ \therefore\ (3x-1)\lg 5 &= \lg 90 \quad \{\lg(a^n) = n\lg a\} \\ \therefore\ 3x - 1 &= \frac{\lg 90}{\lg 5} \\ \therefore\ x &= \tfrac{1}{3}\left(1 + \frac{\lg 90}{\lg 5}\right) \\ \therefore\ x &\approx 1.27 \end{aligned}$$

Example 21

Self Tutor

Find x exactly:

a $e^x = 30$ **b** $3e^{\frac{x}{2}} = 21$

a
$$\begin{aligned} e^x &= 30 \\ \therefore\ x &= \ln 30 \end{aligned}$$

b
$$\begin{aligned} 3e^{\frac{x}{2}} &= 21 \\ \therefore\ e^{\frac{x}{2}} &= 7 \\ \therefore\ \frac{x}{2} &= \ln 7 \\ \therefore\ x &= 2\ln 7 \end{aligned}$$

EXERCISE 5F

1 Solve for x, giving your answer correct to 3 significant figures:

a $2^x = 10$ **b** $3^x = 20$ **c** $4^x = 100$

d $\left(\frac{1}{2}\right)^x = 0.0625$ **e** $\left(\frac{3}{4}\right)^x = 0.1$ **f** $10^x = 0.000\,01$

2 Solve for x, giving your answer correct to 3 significant figures:

a $5^{2x} = 100$ **b** $2^{4x} = 75$ **c** $(0.8)^{3x} = 0.1$

d $3^{x-1} = 200$ **e** $4^{x+2} = 2.5$ **f** $6^{2x-1} = 800$

g $7^{2x+3} = 1000$ **h** $\left(3^{x+1}\right)^2 = 480$ **i** $\left(2^{x-3}\right)^{\frac{1}{2}} = 10$

3 Solve for x, giving an exact answer:

a $e^x = 10$ **b** $e^x = 1000$ **c** $2e^x = 0.3$

d $e^{\frac{x}{2}} = 5$ **e** $e^{2x} = 18$ **f** $e^{-\frac{x}{2}} = 1$

4 **a** Solve $e^{2x} = 300$ exactly.

b Use your calculator to find the solution correct to 2 decimal places.

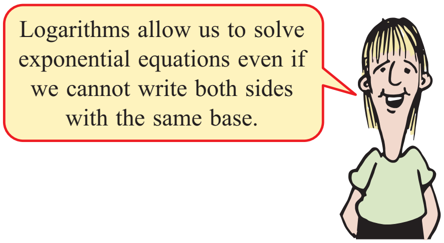
Example 22

Self Tutor

A farmer monitoring an insect plague notices that the area affected by the insects is given by $A = 1000 \times 2^{0.7n}$ hectares, where n is the number of weeks after the initial observation. How long will it take for the affected area to reach 5000 hectares?

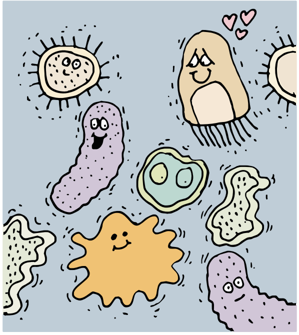
$$\begin{aligned}
\text{When} \quad A &= 5000, \\
1000 \times 2^{0.7n} &= 5000 \\
\therefore \quad 2^{0.7n} &= 5 \\
\therefore \quad \lg 2^{0.7n} &= \lg 5 \\
\therefore \quad 0.7n \lg 2 &= \lg 5 \\
\therefore \quad n &= \frac{\lg 5}{0.7 \times \lg 2} \\
\therefore \quad n &\approx 3.32
\end{aligned}$$

$\therefore$ it takes about 3 weeks and 2 days.



5 Solve for x, giving an exact answer:

a $4 \times 2^{-x} = 0.12$ **b** $300 \times 5^{0.1x} = 1000$ **c** $32 \times 3^{-0.25x} = 4$



6 The weight W of bacteria in a culture t hours after establishment is given by $W = 20 \times 2^{0.15t}$ grams. Find, using logarithms, the time for the weight of the culture to reach:

a 30 grams **b** 100 grams.

7 The mass M of bacteria in a culture t hours after establishment is given by $M = 25 \times e^{0.1t}$ grams.

a Show that the time required for the mass of the culture to reach 50 grams is $10 \ln 2$ hours.

b Find the time required correct to 2 decimal places.

8 The weight of radioactive uranium remaining after t years is given by the formula $W(t) = 50 \times 2^{-0.0002t}$ grams, $t \geqslant 0$.

a Find the initial weight of the uranium.

b Find the time required for the weight to fall to 8 grams.

Example 23

Self Tutor

Find algebraically the exact points of intersection of $y = e^x - 3$ and $y = 1 - 3e^{-x}$.

The functions meet where

GRAPHING PACKAGE

$$\begin{aligned} e^x - 3 &= 1 - 3e^{-x} \\ \therefore\ e^x - 4 + 3e^{-x} &= 0 \\ \therefore\ e^{2x} - 4e^x + 3 &= 0 \quad \{\text{multiplying each term by } e^x\} \\ \therefore\ (e^x - 1)(e^x - 3) &= 0 \\ \therefore\ e^x &= 1 \text{ or } 3 \\ \therefore\ x &= \ln 1 \text{ or } \ln 3 \\ \therefore\ x &= 0 \text{ or } \ln 3 \end{aligned}$$

When $x = 0$, $y = e^0 - 3 = -2$

When $x = \ln 3$, $y = e^{\ln 3} - 3 = 0$

$\therefore$ the functions meet at $(0, -2)$ and at $(\ln 3, 0)$.

9 Solve for x:

a $e^{2x} = 2e^x$

b $e^x = e^{-x}$

c $e^{2x} - 5e^x + 6 = 0$

d $e^x + 2 = 3e^{-x}$

e $1 + 12e^{-x} = e^x$

f $e^x + e^{-x} = 3$

10 Find algebraically the point(s) of intersection of:

a $y = e^x$ and $y = e^{2x} - 6$

b $y = 2e^x + 1$ and $y = 7 - e^x$

c $y = 3 - e^x$ and $y = 5e^{-x} - 3$

G THE CHANGE OF BASE RULE

A logarithm in base b can be written with a different base c using the **change of base rule**:

$$\log_b a = \frac{\log_c a}{\log_c b} \quad \text{for } a, b, c > 0 \text{ and } b, c \neq 1.$$

Proof: If $\log_b a = x$, then $b^x = a$

$$\begin{aligned} \therefore\ \log_c b^x &= \log_c a \quad \{\text{taking logarithms in base } c\} \\ \therefore\ x \log_c b &= \log_c a \quad \{\text{power law of logarithms}\} \\ \therefore\ x &= \frac{\log_c a}{\log_c b} \\ \therefore\ \log_b a &= \frac{\log_c a}{\log_c b} \end{aligned}$$

We can use this rule to write logarithms in base 10 or base e. This is useful in helping us evaluate them on our calculator.

Example 24 Self Tutor

Evaluate $\log_2 9$ by:

a changing to base 10 **b** changing to base e.

a $\log_2 9 = \dfrac{\log_{10} 9}{\log_{10} 2} \approx 3.17$ **b** $\log_2 9 = \dfrac{\ln 9}{\ln 2} \approx 3.17$

The rule can also be used to solve equations involving logarithms with different bases.

Example 25 Self Tutor

Solve for x: $\log_2 x = \log_8 15$

$$\begin{aligned} \log_2 x &= \log_8 15 \\ \therefore\ \log_2 x &= \frac{\log_2 15}{\log_2 8} \quad \{\text{writing RHS with base 2}\} \\ \therefore\ \log_2 x &= \frac{\log_2 15}{3} \\ \therefore\ \log_2 x &= \log_2 15^{\frac{1}{3}} \\ \therefore\ x &= \sqrt[3]{15} \end{aligned}$$

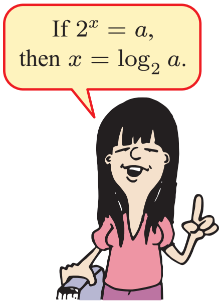
EXERCISE 5G

1 Use the rule $\log_b a = \dfrac{\log_{10} a}{\log_{10} b}$ to evaluate, correct to 3 significant figures:

a $\log_3 12$ **b** $\log_{\frac{1}{2}} 1250$ **c** $\log_3(0.067)$ **d** $\log_{0.4}(0.006\,984)$

2 Use the rule $\log_b a = \dfrac{\ln a}{\ln b}$ to solve, correct to 3 significant figures:

a $2^x = 0.051$ **b** $4^x = 213.8$ **c** $3^{2x+1} = 4.069$



3 Write:

a $\log_9 26$ in the form $a \log_3 b$, where $a, b \in \mathbb{Q}$

b $\log_2 11$ in the form $a \log_4 b$, where $a, b \in \mathbb{Z}$

c $\dfrac{6}{\log_7 25}$ in the form $a \log_5 b$, where $a, b \in \mathbb{Z}$.

4 Solve for x:

a $\log_3 x = \log_{27} 50$ **b** $\log_2 x = \log_4 13$ **c** $\log_{25} x = \log_5 7$

d $\log_3 \sqrt{x} + \log_9 x = \log_3 5$ **e** $\log_8 x^2 - \log_2 \sqrt[3]{x} = 1$ **f** $\log_4 x^3 + \log_2 \sqrt{x} = 8$

5 **a** Show that $\log_a b = \dfrac{1}{\log_b a}$.

b Solve for x:

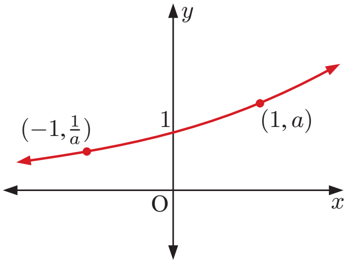
i $\log_3 x = 4 \log_x 3$ **ii** $\log_2 x - 4 = 5 \log_x 2$ **iii** $2 \log_4 x + 3 \log_x 4 = 7$

H GRAPHS OF LOGARITHMIC FUNCTIONS

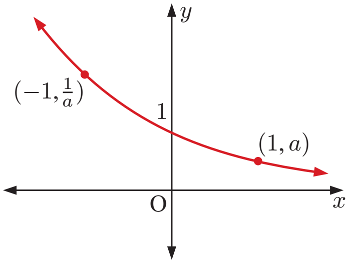
Consider the general exponential function $f(x)=a^x$, $a>0$, $a\neq 1$.

The graph of $y=a^x$ is:

For $a>1$:



For $0<a<1$:



The **horizontal asymptote** for all of these functions is the x-axis $y=0$.

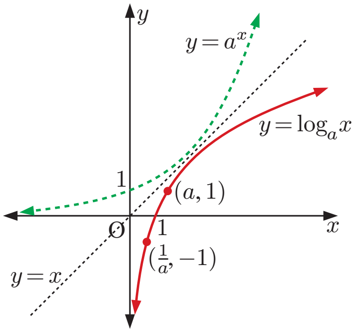
The inverse function f^{-1} is given by $x=a^y$, so $y=\log_a x$.

If $\boldsymbol{f(x)=a^x}$ where $\boldsymbol{a>0,\ a\neq 1}$, then $\boldsymbol{f^{-1}(x)=\log_a x}$.

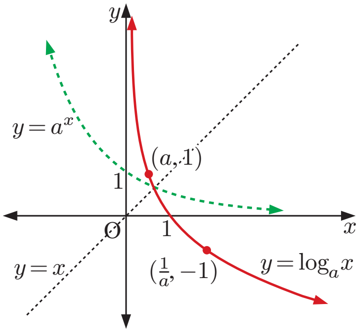
Since $f^{-1}(x)=\log_a x$ is an inverse function, it is a reflection of $f(x)=a^x$ in the line $y=x$. We may therefore deduce the following properties:

Function	$f(x)=a^x$	$f^{-1}(x)=\log_a x$
Domain	$\{x : x\in\mathbb{R}\}$	$\{x : x>0\}$
Range	$\{y : y>0\}$	$\{y : y\in\mathbb{R}\}$
Asymptote	horizontal $y=0$	vertical $x=0$

The graph of $y=\log_a x$ for $a>1$:



The graph of $y=\log_a x$ for $0<a<1$:



The **vertical asymptote** of $y=\log_a x$ is the y-axis $x=0$.

Since we can only find logarithms of positive numbers, the domain of $f^{-1}(x)=\log_a x$ is $\{x \mid x>0\}$.

In general, $y=\log_a(g(x))$ is defined when $g(x)>0$.

Example 26

Self Tutor

Consider the function $f(x) = \log_2(x-1)+1$.

a Find the domain and range of f.
b Find any asymptotes and axes intercepts.
c Sketch the graph of f showing all important features.
d Find f^{-1}.

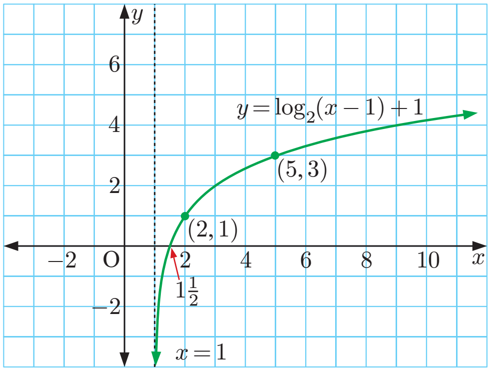
a $x - 1 > 0$ when $x > 1$
So, the domain is $\{x : x > 1\}$ and the range is $y \in \mathbb{R}$.

b As $x \to 1$ from the right, $y \to -\infty$, so the vertical asymptote is $x = 1$.
As $x \to \infty$, $y \to \infty$.
When $x = 0$, y is undefined, so there is no y-intercept.
When $y = 0$, $\log_2(x-1) = -1$

$$\therefore \quad x - 1 = 2^{-1}$$
$$\therefore \quad x = 1\tfrac{1}{2}$$

So, the x-intercept is $1\frac{1}{2}$.

c



d f is defined by $y = \log_2(x-1)+1$

$\therefore$ f^{-1} is defined by $x = \log_2(y-1)+1$

$$\therefore \quad x - 1 = \log_2(y-1)$$
$$\therefore \quad y - 1 = 2^{x-1}$$
$$\therefore \quad y = 2^{x-1}+1$$
$$\therefore \quad f^{-1}(x) = 2^{x-1}+1$$

which has the horizontal asymptote $y = 1$ ✓
Its domain is $\{x : x \in \mathbb{R}\}$, and
its range is $\{y : y > 1\}$.

EXERCISE 5H

1 For the following functions f:

- **i** Find the domain and range.
- **ii** Find any asymptotes and axes intercepts.
- **iii** Sketch the graph of $y = f(x)$ showing all important features.
- **iv** Solve $f(x) = -1$ algebraically and check the solution on your graph.
- **v** Find f^{-1}.

a $f : x \mapsto \log_3(x+1), \quad x > -1$
b $f : x \mapsto 1 - \log_3(x+1), \quad x > -1$
c $f : x \mapsto \log_5(x-2) - 2, \quad x > 2$
d $f : x \mapsto 1 - \log_5(x-2), \quad x > 2$
e $f : x \mapsto 1 - 2\log_2 x, \quad x > 0$

Example 27 Self Tutor

Consider the function $f : x \mapsto e^{x-3}$.

a Find the equation defining f^{-1}.

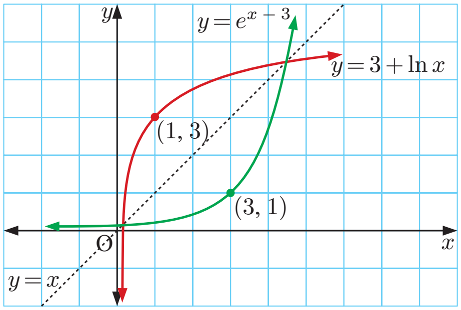
b Sketch the graphs of f and f^{-1} on the same set of axes.

c State the domain and range of f and f^{-1}.

d Find any asymptotes and intercepts of f and f^{-1}.

a
$$f(x) = e^{x-3}$$
$$\therefore \quad f^{-1} \text{ is } x = e^{y-3}$$
$$\therefore \quad y - 3 = \ln x$$
$$\therefore \quad y = 3 + \ln x$$
$$\text{So, } f^{-1}(x) = 3 + \ln x$$

b



c

Function	f	f^{-1}
Domain	$x \in \mathbb{R}$	$x > 0$
Range	$y > 0$	$y \in \mathbb{R}$

d For f, the horizontal asymptote is $y = 0$, and the y-intercept is e^{-3}.

For f^{-1}, the vertical asymptote is $x = 0$, and the x-intercept is e^{-3}.

2 For the following functions f:

- **i** Find the equation of f^{-1}.
- **ii** Sketch the graphs of f and f^{-1} on the same set of axes.
- **iii** State the domain and range of f and f^{-1}.
- **iv** Find any asymptotes and intercepts of f and f^{-1}.

a $f(x) = e^x + 5$

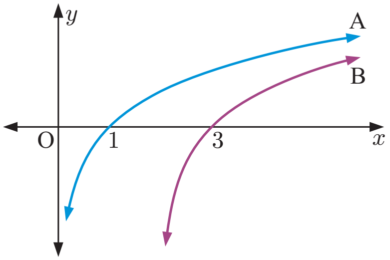
b $f(x) = e^{x+1} - 3$

c $f(x) = \ln x - 4, \quad x > 0$

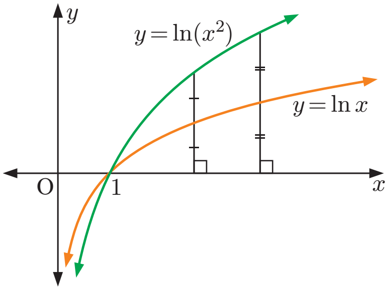
d $f(x) = \ln(x - 1) + 2, \quad x > 1$

3 Consider the graphs A and B. One of them is the graph of $y = \ln x$ and the other is the graph of $y = \ln(x - 2)$.

- **a** Identify which is which. Give evidence for your answer.
- **b** Copy the graphs onto a new set of axes and add to them the graph of $y = \ln(x + 2)$.
- **c** Find the equation of the vertical asymptote for each graph.



4 Kelly said that in order to graph $y = \ln(x^2)$, $x > 0$, you could first graph $y = \ln x$ and then double the distance of each point on the graph from the x-axis.
Is Kelly correct? Explain your answer.



5 Consider the function $f : x \mapsto e^{x+3} + 2$.

- **a** Find the defining equation for f^{-1}.
- **b** Find the values of x for which:
  - **i** $f(x) < 2.1$
  - **ii** $f(x) < 2.01$
  - **iii** $f(x) < 2.001$
  - **iv** $f(x) < 2.0001$

  Hence conjecture the horizontal asymptote for the graph of f.
- **c** Determine the equation of the horizontal asymptote of $f(x)$ by discussing the behaviour of $f(x)$ as $x \to \pm\infty$.
- **d** Hence, determine the vertical asymptote and the domain of f^{-1}.

6 Consider $f(x) = \log_2(x+3)$.

- **a** Find: **i** $f(5)$ **ii** $f(x^2)$ **iii** $f(2x-1)$
- **b** State the domain of $f(x)$.
- **c** Solve $f(x^2+4) = 5$.

7 Suppose $f(x) = e^{3x} + 1$.

- **a** State the range of $f(x)$.
- **b** Find $f^{-1}(x)$.
- **c** Find $f^{-1}(10)$.
- **d** State the domain of $f^{-1}(x)$.
- **e** Find $(f \circ f^{-1})(x)$ and $(f^{-1} \circ f)(x)$.

8 Suppose $f : x \mapsto e^{2x}$ and $g : x \mapsto 2x - 1$.

- **a** Find: **i** $(f^{-1} \circ g)(x)$ **ii** $(g \circ f)^{-1}(x)$
- **b** Solve $(f^{-1} \circ g)(x) = \ln 5$.

9 Consider $f : x \mapsto 10e^{-x}$ and $g : x \mapsto \ln(x-3)$.

- **a** Find $f(1)$ and $g(6)$.
- **b** Find the x-intercept of $g(x)$.
- **c** Find $fg(x)$.
- **d** Solve $f(x) = g^{-1}(x)$.

10 Let $f(x) = \ln(x+6)$ and $g(x) = x - \ln 3$.

- **a** State the domain of $f(x)$.
- **b** Find $f^{-1}(x)$.
- **c** Find the axes intercepts of $f(x)$.
- **d** Solve $gf(x) = f(x^2 - 12)$.

Activity

Click on the icon to obtain a card game for logarithmic functions.

CARD GAME

Review set 5A

1 Find the following, showing all working:

- **a** $\log_4 64$
- **b** $\log_2 256$
- **c** $\log_2(0.25)$
- **d** $\log_{25} 5$
- **e** $\log_8 1$
- **f** $\log_{81} 3$
- **g** $\log_9(0.\overline{1})$
- **h** $\log_k \sqrt{k}$

2 Find:

a $\lg \sqrt{10}$ **b** $\lg \frac{1}{\sqrt[3]{10}}$ **c** $\lg(10^a \times 10^{b+1})$

3 Simplify:

a $4\ln 2 + 2\ln 3$ **b** $\frac{1}{2}\ln 9 - \ln 2$ **c** $2\ln 5 - 1$ **d** $\frac{1}{4}\ln 81$

4 Find:

a $\ln(e\sqrt{e})$ **b** $\ln\left(\frac{1}{e^3}\right)$ **c** $\ln(e^{2x})$ **d** $\ln\left(\frac{e}{e^x}\right)$

5 Write as a single logarithm:

a $\lg 16 + 2\lg 3$ **b** $\log_2 16 - 2\log_2 3$ **c** $2 + \log_4 5$

6 Write as logarithmic equations:

a $P = 3 \times 7^x$ **b** $m = \frac{n^3}{5}$

7 Solve for x:

a $\log_2(x+5) - \log_2(x-2) = 3$ **b** $\lg x + \lg(x+15) = 2$

8 Show that $\log_3 7 \times 2\log_7 x = 2\log_3 x$.

9 Write the following equations without logarithms:

a $\lg T = 2\lg x - \lg 5$ **b** $\log_2 K = x + \log_2 3$

10 Write in the form $a\ln k$ where a and k are positive whole numbers and k is prime:

a $\ln 32$ **b** $\ln 125$ **c** $\ln 729$

11 Copy and complete:

Function	$y = \log_2 x$	$y = \ln(x+5)$
Domain		
Range		

12 If $A = \log_5 2$ and $B = \log_5 3$, write in terms of A and B:

a $\log_5 36$ **b** $\log_5 54$ **c** $\log_5(8\sqrt{3})$ **d** $\log_5(20.25)$ **e** $\log_5(0.\overline{8})$

13 Solve for x:

a $3e^x - 5 = -2e^{-x}$ **b** $2\ln x - 3\ln\left(\frac{1}{x}\right) = 10$

14 Solve for x, giving your answer to 2 decimal places:

a $7^x = 120$ **b** $6 \times 2^{3x} = 300$

15 A population of seals is given by $P = 20 \times 2^{\frac{t}{3}}$ where t is the time in years, $t \geqslant 0$. Find the time required for the population to reach 100.

16 Consider $f : x \mapsto 5e^{-x} + 1$.

a State the range of f.

b Find: **i** $f^{-1}(x)$ **ii** $f^{-1}(2)$

c State the domain of f^{-1}.

d Solve $f^{-1}(x) = 0$.

e Sketch the graphs of f, f^{-1}, and $y = x$ on the same set of axes.

Review set 5B

1 Without using a calculator, find the base 10 logarithms of:

a $\sqrt{1000}$ **b** $\frac{10}{\sqrt[3]{10}}$ **c** $\frac{10^a}{10^{-b}}$

2 Write in the form 10^x giving x correct to 4 decimal places:

a 32 **b** 0.0013 **c** 8.963×10^{-5}

3 Find x if:

a $\log_2 x = -3$ **b** $\log_5 x \approx 2.743$ **c** $\log_3 x \approx -3.145$

4 Write the following equations without logarithms:

a $\log_2 k \approx 1.699 + x$ **b** $\log_a Q = 3\log_a P + \log_a 5$ **c** $\lg A = x \lg 2 + \lg 6$

5 Solve for x, giving exact answers:

a $5^x = 7$ **b** $20 \times 2^{2x+1} = 640$

6 Find the exact value of $\log_{12} 3 - 2\log_{12} 6$.

7 Write $\log_8 30$ in the form $a \log_2 b$, where $a, b \in \mathbb{Q}$.

8 Solve for x:

a $\log_4 x + \log_4(2x - 8) = 3$ **b** $\log_x 135 = 3 + \log_x 5$

9 Consider $f(x) = e^x$ and $g(x) = \ln(x + 4)$, $x > -4$. Find:

a $(f \circ g)(5)$ **b** $(g \circ f)(0)$

10 Write as a single logarithm:

a $\ln 60 - \ln 20$ **b** $\ln 4 + \ln 1$ **c** $\ln 200 - \ln 8 + \ln 5$

11 Write as logarithmic equations:

a $M = 5 \times 6^x$ **b** $T = \frac{5}{\sqrt{l}}$ **c** $G = \frac{4}{c}$

12 Solve exactly for x:

a $e^{2x} = 3e^x$ **b** $e^{2x} - 7e^x + 12 = 0$

13 Consider the function $g : x \mapsto \log_3(x + 2) - 2$.

a Find the domain and range.

b Find any asymptotes and axes intercepts for the graph of the function.

c Find the defining equation for g^{-1}.

d Sketch the graphs of g, g^{-1}, and $y = x$ on the same axes.

14 The weight of a radioactive isotope remaining after t weeks is given by $W = 8000 \times e^{-\frac{t}{20}}$ grams. Find the time for the weight to halve.

15 Solve for x:

a $\log_2 x + \log_4 x^4 = \log_2 125$ **b** $\log_2 x = 25 \log_x 2$ **c** $\log_3 x + 8 \log_x 3 = 6$

16 Consider $f(x) = 5e^{2x}$ and $g(x) = \ln(x - 4)$.

a State the domain and range of g. **b** Find the axes intercepts of g.

c Find the exact solution to $fg(x) = 30$. **d** Solve $f(x) = g^{-1}(x)$.

6

Polynomials

Contents:

Opening problem

To determine whether 7 is a **factor** of 56, we divide 56 by 7. The result is exactly 8. Since there is no remainder, 7 is a factor of 56.

Things to think about:

- **a** Can we perform a similar test for *algebraic* factors? For example, how can we determine whether $x-3$ is a factor of x^3-4x^2+2x+3?
- **b** Given that $x-3$ is a factor of x^3-4x^2+2x+3, what does this tell us about the graph of $f(x)=x^3-4x^2+2x+3$?

Up to this point we have studied linear and quadratic functions at some depth, with perhaps occasional reference to cubic functions. These are part of a larger family of functions called the **polynomials**.

A REAL POLYNOMIALS

A **polynomial function** is a function of the form

$$P(x)=a_nx^n+a_{n-1}x^{n-1}+....+a_2x^2+a_1x+a_0, \quad a_1, ..., a_n \text{ constant}, \ a_n\neq 0.$$

We say that:
x is the **variable**
a_0 is the **constant term**
a_n is the **leading coefficient** and is non-zero
a_r is the **coefficient of x^r** for $r=0, 1, 2, ..., n$
n is the **degree** of the polynomial, being the highest power of the variable.

In **summation notation**, we write $P(x)=\sum_{r=0}^{n}a_rx^r$,

which reads: "the sum from $r=0$ to n, of a_rx^r".

A **real polynomial** $P(x)$ is a polynomial for which $a_r\in\mathbb{R}$, $r=0, 1, 2, ..., n$.

The low degree members of the polynomial family have special names, some of which you are already familiar with. For these polynomials, we commonly write their coefficients as a, b, c, ...

Polynomial function	*Degree*	*Name*
$ax+b, \quad a\neq 0$	1	linear
$ax^2+bx+c, \quad a\neq 0$	2	quadratic
$ax^3+bx^2+cx+d, \quad a\neq 0$	3	cubic
$ax^4+bx^3+cx^2+dx+e, \quad a\neq 0$	4	quartic

ADDITION AND SUBTRACTION

To **add** or **subtract** two polynomials, we collect 'like' terms.

Example 1

Self Tutor

If $P(x) = x^3 - 2x^2 + 3x - 5$ and $Q(x) = 2x^3 + x^2 - 11$, find:

a $P(x) + Q(x)$

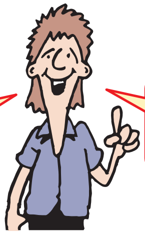
b $P(x) - Q(x)$

a

$$\begin{array}{rl} & P(x) + Q(x) \\ = & x^3 - 2x^2 + 3x - 5 \\ & \underline{+2x^3 + x^2 - 11} \\ = & 3x^3 - x^2 + 3x - 16 \end{array}$$

b

$$\begin{array}{rl} & P(x) - Q(x) \\ = & x^3 - 2x^2 + 3x - 5 - (2x^3 + x^2 - 11) \\ = & x^3 - 2x^2 + 3x - 5 \\ & \underline{-2x^3 - x^2 + 11} \\ = & -x^3 - 3x^2 + 3x + 6 \end{array}$$



SCALAR MULTIPLICATION

To **multiply** a polynomial by a **scalar** (constant) we multiply each term by the scalar.

Example 2

Self Tutor

If $P(x) = x^4 - 2x^3 + 4x + 7$, find:

a $3P(x)$

b $-2P(x)$

a

$$\begin{aligned} & 3P(x) \\ &= 3(x^4 - 2x^3 + 4x + 7) \\ &= 3x^4 - 6x^3 + 12x + 21 \end{aligned}$$

b

$$\begin{aligned} & -2P(x) \\ &= -2(x^4 - 2x^3 + 4x + 7) \\ &= -2x^4 + 4x^3 - 8x - 14 \end{aligned}$$

POLYNOMIAL MULTIPLICATION

To **multiply** two polynomials, we multiply each term of the first polynomial by each term of the second polynomial, and then collect like terms.

Example 3

Self Tutor

If $P(x) = x^3 - 2x + 4$ and $Q(x) = 2x^2 + 3x - 5$, find $P(x)Q(x)$.

$$\begin{aligned} P(x)Q(x) &= (x^3 - 2x + 4)(2x^2 + 3x - 5) \\ &= x^3(2x^2 + 3x - 5) - 2x(2x^2 + 3x - 5) + 4(2x^2 + 3x - 5) \\ &= 2x^5 + 3x^4 - 5x^3 \\ &\qquad\qquad\quad - 4x^3 - 6x^2 + 10x \\ &\qquad\qquad\qquad\quad\ \underline{+ 8x^2 + 12x - 20} \\ &= 2x^5 + 3x^4 - 9x^3 + 2x^2 + 22x - 20 \end{aligned}$$

EXERCISE 6A.1

1 If $P(x) = x^2 + 2x + 3$ and $Q(x) = 4x^2 + 5x + 6$, find in simplest form:

a $3P(x)$ **b** $P(x) + Q(x)$ **c** $P(x) - 2Q(x)$ **d** $P(x)Q(x)$

2 If $f(x) = x^2 - x + 2$ and $g(x) = x^3 - 3x + 5$, find in simplest form:

a $f(x) + g(x)$ **b** $g(x) - f(x)$ **c** $2f(x) + 3g(x)$

d $g(x) + xf(x)$ **e** $f(x)\,g(x)$ **f** $[f(x)]^2$

3 Expand and simplify:

a $(x^2 - 2x + 3)(2x + 1)$ **b** $(x - 1)^2(x^2 + 3x - 2)$ **c** $(x + 2)^3$

d $(2x^2 - x + 3)^2$ **e** $(2x - 1)^4$ **f** $(3x - 2)^2(2x + 1)(x - 4)$

4 Find the following products:

a $(2x^2 - 3x + 5)(3x - 1)$ **b** $(4x^2 - x + 2)(2x + 5)$

c $(2x^2 + 3x + 2)(5 - x)$ **d** $(x - 2)^2(2x + 1)$

e $(x^2 - 3x + 2)(2x^2 + 4x - 1)$ **f** $(3x^2 - x + 2)(5x^2 + 2x - 3)$

g $(x^2 - x + 3)^2$ **h** $(2x^2 + x - 4)^2$

i $(2x + 5)^3$ **j** $(x^3 + x^2 - 2)^2$

Discussion

Suppose $f(x)$ is a polynomial of degree m, and $g(x)$ is a polynomial of degree n.

What is the degree of:

- $f(x) + g(x)$
- $5f(x)$
- $[f(x)]^2$
- $f(x)g(x)$?

DIVISION OF POLYNOMIALS

The division of polynomials is only useful if we divide a polynomial of degree n by another of degree n or less.

Division by linears

Consider $(2x^2 + 3x + 4)(x + 2) + 7$.

If we expand this expression we obtain $(2x^2 + 3x + 4)(x + 2) + 7 = 2x^3 + 7x^2 + 10x + 15$.

Dividing both sides by $(x + 2)$, we obtain

$$\begin{aligned}\frac{2x^3 + 7x^2 + 10x + 15}{x + 2} &= \frac{(2x^2 + 3x + 4)(x + 2) + 7}{x + 2} \\ &= \frac{(2x^2 + 3x + 4)(x + 2)}{x + 2} + \frac{7}{x + 2} \\ &= 2x^2 + 3x + 4 + \frac{7}{x + 2}\end{aligned}$$

where $x + 2$ is the divisor,
$2x^2 + 3x + 4$ is the quotient,
and 7 is the remainder.


The division of polynomials is not required for the syllabus, but is useful for understanding the Remainder and Factor theorems.

If $P(x)$ is divided by $ax+b$ until a constant remainder R is obtained, then

$$\frac{P(x)}{ax+b} = Q(x) + \frac{R}{ax+b} \quad \text{where} \quad \begin{array}{rl} ax+b & \text{is the } \textbf{divisor}, \ D(x), \\ Q(x) & \text{is the } \textbf{quotient}, \\ \text{and} \quad R & \text{is the } \textbf{remainder}. \end{array}$$

Notice that $P(x) = Q(x) \times (ax+b) + R$.

Division algorithm

We can divide a polynomial by another polynomial using an algorithm similar to that used for division of whole numbers:

Step 1: What do we multiply x by to get $2x^3$?
The answer is $2x^2$,
and $2x^2(x+2) = \underbrace{2x^3+4x^2}$.

Step 2: Subtract $2x^3+4x^2$ from $2x^3+7x^2$.
The answer is $3x^2$.

Step 3: Bring down the $10x$ to obtain $3x^2+10x$.

Return to *Step 1* with the question:
"What must we multiply x by to get $3x^2$?"
The answer is $3x$, and $3x(x+2) = 3x^2+6x$ ....

We continue the process until we are left with a constant.

$$\begin{array}{r|r}
 & 2x^2 + 3x + 4 \\ \cline{2-2}
x+2 & 2x^3+7x^2+10x+15 \\
 & -(2x^3+4x^2) \qquad\quad\; \\ \cline{2-2}
 & 3x^2+10x \\
 & -(3x^2+6x) \\ \cline{2-2}
 & 4x+15 \\
 & -(4x+8) \\ \cline{2-2}
 & 7
\end{array}$$

So, $\dfrac{2x^3+7x^2+10x+15}{x+2} = 2x^2+3x+4+\dfrac{7}{x+2}$

Example 4

Self Tutor

Find the quotient and remainder for $\dfrac{x^3-x^2-3x-5}{x-3}$.

Hence write x^3-x^2-3x-5 in the form $Q(x)\times(x-3)+R$.

$$\begin{array}{r|r}
 & x^2+2x+3 \\ \cline{2-2}
x-3 & x^3-x^2-3x-5 \\
 & -(x^3-3x^2) \qquad\quad \\ \cline{2-2}
 & 2x^2-3x \\
 & -(2x^2-6x) \\ \cline{2-2}
 & 3x-5 \\
 & -(3x-9) \\ \cline{2-2}
 & 4
\end{array}$$

The quotient is x^2+2x+3 and the remainder is 4.

$\therefore \dfrac{x^3-x^2-3x-5}{x-3} = x^2+2x+3+\dfrac{4}{x-3}$

$\therefore x^3-x^2-3x-5 = (x^2+2x+3)(x-3)+4.$

Check your answer by expanding the RHS.

Example 5

Self Tutor

Perform the division $\dfrac{x^4+2x^2-1}{x+3}$.

Hence write x^4+2x^2-1 in the form $Q(x)\times(x+3)+R$.

$$
\begin{array}{rr}
 & x^3 - 3x^2 + 11x - 33 \\ \cline{2-2}
x+3 & \big)\; x^4 + 0x^3 + 2x^2 + 0x - 1 \\
 & -(x^4 + 3x^3) \qquad\qquad\qquad\quad \\ \cline{2-2}
 & -3x^3 + 2x^2 \qquad\qquad\quad \\
 & -(-3x^3 - 9x^2) \qquad\qquad \\ \cline{2-2}
 & 11x^2 + 0x \qquad\quad \\
 & -(11x^2 + 33x) \qquad \\ \cline{2-2}
 & -33x - 1 \\
 & -(-33x - 99) \\ \cline{2-2}
 & 98
\end{array}
$$

$\therefore\ \dfrac{x^4+2x^2-1}{x+3} = x^3-3x^2+11x-33+\dfrac{98}{x+3}$

$\therefore\ x^4+2x^2-1=(x^3-3x^2+11x-33)(x+3)+98$

Notice the insertion of $0x^3$ and $0x$.

EXERCISE 6A.2

1 Find the quotient and remainder for the following, and hence write the division in the form $P(x)=Q(x)\,D(x)+R$, where $D(x)$ is the divisor.

a $\dfrac{x^2+2x-3}{x+2}$ **b** $\dfrac{x^2-5x+1}{x-1}$ **c** $\dfrac{2x^3+6x^2-4x+3}{x-2}$

2 Perform the following divisions, and hence write the division in the form $P(x)=Q(x)\,D(x)+R$.

a $\dfrac{x^2-3x+6}{x-4}$ **b** $\dfrac{x^2+4x-11}{x+3}$ **c** $\dfrac{2x^2-7x+2}{x-2}$

d $\dfrac{2x^3+3x^2-3x-2}{2x+1}$ **e** $\dfrac{3x^3+11x^2+8x+7}{3x-1}$ **f** $\dfrac{2x^4-x^3-x^2+7x+4}{2x+3}$

3 Perform the divisions:

a $\dfrac{x^2+5}{x-2}$ **b** $\dfrac{2x^2+3x}{x+1}$ **c** $\dfrac{3x^2+2x-5}{x+2}$

d $\dfrac{x^3+2x^2-5x+2}{x-1}$ **e** $\dfrac{2x^3-x}{x+4}$ **f** $\dfrac{x^3+x^2-5}{x-2}$

DIVISION BY QUADRATICS

As with division by linears, we can use the **division algorithm** to divide polynomials by quadratics. The division process stops when the remainder has degree less than that of the divisor, so

If $P(x)$ is divided by ax^2+bx+c then

$$\frac{P(x)}{ax^2+bx+c}=Q(x)+\frac{ex+f}{ax^2+bx+c}$$

where ax^2+bx+c is the **divisor**,
$Q(x)$ is the **quotient**,
and $ex+f$ is the **remainder**.

The remainder will be linear if $e\neq 0$, and constant if $e=0$.

Example 6

Self Tutor

Find the quotient and remainder for $\dfrac{x^4+4x^3-x+1}{x^2-x+1}$.

Hence write x^4+4x^3-x+1 in the form $Q(x)\times(x^2-x+1)+R(x)$.

$$\begin{array}{r r}
 & x^2+5x+4 \\ \cline{2-2}
x^2-x+1 & \big)\; x^4+4x^3+0x^2-x+1 \\
 & -(x^4-x^3+x^2) \\ \cline{2-2}
 & 5x^3-x^2-x \\
 & -(5x^3-5x^2+5x) \\ \cline{2-2}
 & 4x^2-6x+1 \\
 & -(4x^2-4x+4) \\ \cline{2-2}
 & -2x-3
\end{array}$$

The quotient is x^2+5x+4
and the remainder is $-2x-3$.

$\therefore\ x^4+4x^3-x+1$
$=(x^2+5x+4)(x^2-x+1)-2x-3$

EXERCISE 6A.3

1 Find the quotient and remainder for:

a $\dfrac{x^3+2x^2+x-3}{x^2+x+1}$ **b** $\dfrac{3x^2-x}{x^2-1}$ **c** $\dfrac{3x^3+x-1}{x^2+1}$ **d** $\dfrac{x-4}{x^2+2x-1}$

2 Carry out the following divisions and also write each in the form $P(x)=Q(x)\,D(x)+R(x)$:

a $\dfrac{x^2-x+1}{x^2+x+1}$ **b** $\dfrac{x^3}{x^2+2}$ **c** $\dfrac{x^4+3x^2+x-1}{x^2-x+1}$

d $\dfrac{2x^3-x+6}{(x-1)^2}$ **e** $\dfrac{x^4}{(x+1)^2}$ **f** $\dfrac{x^4-2x^3+x+5}{(x-1)(x+2)}$

3 Suppose $P(x)=(x-2)(x^2+2x+3)+7$. Find the quotient and remainder when $P(x)$ is divided by $x-2$.

4 Suppose $f(x)=(x-1)(x+2)(x^2-3x+5)+15-10x$. Find the quotient and remainder when $f(x)$ is divided by x^2+x-2.

B ZEROS, ROOTS, AND FACTORS

A **zero** of a polynomial is a value of the variable which makes the polynomial equal to zero.

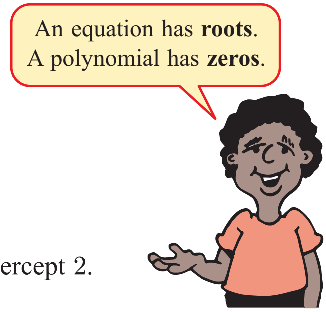
α is a **zero** of polynomial $P(x) \Leftrightarrow P(\alpha) = 0$.

The **roots** of a polynomial **equation** are the solutions to the equation.

α is a **root** (or **solution**) of $P(x) = 0 \Leftrightarrow P(\alpha) = 0$.

The **roots** of $P(x) = 0$ are the **zeros** of $P(x)$ and the x-intercepts of the graph of $y = P(x)$.

Consider
$$\begin{aligned} P(x) &= x^3 + 2x^2 - 3x - 10 \\ \therefore\ P(2) &= 2^3 + 2(2)^2 - 3(2) - 10 \\ &= 8 + 8 - 6 - 10 \\ &= 0 \end{aligned}$$



This tells us:

- 2 is a zero of $x^3 + 2x^2 - 3x - 10$
- 2 is a root of $x^3 + 2x^2 - 3x - 10 = 0$
- the graph of $y = x^3 + 2x^2 - 3x - 10$ has the x-intercept 2.

If $P(x) = (x + 1)(2x - 1)(x + 2)$, then $(x + 1)$, $(2x - 1)$, and $(x + 2)$ are its **linear factors**.

Likewise $P(x) = (x + 3)^2(2x + 3)$ has been factorised into 3 linear factors, one of which is repeated.

$x - \alpha$ is a **factor** of the polynomial $P(x) \Leftrightarrow$ there exists a polynomial $Q(x)$ such that $P(x) = (x - \alpha)Q(x)$.

Example 7

Self Tutor

Find the zeros of:

a $x^2 - 6x + 2$

b $x^3 - 5x$

a We wish to find x such that

$$x^2 - 6x + 2 = 0$$
$$\therefore\ x = \frac{6 \pm \sqrt{36 - 4(1)(2)}}{2}$$
$$\therefore\ x = \frac{6 \pm \sqrt{28}}{2}$$
$$\therefore\ x = \frac{6 \pm 2\sqrt{7}}{2}$$
$$\therefore\ x = 3 \pm \sqrt{7}$$

The zeros are $3 - \sqrt{7}$ and $3 + \sqrt{7}$.

b We wish to find x such that

$$x^3 - 5x = 0$$
$$\therefore\ x(x^2 - 5) = 0$$
$$\therefore\ x(x + \sqrt{5})(x - \sqrt{5}) = 0$$
$$\therefore\ x = 0 \text{ or } \pm\sqrt{5}$$

The zeros are $-\sqrt{5}$, 0, and $\sqrt{5}$.

EXERCISE 6B.1

1 Find the zeros of:

a $2x^2 - 5x - 12$ **b** $x^2 + 6x - 1$ **c** $x^2 - 10x + 6$

d $x^3 - 4x$ **e** $x^3 - 11x$ **f** $x^4 - 6x^2 + 8$

2 Find the roots of:

a $5x^2 = 3x + 2$ **b** $(2x+1)(x^2-3) = 0$ **c** $(3x-1)(x^2+x-6) = 0$

d $-2x(x^2 - 2x - 2) = 0$ **e** $x^3 = 7x$ **f** $x^4 = 7x^2 - 10$

Example 8 Self Tutor

Factorise:

a $2x^3 + 5x^2 - 3x$ **b** $x^2 + 4x - 1$

a
$$\begin{aligned} &2x^3 + 5x^2 - 3x \\ &= x(2x^2 + 5x - 3) \\ &= x(2x-1)(x+3) \end{aligned}$$

b $x^2 + 4x - 1$ is zero when
$$\begin{aligned} x &= \frac{-4 \pm \sqrt{16 - 4(1)(-1)}}{2} \\ \therefore\ x &= \frac{-4 \pm \sqrt{20}}{2} \\ \therefore\ x &= \frac{-4 \pm 2\sqrt{5}}{2} \\ \therefore\ x &= -2 \pm \sqrt{5} \end{aligned}$$

$$\begin{aligned} \therefore\ x^2 + 4x - 1 &= (x - [-2 + \sqrt{5}])(x - [-2 - \sqrt{5}]) \\ &= (x + 2 - \sqrt{5})(x + 2 + \sqrt{5}) \end{aligned}$$

3 Find the linear factors of:

a $2x^2 - 7x - 15$ **b** $x^3 - 11x^2 + 28x$ **c** $x^2 - 6x + 3$

d $x^3 + 2x^2 - 4x$ **e** $6x^3 - x^2 - 2x$ **f** $x^4 - 6x^2 + 5$

4 If $P(x) = a(x - \alpha)(x - \beta)(x - \gamma)$ then α, β, and γ are its zeros.
Verify this statement by finding $P(\alpha)$, $P(\beta)$, and $P(\gamma)$.

Example 9 Self Tutor

Find *all* cubic polynomials with zeros $\frac{1}{2}$ and $-3 \pm \sqrt{2}$.

The zeros $-3 \pm \sqrt{2}$ have sum $= -3 + \sqrt{2} - 3 - \sqrt{2} = -6$ and
product $= (-3 + \sqrt{2})(-3 - \sqrt{2}) = 7$

$\therefore$ they come from the quadratic factor $x^2 + 6x + 7$

$\frac{1}{2}$ comes from the linear factor $2x - 1$.

$\therefore$ $P(x) = a(2x - 1)(x^2 + 6x + 7)$, $a \neq 0$.

5 Find *all* cubic polynomials with zeros:

a -3, 4, 5 **b** ± 2, 3 **c** 3, $1 \pm \sqrt{5}$ **d** -1, $-2 \pm \sqrt{2}$

Example 10 Self Tutor

Find *all* quartic polynomials with zeros 2, $-\frac{1}{3}$, and $-1 \pm \sqrt{5}$.

The zeros $-1 \pm \sqrt{5}$ have sum $= -1 + \sqrt{5} - 1 - \sqrt{5} = -2$ and
product $= (-1+\sqrt{5})(-1-\sqrt{5}) = -4$

$\therefore$ they come from the quadratic factor $x^2 + 2x - 4$.

The zeros 2 and $-\frac{1}{3}$ come from the linear factors $x - 2$ and $3x + 1$.

$\therefore \quad P(x) = a(x-2)(3x+1)(x^2+2x-4), \quad a \neq 0.$

6 Find *all* quartic polynomials with zeros of:

a $\pm 1, \ \pm\sqrt{2}$ **b** $2, \ -\frac{1}{5}, \ \pm\sqrt{3}$ **c** $-3, \ \frac{1}{4}, \ 1 \pm \sqrt{2}$ **d** $2 \pm \sqrt{5}, \ -2 \pm \sqrt{7}$

POLYNOMIAL EQUALITY

Two polynomials are **equal** if and only if they have the **same degree** (order), and corresponding terms have equal coefficients.

If we know that two polynomials are **equal** then we can **equate coefficients** to find unknown coefficients.

For example, if $2x^3 + 3x^2 - 4x + 6 = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$, then
$a = 2$, $b = 3$, $c = -4$, and $d = 6$.

Example 11 Self Tutor

Find constants a, b, and c given that:

$$6x^3 + 7x^2 - 19x + 7 = (2x-1)(ax^2 + bx + c) \quad \text{for all } x.$$

$$6x^3 + 7x^2 - 19x + 7 = (2x-1)(ax^2+bx+c)$$
$$\therefore \quad 6x^3 + 7x^2 - 19x + 7 = 2ax^3 + 2bx^2 + 2cx - ax^2 - bx - c$$
$$\therefore \quad 6x^3 + 7x^2 - 19x + 7 = 2ax^3 + (2b-a)x^2 + (2c-b)x - c$$

Since this is true for all x, we equate coefficients:

$$\therefore \quad \underbrace{2a = 6}_{x^3\text{ s}} \qquad \underbrace{2b - a = 7}_{x^2\text{ s}} \qquad \underbrace{2c - b = -19}_{x\text{ s}} \qquad \text{and} \qquad \underbrace{7 = -c}_{\text{constants}}$$

$$\therefore \quad a = 3 \quad \text{and} \quad c = -7 \quad \text{and consequently} \quad \underbrace{2b - 3 = 7 \quad \text{and} \quad -14 - b = -19}_{\substack{\therefore\ b = 5 \\ \text{in both equations}}}$$

So, $a = 3$, $b = 5$, and $c = -7$.

Example 12

Self Tutor

Find constants a and b if $z^4+9=(z^2+az+3)(z^2+bz+3)$ for all z.

$z^4+9=(z^2+az+3)(z^2+bz+3)$ for all z

$\therefore$ $z^4+9=z^4+bz^3+3z^2$
$+az^3+abz^2+3az$
$+3z^2+3bz+9$

$\therefore$ $z^4+9=z^4+(a+b)z^3+(ab+6)z^2+(3a+3b)z+9$ for all z

Equating coefficients gives $\begin{cases} a+b=0 & \text{.... (1)} \quad \{z^3\text{s}\} \\ ab+6=0 & \text{.... (2)} \quad \{z^2\text{s}\} \\ 3a+3b=0 & \text{.... (3)} \quad \{z\,\text{s}\} \end{cases}$

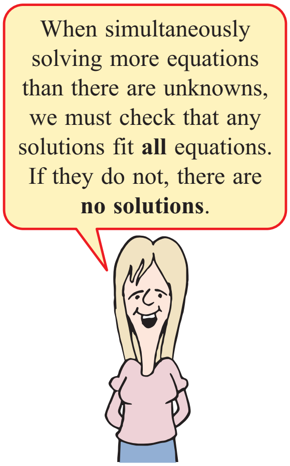
From (1) and (3) we see that $b=-a$

$\therefore$ in (2), $a(-a)+6=0$

$\therefore$ $a^2=6$

$\therefore$ $a=\pm\sqrt{6}$ and so $b=\mp\sqrt{6}$

$\therefore$ $a=\sqrt{6},\ b=-\sqrt{6}$ *or* $a=-\sqrt{6},\ b=\sqrt{6}$



EXERCISE 6B.2

1 Find constants a, b, and c given that:

a $2x^2+4x+5=ax^2+[2b-6]x+c$ for all x

b $2x^3-x^2+6=(x-1)^2(2x+a)+bx+c$ for all x

c $6x^3-13x^2+7x+4=(3x+1)(ax^2+bx+c)$ for all x.

2 Find constants a and b if:

a $z^4+4=(z^2+az+2)(z^2+bz+2)$ for all z

b $2z^4+5z^3+4z^2+7z+6=(z^2+az+2)(2z^2+bz+3)$ for all z.

3 **a** Given that $x^3+9x^2+11x-21=(x+3)(ax^2+bx+c)$, find the values of a, b, and c.

b Hence, fully factorise $x^3+9x^2+11x-21$.

4 **a** Given that $4x^3+12x^2+3x-5=(2x-1)(px^2+qx+r)$, find the values of p, q, and r.

b Hence, find the solutions to $4x^3+12x^2+3x-5=0$.

5 **a** Given that $3x^3+10x^2-7x+4=(x+4)(ax^2+bx+c)$, find the values of a, b, and c.

b Hence, show that $3x^3+10x^2-7x+4$ has only one real zero.

6 Suppose $3x^3+kx^2-7x-2=(3x+2)(ax^2+bx+c)$.

a Find the values of a, b, c, and k.

b Hence, find the roots of $3x^3+kx^2-7x-2=0$.

7 **a** Find real numbers a and b such that $x^4-4x^2+8x-4=(x^2+ax+2)(x^2+bx-2)$.

b Hence, find the real roots of $x^4+8x=4x^2+4$.

Example 13

Self Tutor

$x+3$ is a factor of $P(x)=x^3+ax^2-7x+6$. Find $a\in\mathbb{R}$ and the other factors.

Since $x+3$ is a factor,

The coefficient of x^3 is $1\times 1=1$

This must be 2 so the constant term is $3\times 2=6$

$$\begin{aligned} x^3+ax^2-7x+6 &= (x+3)(x^2+bx+2) \quad \text{for some constant } b \\ &= x^3+bx^2+\ \ 2x \\ &\qquad\ \ +3x^2+3bx+6 \\ &= x^3+(b+3)x^2+(3b+2)x+6 \end{aligned}$$

Equating coefficients gives $3b+2=-7$ and $a=b+3$

$\therefore\ b=-3$ and $a=0$

$$\begin{aligned} \therefore\ P(x) &= (x+3)(x^2-3x+2) \\ &= (x+3)(x-1)(x-2) \end{aligned}$$

The other factors are $(x-1)$ and $(x-2)$.

8 $2x-3$ is a factor of $2x^3+3x^2+ax+3$. Find $a\in\mathbb{R}$ and all zeros of the cubic.

Example 14

Self Tutor

$2x+3$ and $x-1$ are factors of $2x^4+ax^3-3x^2+bx+3$.

Find constants a and b and all zeros of the polynomial.

Since $2x+3$ and $x-1$ are factors,

The coefficient of x^4 is $2\times 1\times 1=2$

This must be -1 so the constant term is $3\times -1\times -1=3$

$$\begin{aligned} 2x^4+ax^3-3x^2+bx+3 &= (2x+3)(x-1)(x^2+cx-1) \quad \text{for some } c \\ &= (2x^2+x-3)(x^2+cx-1) \\ &= 2x^4+2cx^3-2x^2 \\ &\qquad +\ \ x^3+cx^2-\ \ x \\ &\qquad\qquad\quad -3x^2-3cx+3 \\ &= 2x^4+(2c+1)x^3+(c-5)x^2+(-1-3c)x+3 \end{aligned}$$

Equating coefficients gives $2c+1=a$, $c-5=-3$, and $-1-3c=b$

$\therefore\ c=2$

$\therefore\ a=5$ and $b=-7$

$\therefore\ P(x)=(2x+3)(x-1)(x^2+2x-1)$

Now x^2+2x-1 has zeros $\dfrac{-2\pm\sqrt{4-4(1)(-1)}}{2}=\dfrac{-2\pm 2\sqrt{2}}{2}=-1\pm\sqrt{2}$

$\therefore\ P(x)$ has zeros $-\frac{3}{2}$, 1, and $-1\pm\sqrt{2}$.

9 $2x+1$ and $x-2$ are factors of $P(x)=2x^4+ax^3+bx^2+18x+8$.

a Find a and b. **b** Hence, solve $P(x)=0$.

10 x^3+3x^2-9x+c, $c\in\mathbb{R}$, has two identical linear factors. Prove that c is either 5 or -27, and factorise the cubic into linear factors in each case.

C THE REMAINDER THEOREM

Consider the cubic polynomial $P(x)=x^3+5x^2-11x+3$.

If we divide $P(x)$ by $x-2$, we find that

$$\frac{x^3+5x^2-11x+3}{x-2}=x^2+7x+3+\frac{\overset{\text{remainder}}{9}}{x-2}$$

So, when $P(x)$ is divided by $x-2$, the remainder is 9.

Notice also that $P(2)=8+20-22+3$
$=9$, which is the remainder.

By considering other examples like the one above, we formulate the **Remainder theorem**.

The Remainder Theorem

When a polynomial $P(x)$ is divided by $x-k$ until a constant remainder R is obtained, then $R=P(k)$.

Proof:

By the division algorithm, $P(x)=Q(x)(x-k)+R$
Letting $x=k$, $P(k)=Q(k)\times 0+R$
$\therefore$ $P(k)=R$

When using the Remainder theorem, it is important to realise that the following statements are equivalent:

- $P(x)=(x-k)Q(x)+R$
- $P(k)=R$
- $P(x)$ divided by $x-k$ leaves a remainder of R.

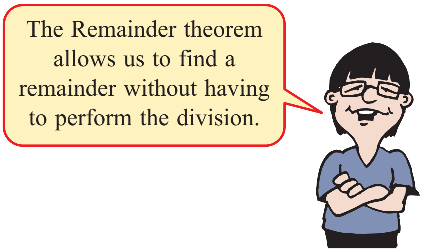
Example 15 **Self Tutor**

Use the Remainder theorem to find the remainder when x^4-3x^3+x-4 is divided by $x+2$.

If $P(x)=x^4-3x^3+x-4$, then

$$\begin{aligned}P(-2)&=(-2)^4-3(-2)^3+(-2)-4\\&=16+24-2-4\\&=34\end{aligned}$$

$\therefore$ when x^4-3x^3+x-4 is divided by $x+2$, the remainder is 34. {Remainder theorem}



Example 16

Self Tutor

When $2x^3 + 2x^2 + ax + b$ is divided by $x + 3$, the remainder is -11.

When the same polynomial is divided by $x - 2$, the remainder is 9.

Find a and b.

Let $P(x) = 2x^3 + 2x^2 + ax + b$

Now $P(-3) = -11$ and $P(2) = 9$ {Remainder theorem}

$$\begin{aligned}
\text{So, } \quad 2(-3)^3 + 2(-3)^2 + a(-3) + b &= -11 \\
\therefore \quad -54 + 18 - 3a + b &= -11 \\
\therefore \quad -3a + b &= 25 \quad \text{.... (1)}
\end{aligned}$$

$$\begin{aligned}
\text{and } \quad 2(2)^3 + 2(2)^2 + a(2) + b &= 9 \\
\therefore \quad 16 + 8 + 2a + b &= 9 \\
\therefore \quad 2a + b &= -15 \quad \text{.... (2)}
\end{aligned}$$

$$\begin{aligned}
\text{Solving simultaneously:} \quad 3a - b &= -25 \quad \{-1 \times (1)\} \\
2a + b &= -15 \quad \{(2)\} \\
\text{Adding,} \quad 5a \quad &= -40 \\
\therefore \quad a &= -8
\end{aligned}$$

Substituting $a = -8$ in (2) gives $2(-8) + b = -15$

$$\therefore \quad b = 1$$

EXERCISE 6C

1 For $P(x)$ a real polynomial, write two equivalent statements for each of:

- **a** If $P(2) = 7$, then ......
- **b** If $P(x) = Q(x)(x + 3) - 8$, then ......
- **c** If $P(x)$ divided by $x - 5$ has a remainder of 11 then ......

2 Without performing division, find the remainder when:

- **a** $x^3 + 2x^2 - 7x + 8$ is divided by $x - 1$
- **b** $2x^3 + x^2 - 5x + 11$ is divided by $x + 3$
- **c** $x^4 - 2x^2 + 3x - 1$ is divided by $x + 2$.

3 Use the Remainder theorem to find the remainder when $x^3 - x^2 - 3x - 5$ is divided by $x - 3$. Check that your answer is the same as when this long division was performed on page **159**.

4 Find $a \in \mathbb{R}$ such that:

- **a** when $x^3 - 2x + a$ is divided by $x - 2$, the remainder is 7
- **b** when $2x^3 + x^2 + ax - 5$ is divided by $x + 1$, the remainder is -8.

5 When $x^3 + 2x^2 + ax + b$ is divided by $x - 1$ the remainder is 4, and when divided by $x + 2$ the remainder is 16. Find a and b.

6 When $x^3 + 4x^2 + ax + b$ is divided by $x - 2$ the remainder is 20, and when divided by $x + 5$ the remainder is 6. Find a and b.

7 Consider $f(x) = 2x^3 + ax^2 - 3x + b$. When $f(x)$ is divided by $x + 1$, the remainder is 7. When $f(x)$ is divided by $x - 2$, the remainder is 28. Find the remainder when $f(x)$ is divided by $x + 3$.

8 **a** Suppose a polynomial $P(x)$ is divided by $2x - 1$ until a constant remainder R is obtained. Show that $R = P(\frac{1}{2})$.

Hint: $P(x) = Q(x)(2x - 1) + R$.

b Find the remainder when:

i $4x^2 - 10x + 1$ is divided by $2x - 1$ **ii** $2x^3 - 5x^2 + 8$ is divided by $2x - 1$

iii $4x^3 + 7x - 3$ is divided by $2x + 1$.

9 When $2x^3 + ax^2 + bx + 4$ is divided by $x + 1$ the remainder is -5, and when divided by $2x - 1$ the remainder is 10. Find a and b.

10 When $P(z)$ is divided by $z^3 - 3z + 2$ the remainder is $4z - 7$.

Find the remainder when $P(z)$ is divided by: **a** $z - 1$ **b** $z - 2$.

D THE FACTOR THEOREM

For any polynomial $P(x)$, k is a zero of $P(x)$ $\Leftrightarrow$ $x - k$ is a factor of $P(x)$.

Proof:

$$\begin{aligned} k \text{ is a zero of } P(x) &\Leftrightarrow P(k) = 0 && \{\text{definition of a zero}\} \\ &\Leftrightarrow R = 0 && \{\text{Remainder theorem}\} \\ &\Leftrightarrow P(x) = Q(x)(x - k) && \{\text{division algorithm}\} \\ &\Leftrightarrow x - k \text{ is a factor of } P(x) && \{\text{definition of a factor}\} \end{aligned}$$

The **Factor theorem** says that if 2 is a zero of $P(x)$ then $x - 2$ is a factor of $P(x)$, and vice versa.

We can use the Factor theorem to determine whether $x - k$ is a factor of a polynomial, without having to perform the long division.

Example 17

Self Tutor

Determine whether:

a $x - 2$ is a factor of $x^3 + 3x^2 - 13x + 6$ **b** $x + 3$ is a factor of $x^3 - 8x + 7$.

a Let $P(x) = x^3 + 3x^2 - 13x + 6$

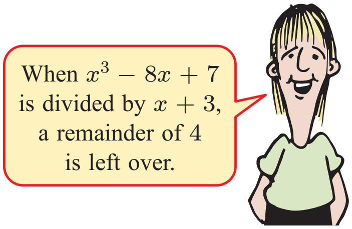
$$\begin{aligned} \therefore\ P(2) &= (2)^3 + 3(2)^2 - 13(2) + 6 \\ &= 8 + 12 - 26 + 6 \\ &= 0 \end{aligned}$$

Since $P(2) = 0$, $x - 2$ is a factor of $x^3 + 3x^2 - 13x + 6$. {Factor theorem}

b Let $P(x) = x^3 - 8x + 7$

$$\begin{aligned} \therefore\ P(-3) &= (-3)^3 - 8(-3) + 7 \\ &= -27 + 24 + 7 \\ &= 4 \end{aligned}$$

Since $P(-3) \neq 0$, $x + 3$ is *not* a factor of $x^3 - 8x + 7$. {Factor theorem}



Example 18

🔊 Self Tutor

$x-2$ is a factor of $P(x)=x^3+kx^2-3x+6$.

Find k, and write $P(x)$ as a product of linear factors.

Since $x-2$ is a factor, $P(2)=0$ {Factor theorem}

$$\begin{aligned} \therefore\ (2)^3+k(2)^2-3(2)+6&=0\\ \therefore\ 8+4k&=0\\ \therefore\ k&=-2 \end{aligned}$$

The coefficient of x^3 is $1\times 1=1$

The constant term is $-2\times -3=6$

$$\begin{aligned} \text{So,}\quad P(x)=x^3-2x^2-3x+6&=(x-2)(x^2+bx-3)\\ &=x^3+(b-2)x^2+(-2b-3)x+6 \end{aligned}$$

Equating x^2 s: $b-2=-2$

$\therefore\ b=0$

$$\begin{aligned} \text{So,}\quad P(x)&=(x-2)(x^2-3)\\ &=(x-2)(x+\sqrt{3})(x-\sqrt{3}) \end{aligned}$$

Example 19

🔊 Self Tutor

$2x-1$ is a factor of $f(x)=4x^3-4x^2+ax+b$, and the remainder when $f(x)$ is divided by $x-1$ is -1. Find the values of a and b.

$2x-1$ is a factor of $f(x)$, so $f(\frac{1}{2})=0$

$$\begin{aligned} \therefore\ 4(\tfrac{1}{2})^3-4(\tfrac{1}{2})^2+a(\tfrac{1}{2})+b&=0\\ \therefore\ \tfrac{1}{2}a+b&=\tfrac{1}{2} \quad \text{.... (1)} \end{aligned}$$

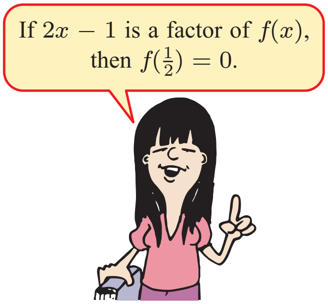
Also, $f(1)=-1$ {Remainder theorem}

$$\begin{aligned} \therefore\ 4(1)^3-4(1)^2+a(1)+b&=-1\\ \therefore\ a+b&=-1 \quad \text{.... (2)} \end{aligned}$$

Solving simultaneously:

$$\begin{array}{rrcl} & -a-2b &=& -1 \quad \{-2\times(1)\}\\ & a+b &=& -1 \quad \{(2)\}\\ \hline \text{Adding,} & -b &=& -2 \end{array}$$

$\therefore\ b=2$ and $a=-3$



EXERCISE 6D

1 Use the Factor theorem to determine whether:

a $x-1$ is a factor of $4x^3-7x^2+5x-2$

b $x-3$ is a factor of $x^4-x^3-4x^2-15$

c $x+2$ is a factor of $3x^3+5x^2-6x-8$

d $x+4$ is a factor of $2x^3+6x^2+4x+16$.

2 **a** Find c given that $x+1$ is a factor of $5x^3-3x^2+cx+10$.

b Find c given that $x-3$ is a factor of $x^4-2x^3+cx^2-4x+3$.

c Find b given that $x+2$ is a factor of $x^6+bx^5-2x^3-5x+6$.

3 $x+2$ is a factor of $P(x)=2x^3+x^2+kx-4$.
Find k, and hence write $P(x)$ as a product of linear factors.

4 $x-3$ is a factor of $P(x)=3x^3+kx^2-5x+6$.

a Find k. **b** Write $P(x)$ in the form $P(x)=(x-3)(ax^2+bx+c)$.

c Find all solutions to $P(x)=0$.

5 $2x^3+ax^2+bx+5$ has factors $x-1$ and $x+5$. Find a and b.

6 $x-2$ is a factor of $f(x)=x^3+ax^2-11x+b$. The remainder when $f(x)$ is divided by $x+1$ is 15. Find a and b.

7 $x+3$ is a factor of $P(x)=2x^3+9x^2+ax+b$. When $P(x)$ is divided by $x+4$, the remainder is -18.

a Find a and b.

b Find the remainder when $P(x)$ is divided by $x-2$.

c Write $P(x)$ in the form $P(x)=(x+3)(px^2+qx+r)$.

d Find the zeros of $P(x)$.

8 $2x-1$ is a factor of $P(x)=2x^3+ax^2-8x+b$. When $P(x)$ is divided by $x-1$, the remainder is 3.

a Find a and b.

b Find the irrational roots of $P(x)=0$, giving your answer in the form $x=p\pm\sqrt{q}$ where $p,\ q\in\mathbb{Z}$.

9 **a** Consider $P(x)=x^3-a^3$ where a is real.

i Find $P(a)$. What is the significance of this result?

ii Factorise x^3-a^3 as the product of a real linear and a quadratic factor.

b Now consider $P(x)=x^3+a^3$, where a is real.

i Find $P(-a)$. What is the significance of this result?

ii Factorise x^3+a^3 as the product of a real linear and a quadratic factor.

10 Find the real number a such that $(x-1-a)$ is a factor of $P(x)=x^3-3ax-9$.

E CUBIC EQUATIONS

In **Discovery 4** in **Chapter 3** on page **97**, we considered the sum and products of roots of a quadratic. In particular, we saw that

$$(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta.$$

If we perform a similar expansion for a cubic, we find that

$$(x-\alpha)(x-\beta)(x-\gamma)=x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma.$$

In both cases, the *product of the roots* has the same size as the constant term in the expanded polynomial.

If the leading coefficient of the polynomial $\neq 1$, then we need to multiply by this as well:

$$a(x-\alpha)(x-\beta)(x-\gamma) = ax^3 - a(\alpha+\beta+\gamma)x^2 + a(\alpha\beta+\beta\gamma+\gamma\alpha)x - a\alpha\beta\gamma.$$

If you think a cubic equation has integer roots, try to find them by factorising the constant term.

Example 20 Self Tutor

Solve for x: $x^3 - 31x - 30 = 0$.

Let $P(x) = x^3 - 31x - 30$.

The constant term is -30, so the product of the roots is 30.

Since $30 = 5 \times 3 \times 2 \times 1$, likely integer roots are $\pm1, \pm2, \pm3, \pm5$. They could also be ±6 since $2 \times 3 = 6$, and so on.

Now $P(1) = -60$, so 1 is not a root.

But $P(-1) = 0$, so -1 is a root, and $(x+1)$ is a factor of $P(x)$.

The coefficient of x^3 is $1 \times 1 = 1$

The constant term is $1 \times -30 = -30$

$$\begin{aligned} \text{So,}\quad P(x) = x^3 + 0x^2 - 31x - 30 &= (x+1)(x^2 + bx - 30) \\ &= x^3 + (b+1)x^2 + (b-30)x - 30 \end{aligned}$$

Equating x^2 s: $b + 1 = 0$

$\therefore\ b = -1$

$$\begin{aligned} \text{Hence}\quad P(x) &= (x+1)(x^2 - x - 30) \\ &= (x+1)(x+5)(x-6) \end{aligned}$$

$\therefore$ the solutions are -1, -5, and 6.

Note that this method only works for those cubics with all integer roots.

EXERCISE 6E

1 Solve for x:

- **a** $x^3 - 6x^2 + 11x - 6 = 0$
- **b** $x^3 - 3x^2 + 4 = 0$
- **c** $x^3 + 2x^2 - x - 2 = 0$
- **d** $x^3 - 6x^2 + 5x + 12 = 0$
- **e** $x^3 + 5x^2 - 16x - 80 = 0$
- **f** $x^3 + 13x^2 + 55x + 75 = 0$

2 Solve for x:

- **a** $2x^3 - 6x^2 - 8x + 24 = 0$
- **b** $2x^3 - 2x^2 - 48x - 72 = 0$
- **c** $3x^3 - 24x^2 - 15x + 252 = 0$

Discussion

Consider the general cubic $p(x) = ax^3 + bx^2 + cx + d$, $a, b, c, d \in \mathbb{R}$.

What happens to $p(x)$ if x gets:

- very large and positive
- very large and negative?

What does this tell you about the number of solutions that $p(x) = 0$ may have?

Review set 6A

1 Given $p(x) = 5x^2 - x + 4$ and $q(x) = 3x^2 + 7x - 1$, find:

  a $p(x) + q(x)$ **b** $2p(x) - q(x)$ **c** $p(x)q(x)$

2 Find the quotient and remainder of:

  a $\dfrac{2x^2 + 11x + 18}{x + 3}$ **b** $\dfrac{x^3 - 6x^2 + 10x - 9}{x - 2}$

3 Find the zeros of:

  a $3x^2 + 2x - 8$ **b** $x^2 + 8x + 11$

4 **a** Given that $x^3 + x^2 - 3x + 9 = (x + 3)(ax^2 + bx + c)$, find the values of a, b, and c.

  b Show that $x^3 + x^2 - 3x + 9$ has only one real zero.

5 Use the Remainder theorem to find the remainder when:

  a $x^3 - 4x^2 + 5x - 1$ is divided by $x - 2$

  b $2x^3 + 6x^2 - 7x + 12$ is divided by $x + 5$.

6 Use the Factor theorem to determine whether:

  a $x + 1$ is a factor of $2x^4 - 9x^2 - 6x - 1$

  b $x - 3$ is a factor of $x^4 - 2x^3 - 4x^2 + 5x - 6$.

7 $2x^2 + kx - 5$ has remainder 3 when divided by $x + 4$. Find k.

8 $ax^3 + 5x^2 - x + b$ has remainder 7 when divided by $x - 1$, and remainder -11 when divided by $x + 2$. Find a and b.

9 Find c given that $x - 2$ is a factor of $x^5 - 2x^4 + cx^3 - 7x^2 + 5x - 6$.

10 $x - 4$ is a factor of $f(x) = x^3 + 2x^2 + ax + b$, and when $f(x)$ is divided by $x + 2$ the remainder is 18.

  a Find a and b. **b** Find all zeros of $f(x)$.

11 Solve for x: $x^3 - x^2 - 17x - 15 = 0$

Review set 6B

1 Expand and simplify:

  a $(3x^3 + 2x - 5)(4x - 3)$ **b** $(2x^2 - x + 3)^2$

2 Carry out the following divisions:

a $\dfrac{x^3}{x+2}$ **b** $\dfrac{x^3}{(x+2)(x+3)}$

3 Find *all* cubic polynomials with zeros $\frac{1}{4}$, $1 \pm \sqrt{5}$.

4 If $f(x) = x^3 - 3x^2 - 9x + b$ has $(x-k)^2$ as a factor, show that there are two possible values of k. For each of these two values of k, find the corresponding value for b, and hence solve $f(x) = 0$.

5 Find the remainder when:

a $x^3 - 5x^2 + 9$ is divided by $x - 2$ **b** $4x^3 + 7x - 11$ is divided by $2x - 1$.

6 When $f(x) = 2x^3 - x^2 + ax - 4$ is divided by $x - 3$, the remainder is 56.

a Find a. **b** Find the remainder when $f(x)$ is divided by $x + 1$.

7 **a** Use the Factor theorem to show that $x - 2$ is a factor of $x^3 - 13x + 18$.

b Write $x^3 - 13x + 18$ in the form $(x-2)(ax^2 + bx + c)$, where $a, b, c \in \mathbb{Z}$.

c Find the real roots of $x^3 + 18 = 13x$.

8 $x - 2$ and $x + 3$ are factors of $ax^3 - 3x^2 - 11x + b$. Find a and b.

9 $x + 1$ is a factor of $f(x) = x^3 + 5x^2 + kx + 4$. Find k, and the zeros of $f(x)$.

10 $2x - 1$ is a factor of $f(x) = 2x^3 - 9x^2 + ax + b$, and when $f(x)$ is divided by $x - 1$ the remainder is -15.

a Find a and b. **b** Write $f(x)$ as a product of linear factors.

11 Solve for x: $2x^3 - 2x^2 - 28x + 48 = 0$

7

Straight line graphs

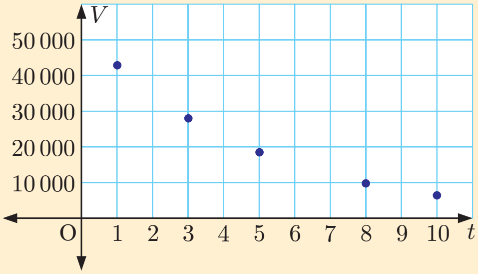
Contents:

Opening problem

This table shows the value $\$V$ of Doug's father's car t years after purchase.

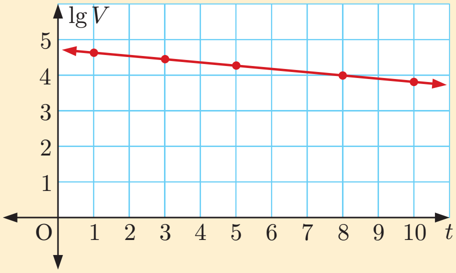
t (years)	1	3	5	8	10
V (dollars)	42 900	28 000	18 500	9800	6400

Doug is trying to work out the equation connecting V and t. When he plots the values on a graph, the result is a curve:



Doug's father suggests that he plots $\lg V$ against t. When Doug does this, the result is a straight line:

t	1	3	5	8	10
$\lg V$	4.63	4.45	4.27	3.99	3.81



Things to think about:

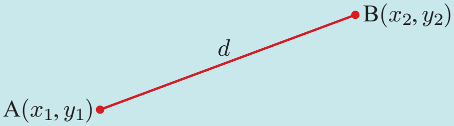
- **a** Is it easier to find the equation of a curve or a straight line?
- **b** How can Doug use the equation of the straight line to determine the relationship between V and t?

BACKGROUND KNOWLEDGE

You should be familiar with the following facts involving points and lines on the coordinate plane:

Distance between two points

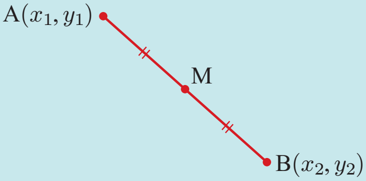
The **distance** between $A(x_1, y_1)$ and $B(x_2, y_2)$ is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.



Midpoint

If A is (x_1, y_1) and B is (x_2, y_2), then the midpoint of AB is

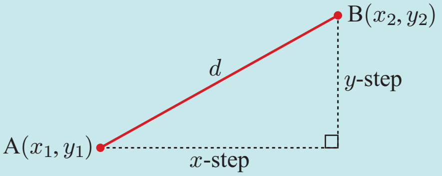
$$M\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$



Gradient

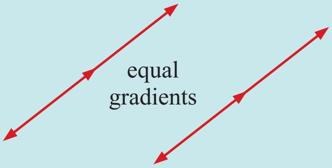
The **gradient** of a line passing through A(x_1, y_1) and B(x_2, y_2) is

$$\frac{y\text{-step}}{x\text{-step}} = \frac{y_2 - y_1}{x_2 - x_1}.$$

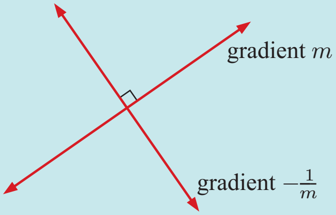


Gradients of parallel and perpendicular lines

- If two lines are **parallel**, then their gradients are **equal**.



- If two lines are **perpendicular**, then their gradients are **negative reciprocals**.
  If the gradient of one line is m, then the gradient of the other line is $-\frac{1}{m}$.

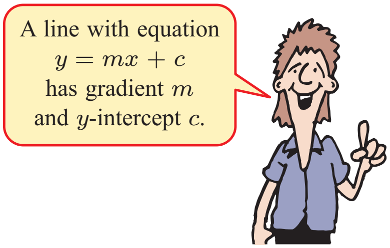


A EQUATIONS OF STRAIGHT LINES

The **equation of a line** is an equation which connects the x and y values for every point on the line.

The equation of a straight line can be written in:

- **gradient-intercept form** $y = mx + c$, or
- **general form** $Ax + By = D$.



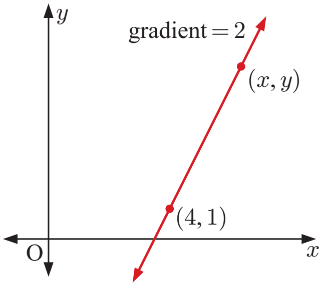
FINDING THE EQUATION OF A LINE

In order to find the equation of a line, we need to know some information.

Suppose we know the gradient of the line is 2, and that the line passes through $(4, 1)$.

We suppose (x, y) is any point on the line.

The gradient between $(4, 1)$ and (x, y) is $\frac{y-1}{x-4}$, and this gradient must equal 2.



So, $\frac{y-1}{x-4} = 2$

$\therefore$ $y - 1 = 2(x-4)$ {multiplying both sides by $(x-4)$}

$\therefore$ $y - 1 = 2x - 8$ {expanding the brackets}

$\therefore$ $y = 2x - 7$ {adding 1 to both sides}

This is the equation of the line in gradient-intercept form.

We can find the equation of a line if we know:

- its **gradient** and the **coordinates of any point** on the line, or
- the **coordinates of two distinct points** on the line.

If a straight line has gradient m and passes through the point (x_1, y_1) then its equation is $\boldsymbol{\frac{y-y_1}{x-x_1} = m}$ or $\boldsymbol{y - y_1 = m(x - x_1)}$.

We can rearrange this equation into either gradient-intercept or general form.

Example 1

Self Tutor

Find, in *gradient-intercept form*, the equation of the line through $(-1, 3)$ with a gradient of 5.

The equation of the line is $y - 3 = 5(x - (-1))$

$\therefore$ $y - 3 = 5x + 5$

$\therefore$ $y = 5x + 8$

Example 2

Self Tutor

Find, in *general form*, the equation of the line with gradient $\frac{3}{4}$ which passes through $(5, -2)$.

The equation of the line is $y - (-2) = \frac{3}{4}(x-5)$

$\therefore$ $4(y+2) = 3(x-5)$

$\therefore$ $4y + 8 = 3x - 15$

$\therefore$ $3x - 4y = 23$

EXERCISE 7A.1

1 Find the gradient and y-intercept of the line with equation:

a $y = 3x + 5$ **b** $y = 4x - 2$ **c** $y = \frac{1}{5}x + \frac{3}{5}$

d $y = -7x - 3$ **e** $y = \dfrac{x+2}{6}$ **f** $y = \dfrac{8-5x}{3}$

2 Find the equation of the line with:

a gradient 1 and y-intercept -2 **b** gradient -1 and y-intercept 4

c gradient 2 and y-intercept 0 **d** gradient $-\frac{1}{2}$ and y-intercept 3.

3 Find, in *gradient-intercept form*, the equation of the line through:

a $(2, -5)$ with gradient 4 **b** $(-1, -2)$ with gradient -3

c $(7, -3)$ with gradient -5 **d** $(1, 4)$ with gradient $\frac{1}{2}$

e $(-1, 3)$ with gradient $-\frac{1}{3}$ **f** $(2, 6)$ with gradient 0.

4 Find, in *general form*, the equation of the line through:

a $(2, 5)$ having gradient $\frac{2}{3}$ **b** $(-1, 4)$ having gradient $\frac{3}{5}$

c $(5, 0)$ having gradient $-\frac{1}{3}$ **d** $(6, -2)$ having gradient $-\frac{2}{7}$

e $(-3, -1)$ having gradient 4 **f** $(5, -3)$ having gradient -2

g $(4, -5)$ having gradient $-3\frac{1}{2}$ **h** $(-7, -2)$ having gradient 6.

Example 3

Self Tutor

Find the equation of the line which passes through the points A$(-1, 5)$ and B$(2, 3)$.

The gradient of the line is $\dfrac{3-5}{2-(-1)} = -\dfrac{2}{3}$.

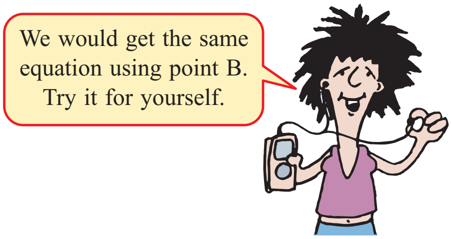
Using point A, the equation is

$$y - 5 = -\tfrac{2}{3}(x - (-1))$$

$$\therefore\ 3(y-5) = -2(x+1)$$

$$\therefore\ 3y - 15 = -2x - 2$$

$$\therefore\ 2x + 3y = 13$$



5 Find, in *gradient-intercept form*, the equation of the line which passes through the points:

a A$(2, 3)$ and B$(4, 8)$ **b** A$(0, 3)$ and B$(-1, 5)$

c A$(-1, -2)$ and B$(4, -2)$ **d** C$(-3, 1)$ and D$(2, 0)$

e P$(5, -1)$ and Q$(-1, -2)$ **f** R$(-1, -3)$ and S$(-4, -1)$.

6 Find, in *general form*, the equation of the line which passes through:

a $(0, 1)$ and $(3, 2)$ **b** $(1, 4)$ and $(0, -1)$ **c** $(2, -1)$ and $(-1, -4)$

d $(0, -2)$ and $(5, 2)$ **e** $(3, 2)$ and $(-1, 0)$ **f** $(-1, -1)$ and $(2, -3)$.

7 Consider the points A(2, 5) and B(−4, 2). Find:

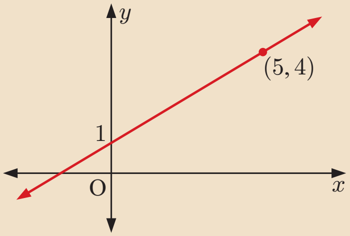
- **a** the distance between A and B
- **b** the midpoint of AB
- **c** the gradient of the line which passes through A and B
- **d** the equation of the line which passes through A and B.

Example 4

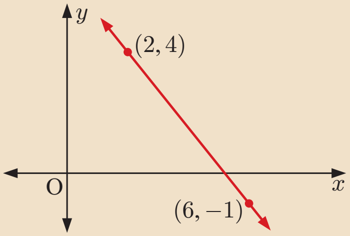
Self Tutor

Find the equation of the line with graph:

a



b



a Two points on the line are (0, 1) and (5, 4).

$\therefore$ the gradient $m = \dfrac{4-1}{5-0} = \frac{3}{5}$

and the y-intercept $c = 1$.

The equation is $y = \frac{3}{5}x + 1$ {gradient-intercept form}

b Two points on the line are (2, 4) and (6, −1).

$\therefore$ the gradient $m = \dfrac{-1-4}{6-2} = -\frac{5}{4}$

Since we do not know the y-intercept we use the general form.

The equation is $y - 4 = -\frac{5}{4}(x-2)$

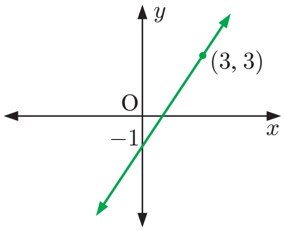
$\therefore\ 4(y-4) = -5(x-2)$

$\therefore\ 4y - 16 = -5x + 10$

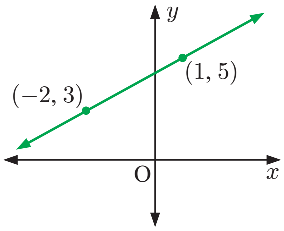
$\therefore\ 5x + 4y = 26$

8 Find the equations of the illustrated lines:

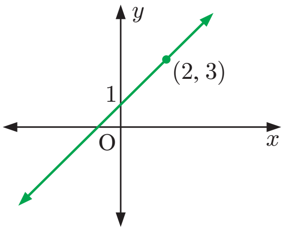
a



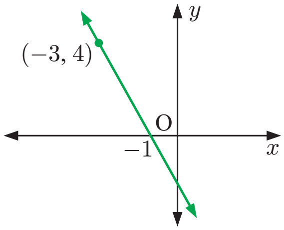
b



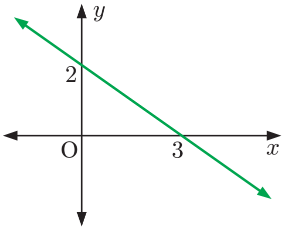
c



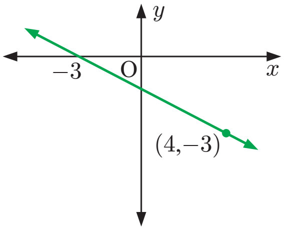
d



e



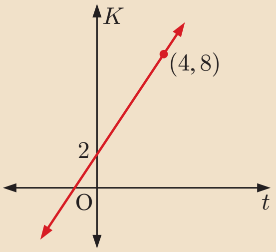
f



Example 5

Self Tutor

Find the equation connecting the variables.



$(0, 2)$ and $(4, 8)$ lie on the straight line.

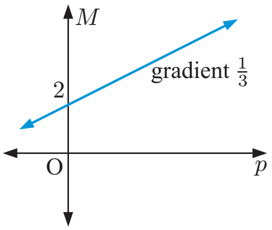
$\therefore$ the gradient $m = \frac{8-2}{4-0} = \frac{6}{4} = \frac{3}{2}$, and the y-intercept $c = 2$.

In this case K is on the vertical axis and t is on the horizontal axis.

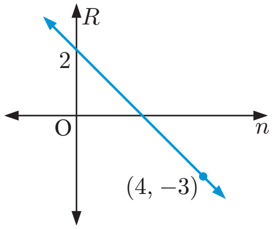
$\therefore$ the equation is $K = \frac{3}{2}t + 2$.

9 Find the equation connecting the variables:

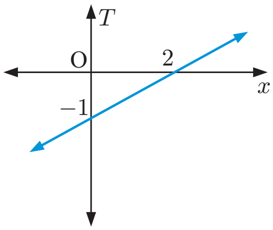
a



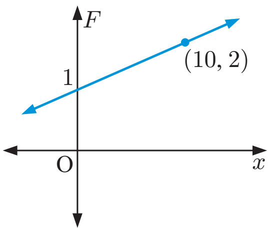
b



c



d



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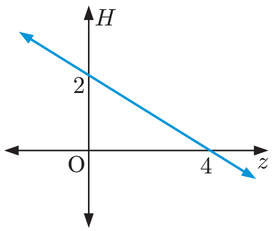

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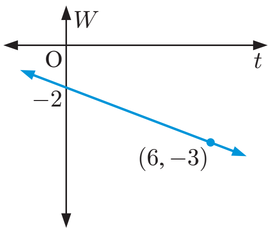


Example 6

Self Tutor

Consider the points A$(-2, 5)$ and B$(1, 3)$. A line perpendicular to AB, passes through B.

a Find the equation of the line.

b Find the coordinates of the point where the line cuts the x-axis.

a The gradient of AB $= \frac{3-5}{1-(-2)} = -\frac{2}{3}$

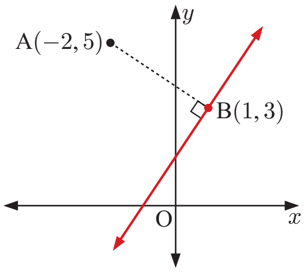
$\therefore$ the perpendicular line has gradient $\frac{3}{2}$, and passes through B$(1, 3)$.

$\therefore$ its equation is $y - 3 = \frac{3}{2}(x-1)$

$\therefore$ $2(y-3) = 3(x-1)$

$\therefore$ $2y - 6 = 3x - 3$

$\therefore$ $3x - 2y = -3$



b The line cuts the x-axis when $y = 0$

$$\therefore\ 3x - 2(0) = -3$$
$$\therefore\ x = -1$$

$\therefore$ the line cuts the x-axis at $(-1, 0)$.

10 Consider the points $P(-3, -2)$ and $Q(1, 6)$. A line perpendicular to PQ, passes through Q.

- **a** Find the equation of the line.
- **b** Find the coordinates of the point where the line cuts the x-axis.

11 Suppose A has coordinates $(-7, 4)$ and B has coordinates $(3, -2)$. A line parallel to AB, passes through $C(5, -1)$.

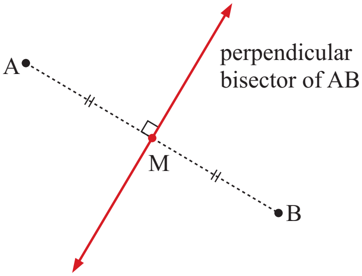
- **a** Find the equation of the line.
- **b** Find the coordinates of the point where the line cuts the y-axis.

12 Suppose P has coordinates $(3, 8)$ and Q has coordinates $(-5, 2)$. The line perpendicular to PQ and passing through P, cuts the x-axis at R and the y-axis at S. Find the area of triangle ORS, where O is the origin.

PERPENDICULAR BISECTORS

We have already seen that the **midpoint** M of the line segment AB is the point on the line segment that is halfway between A and B.

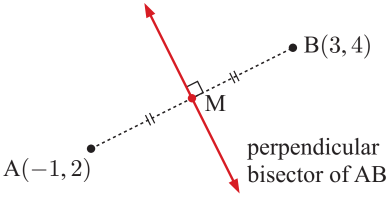
The **perpendicular bisector** of AB is the line which is perpendicular to AB, and which passes through its midpoint M.



Example 7

Self Tutor

Find the equation of the perpendicular bisector of AB given $A(-1, 2)$ and $B(3, 4)$.



The midpoint M of AB is $\left(\dfrac{-1+3}{2}, \dfrac{2+4}{2}\right)$ or $M(1, 3)$.

The gradient of AB is $\dfrac{4-2}{3-(-1)} = \frac{2}{4} = \frac{1}{2}$

$\therefore$ the gradient of the perpendicular bisector is $-\frac{2}{1}$ {the negative reciprocal of $\frac{1}{2}$}

The equation of the perpendicular bisector is $y - 3 = -2(x - 1)$ {using $M(1, 3)$}

$$\therefore\ y - 3 = -2x + 2$$
$$\therefore\ y = -2x + 5$$

EXERCISE 7A.2

1 Consider the points $P(-3,\ 7)$ and $Q(1,\ -5)$. Find:

- **a** the distance between P and Q
- **b** the midpoint of PQ
- **c** the gradient of PQ
- **d** the equation of the perpendicular bisector of PQ.

2 Find the equation of the perpendicular bisector of AB given:

- **a** $A(3,\ -3)$ and $B(1,\ -1)$
- **b** $A(1,\ 3)$ and $B(-3,\ 5)$
- **c** $A(3,\ 1)$ and $B(-3,\ 6)$
- **d** $A(4,\ -2)$ and $B(4,\ 4)$.

3 Consider the points $P(-1,\ 5)$ and $Q(3,\ 7)$. The perpendicular bisector of PQ cuts the x-axis at R. Find the area of triangle PQR.

B INTERSECTION OF STRAIGHT LINES

To find where straight lines meet, we need to solve the equations of the lines simultaneously.

Example 8

Self Tutor

Find where the line:

- **a** $y = 2x - 5$ meets the line $4x + 3y = 15$
- **b** $x + 3y = 5$ meets the line $2x - 5y = -12$.

a Substituting $y = 2x - 5$ into $4x + 3y = 15$ gives

$$\begin{aligned} 4x + 3(2x - 5) &= 15 \\ \therefore\ 4x + 6x - 15 &= 15 \\ \therefore\ 10x &= 30 \\ \therefore\ x &= 3 \quad \text{and} \quad y = 2(3) - 5 = 1 \end{aligned}$$

The lines meet at $(3,\ 1)$.

b $x + 3y = 5$, so $x = 5 - 3y$.

Substituting $x = 5 - 3y$ into $2x - 5y = -12$ gives

$$\begin{aligned} 2(5 - 3y) - 5y &= -12 \\ \therefore\ 10 - 6y - 5y &= -12 \\ \therefore\ -11y &= -22 \\ \therefore\ y &= 2 \quad \text{and} \quad x = 5 - 3(2) = -1 \end{aligned}$$

The lines meet at $(-1,\ 2)$.

EXERCISE 7B

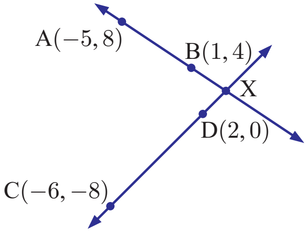
1 Find the intersection point of each pair of lines:

- **a** $y = 4x - 1$ and $2x + y = 5$
- **b** $y = 9 - 2x$ and $4x + 3y = 15$
- **c** $x + 4y = 7$ and $5x - 2y = -31$
- **d** $3x + y = -5$ and $4x - 7y = 10$.

2 Line l_1 has equation $y = 2x + 7$. Line l_2 passes through $(-7, 6)$ and $(3, 0)$.

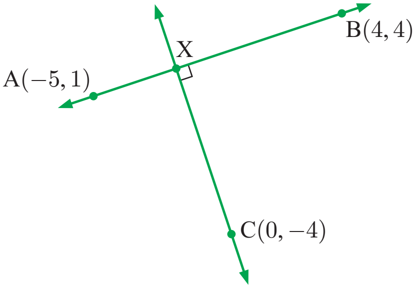
- **a** Find the equation of l_2.
- **b** Find the intersection point of l_1 and l_2.

3 Find the coordinates of X.



4 In the diagram alongside, a line has been drawn through C, perpendicular to the line AB. The point of intersection of the lines is X. Find:

- **a** the equation of AB
- **b** the equation of CX
- **c** the coordinates of X.

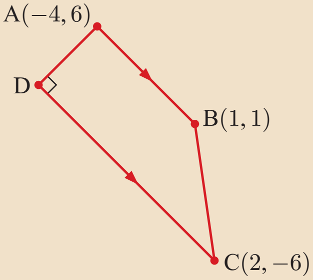


Example 9

Self Tutor

ABCD is a trapezium in which AB is parallel to DC, and $\widehat{ADC} = 90°$. Find:

- **a** the coordinates of D
- **b** the area of the trapezium.



a Point D is the intersection of AD and DC.

The gradient of AB is $\dfrac{1-6}{1-(-4)} = \dfrac{-5}{5} = -1$

$\therefore$ DC also has gradient -1, and has equation

$$y - (-6) = -1(x - 2)$$
$$\therefore\ y + 6 = -x + 2$$
$$\therefore\ y = -x - 4 \quad \text{.... (1)}$$

AD is perpendicular to DC, so its gradient is 1, and its equation is

$$y - 6 = 1(x - (-4))$$
$$\therefore\ y - 6 = x + 4$$
$$\therefore\ y = x + 10 \quad \text{.... (2)}$$

Substituting (1) into (2) gives

$$-x - 4 = x + 10$$
$$\therefore\ -2x = 14$$
$$\therefore\ x = -7 \quad \text{and} \quad y = -(-7) - 4 = 3$$

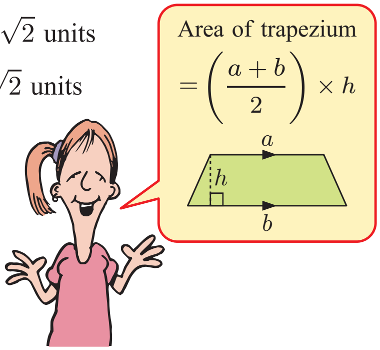
$\therefore$ D is $(-7, 3)$.

b The length of AB $= \sqrt{(1 - -4)^2 + (1-6)^2} = \sqrt{50} = 5\sqrt{2}$ units

The length of DC $= \sqrt{(2 - -7)^2 + (-6-3)^2} = \sqrt{162} = 9\sqrt{2}$ units

The length of AD $= \sqrt{(-7 - -4)^2 + (3-6)^2} = \sqrt{18} = 3\sqrt{2}$ units

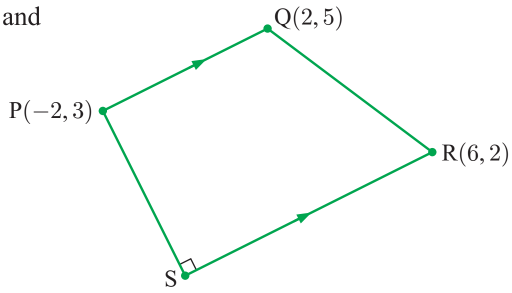
$$\therefore \text{ the area of the trapezium} = \left(\frac{5\sqrt{2}+9\sqrt{2}}{2}\right) \times 3\sqrt{2}$$
$$= 7\sqrt{2} \times 3\sqrt{2}$$
$$= 42 \text{ units}^2$$


Area of trapezium
$$= \left(\frac{a+b}{2}\right) \times h$$

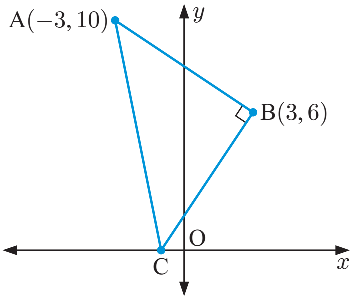
5 PQRS is a trapezium in which PQ is parallel to SR, and $P\widehat{S}R = 90°$. Find:

a the coordinates of S

b the area of the trapezium.



6



ABC is a triangle in which $A\widehat{B}C = 90°$, and C lies on the x-axis. Find:

a the coordinates of C

b the area of the triangle.

7 A trapezium ABCD has vertices A(3, 0), B(−2, −5), C(−4, 1), and D. The side AD is parallel to BC, and the side CD is perpendicular to BC. Find the area of the trapezium.

8 ABC is a triangle in which $A\widehat{B}C = 90°$, AC is parallel to the x-axis, M is the midpoint of AB, and N is the midpoint of BC.

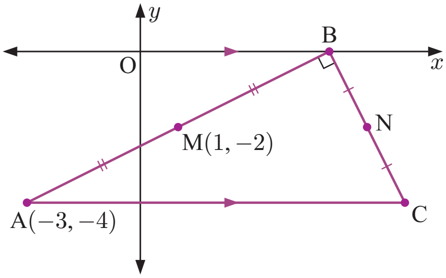
a Find the coordinates of:

i B **ii** C **iii** N

b Show that MN is parallel to AC.

c Find the area of:

i trapezium AMNC **ii** triangle ABC.



C INTERSECTION OF A STRAIGHT LINE AND A CURVE

To find where a straight line intersects a curve, we first rearrange the equation of the line so that x or y is the subject. We then substitute this expression for x or y into the equation of the curve.

While a straight line meets another straight line at most once, a straight line may meet a curve more than once.

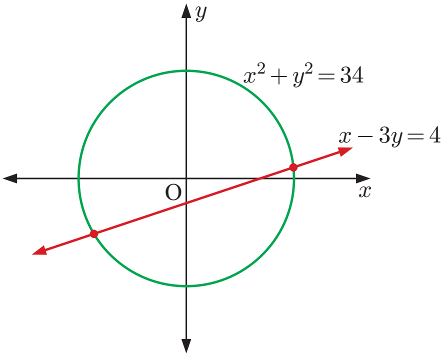
Example 10

Self Tutor

Find the points where the line $x - 3y = 4$ intersects the curve $x^2 + y^2 = 34$.

Substituting $x = 3y + 4$ into $x^2 + y^2 = 34$ gives

$$(3y+4)^2 + y^2 = 34$$
$$\therefore\ 9y^2 + 24y + 16 + y^2 = 34$$
$$\therefore\ 10y^2 + 24y - 18 = 0$$
$$\therefore\ 2(5y^2 + 12y - 9) = 0$$
$$\therefore\ 2(5y-3)(y+3) = 0$$
$$\therefore\ y = \tfrac{3}{5} \text{ or } -3$$



When $y = \frac{3}{5}$, $x = 3(\frac{3}{5}) + 4 = \frac{29}{5}$

When $y = -3$, $x = 3(-3) + 4 = -5$

$\therefore$ the line intersects the curve at $(\frac{29}{5}, \frac{3}{5})$ and $(-5, -3)$.

Example 11

Self Tutor

Find the points where the line $2x + 3y = 5$ intersects the curve $\dfrac{1}{x} - \dfrac{3}{y} = 2$.

If $2x + 3y = 5$, then $y = \dfrac{5-2x}{3}$.

Substituting into $\dfrac{1}{x} - \dfrac{3}{y} = 2$ gives $\dfrac{1}{x} - \dfrac{3}{\frac{5-2x}{3}} = 2$

$$\therefore\ \frac{1}{x} - \frac{9}{5-2x} = 2$$
$$\therefore\ (5-2x) - 9x = 2x(5-2x) \qquad \{\times \text{ both sides by } x(5-2x)\}$$
$$\therefore\ 5 - 11x = 10x - 4x^2$$
$$\therefore\ 4x^2 - 21x + 5 = 0$$
$$\therefore\ (4x-1)(x-5) = 0$$
$$\therefore\ x = \tfrac{1}{4} \text{ or } 5$$

When $x = \frac{1}{4}$, $y = \dfrac{5 - 2(\frac{1}{4})}{3} = \frac{3}{2}$, and when $x = 5$, $y = \dfrac{5-2(5)}{3} = -\frac{5}{3}$.

$\therefore$ the line intersects the curve at $(\frac{1}{4}, \frac{3}{2})$ and $(5, -\frac{5}{3})$.

EXERCISE 7C

1 Find the points where the line $x - 2y = 3$ intersects the curve $x^2 + y^2 = 5$.

2 The line $x + y = 7$ meets the curve $x^2 + y^2 = 29$ at A and B. Find the distance between A and B.

3 The line $2x + y = 5$ meets the curve $x^2 + y^2 = 10$ at P and Q. Find the equation of the perpendicular bisector of PQ.

4 Find the points where the line $x - 2y = 4$ intersects the curve $3x^2 + y^2 + xy + 3y = 8$.

5 The line $y = 2x + 1$ meets the curve $x^2 + y^2 + xy + 16x = 29$ at P and Q. Find the distance between P and Q.

6 The line $3x + y = 1$ intersects the curve $2x^2 + y^2 + 5xy - 7x = -31$ at A and B. Find the equation of the perpendicular bisector of AB.

7 Find the points where the line $x - 2y = 6$ intersects the curve $\frac{4}{x} - \frac{1}{y} = 2$.

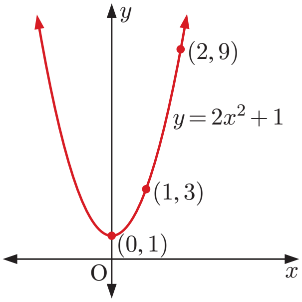
8 The line $3x + 2y = 12$ intersects the curve $\frac{4}{x} + \frac{3}{y} = 3$ at P and Q. Find the midpoint of PQ.

D TRANSFORMING RELATIONSHIPS TO STRAIGHT LINE FORM

Even if x and y are not linearly related, it is sometimes still possible to use a straight line graph to display the relationship. We do this by changing the variables on the axes.

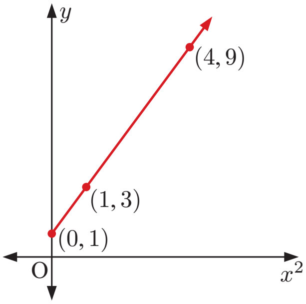
For example, consider the relationship $y = 2x^2 + 1$.

x and y are not linearly related, but x^2 and y *are* linearly related since $y = 2(x^2) + 1$.



We can use a table of values to plot y against x^2:

x	0	1	2
x^2	0	1	4
y	1	3	9



The graph of y against x^2 is a straight line with gradient 2 and y-intercept 1.

Click on the icon to view a demonstration of how the two graphs are related.

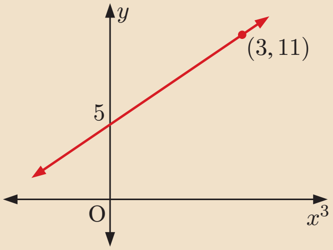
DEMO

Observe that for the graph of y against x^2, the line terminates at $(0, 1)$, since $x^2 \geqslant 0$ for all x. We need to be careful with the domain and range when we transform relationships.

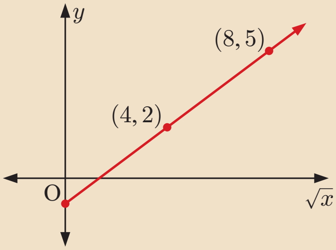
Example 12 Self Tutor

Find y in terms of x:

a



b



a The graph of y against x^3 is linear.

The gradient is $\dfrac{11-5}{3-0} = 2$, and

the y-intercept is 5.

$\therefore$ the equation is $y = 2x^3 + 5$.

b The graph of y against $\sqrt{x}$ is linear.

The gradient is $\dfrac{5-2}{8-4} = \dfrac{3}{4}$.

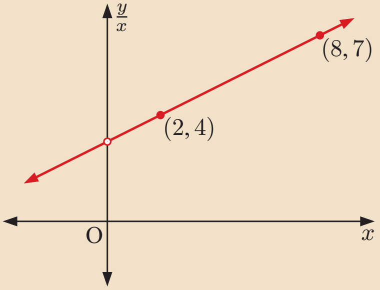
$\therefore$ the equation is

$$y - 2 = \tfrac{3}{4}(\sqrt{x} - 4)$$
$$\therefore \; y - 2 = \tfrac{3}{4}\sqrt{x} - 3$$
$$\therefore \; y = \tfrac{3}{4}\sqrt{x} - 1, \quad x \geqslant 0$$

Example 13 Self Tutor

a Find y in terms of x.

b Find y when $x = 4$.



a The graph of $\dfrac{y}{x}$ against x is linear.

The gradient is $\dfrac{7-4}{8-2} = \dfrac{1}{2}$.

$\therefore$ the equation is $\dfrac{y}{x} - 4 = \dfrac{1}{2}(x-2)$

$$\therefore \; \frac{y}{x} - 4 = \frac{1}{2}x - 1$$
$$\therefore \; \frac{y}{x} = \frac{1}{2}x + 3$$
$$\therefore \; y = \tfrac{1}{2}x^2 + 3x$$

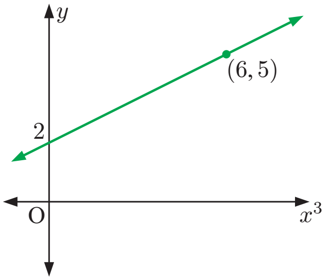
b When $x = 4$, $y = \tfrac{1}{2}(4)^2 + 3(4)$

$= 20$

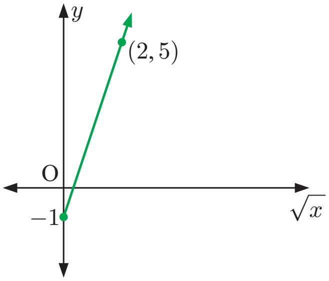
EXERCISE 7D

1 Find y in terms of x:

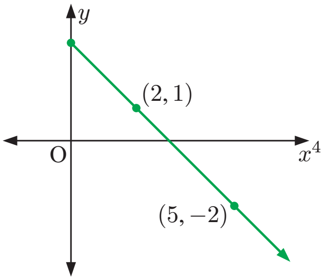
a



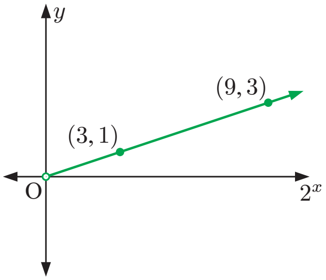
b



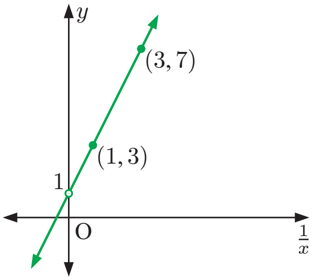
c



d



e



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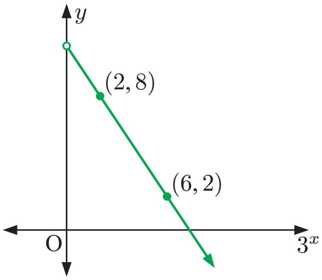
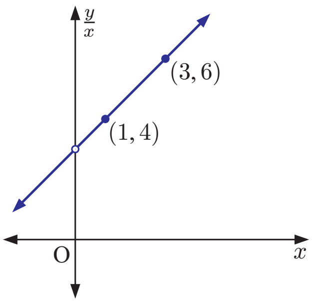


2 For each of the following relations:

i find y in terms of x **ii** find the value of y when $x = 3$.

a



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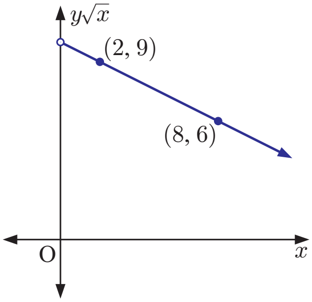

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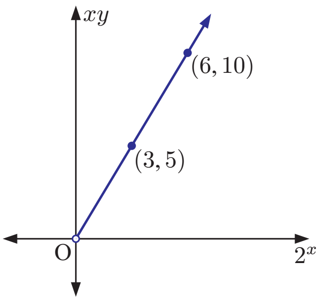

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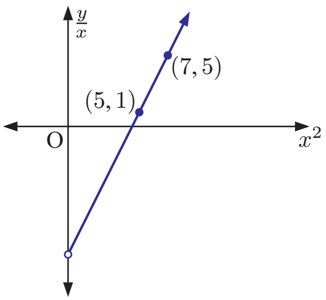

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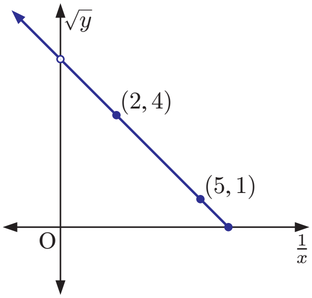
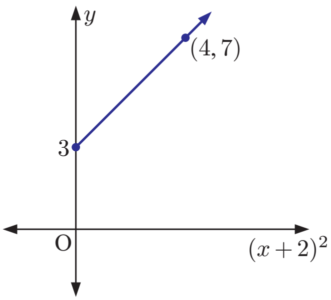


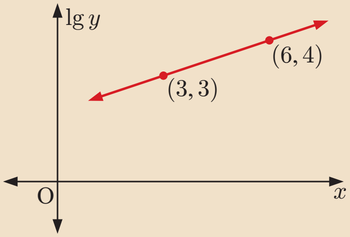
f



Example 14

Self Tutor

Write y in terms of x, giving your answer in the form $y = a \times 10^{bx}$, where $a, b \in \mathbb{Q}$.



The graph of $\lg y$ against x is linear.

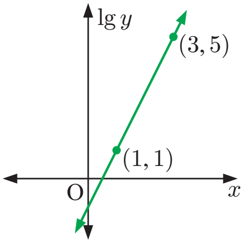
The gradient is $\frac{4-3}{6-3} = \frac{1}{3}$.

$$\therefore \text{ the equation is } \quad \lg y - 3 = \tfrac{1}{3}(x-3)$$
$$\lg y - 3 = \tfrac{1}{3}x - 1$$
$$\therefore \quad \lg y = \tfrac{1}{3}x + 2$$
$$\therefore \quad y = 10^{\frac{1}{3}x+2} \qquad \{\text{if } \lg p = q \text{ then } p = 10^q\}$$
$$\therefore \quad y = 10^{\frac{1}{3}x} \times 10^2$$
$$\therefore \quad y = 100 \times 10^{\frac{1}{3}x}$$

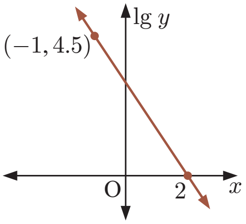
In **Chapter 5**, we saw that a linear relationship between $\lg y$ and x indicates an exponential relationship between y and x.

3 **a** Find $\lg y$ in terms of x.

b Write y in terms of x, giving your answer in the form $y = a \times 10^{bx}$, where $a, b \in \mathbb{Q}$.

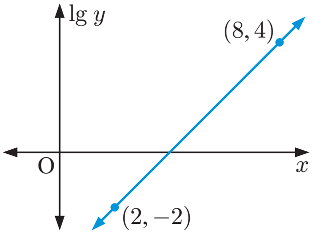


4 Write y in terms of x, giving your answer in the form $y = a \times 10^{bx}$, where $a, b \in \mathbb{Q}$.

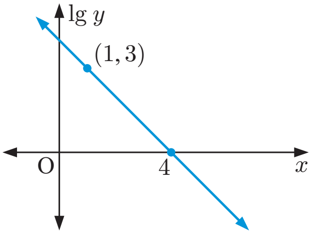


5 Write y in terms of x, giving your answer in the form $y = a \times b^x$, where $a, b \in \mathbb{Q}$.

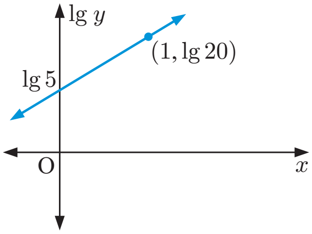
a



b

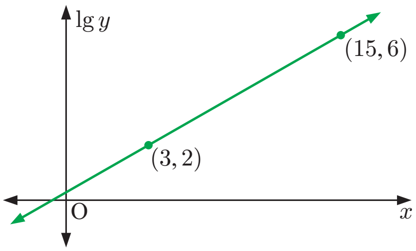


c



6 **a** Write y in terms of x, giving your answer in the form $y = a \times 10^{bx}$, where $a, b \in \mathbb{Q}$.

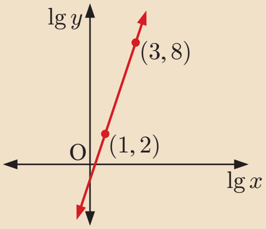
b Find y when $x = 6$.



Example 15

Self Tutor

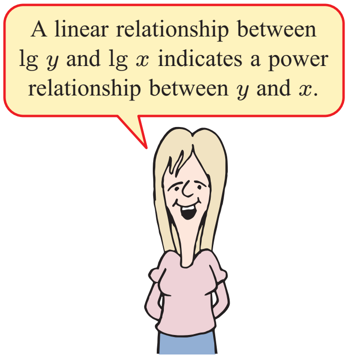
Write y in terms of x, giving your answer in the form $y = a \times x^b$, where $a, b \in \mathbb{Q}$.



The graph of $\lg y$ against $\lg x$ is linear.

The gradient is $\dfrac{8-2}{3-1} = 3$.

$$
\begin{aligned}
\therefore \text{ the equation is } \quad \lg y - 2 &= 3(\lg x - 1)\\
\therefore \quad \lg y - 2 &= 3\lg x - 3\\
\therefore \quad \lg y &= 3\lg x - 1\\
\therefore \quad \lg y &= \lg x^3 - \lg 10\\
\therefore \quad \lg y &= \lg\left(\frac{x^3}{10}\right)\\
\therefore \quad y &= \tfrac{1}{10} \times x^3
\end{aligned}
$$

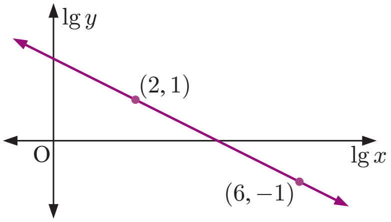


A linear relationship between lg y and lg x indicates a power relationship between y and x.

7 Consider the graph alongside.

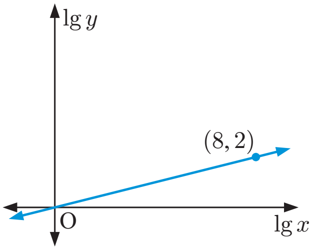
a Write an equation for the line in the form $\lg y = m \lg x + c$.

b Hence write y in terms of x.

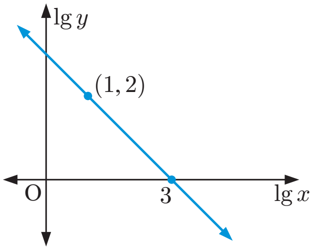


8 Write y in terms of x:

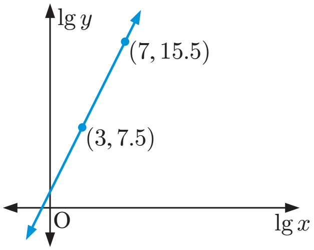
a



b

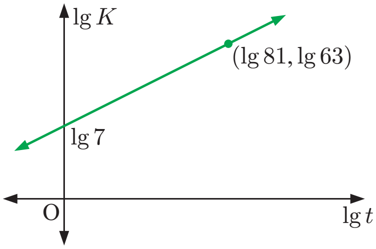


c



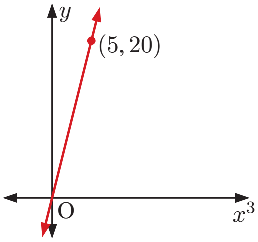
9 **a** Write K in terms of t.

b Hence find K when $t = 9$.

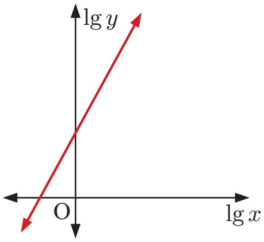


10

Graph A



Graph B



The relationship between x and y in *Graph A* can also be plotted as a straight line in *Graph B*. For the straight line in *Graph B*, find the:

a gradient **b** intercept on the vertical axis.

E FINDING RELATIONSHIPS FROM DATA

We have seen how the transformation of variables may allow us to display a non-linear relationship using a straight line graph. It is particularly useful to do this if we are trying to use a function to model data.

Exponential, power, and logarithmic models can be transformed to straight line graphs.

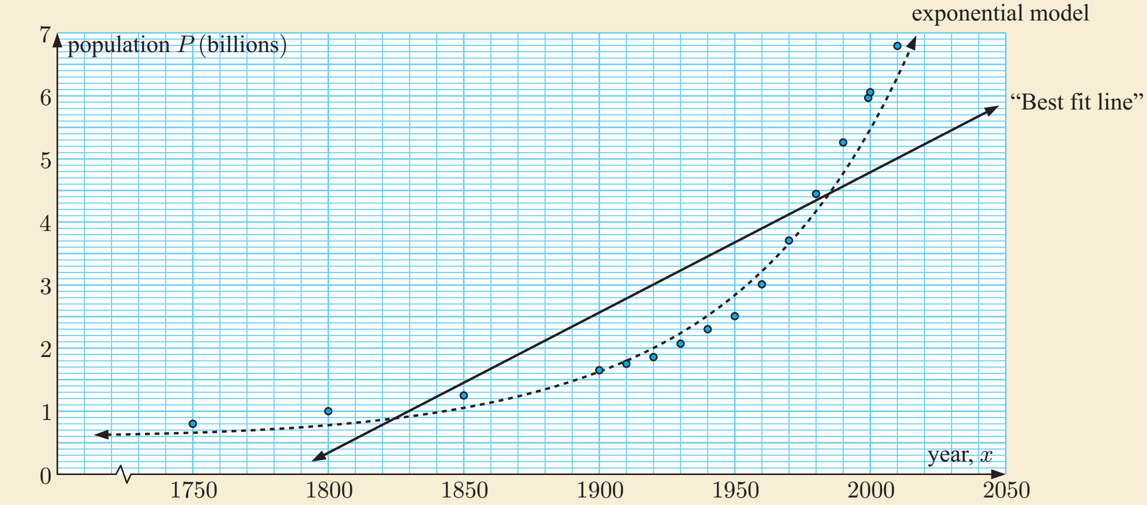
Case study — Exponential growth and decay

Logarithms are particularly important in science. Many physical processes are modelled accurately by exponential laws.

For example, the United Nations published the following data on world population:

Year	*Population P* (in billions)	lg *P*
1750	0.79	−0.236
1800	0.98	−0.0202
1850	1.26	0.231
1900	1.65	0.501
1910	1.75	0.560
1920	1.86	0.621
1930	2.07	0.728
1940	2 30	0 833
1950	2.52	0.924
1960	3.02	1.11
1970	3.70	1.31
1980	4.44	1.49
1990	5.27	1.66
1999	5.98	1.79
2000	6.06	1.80
2010	6 79	1 92

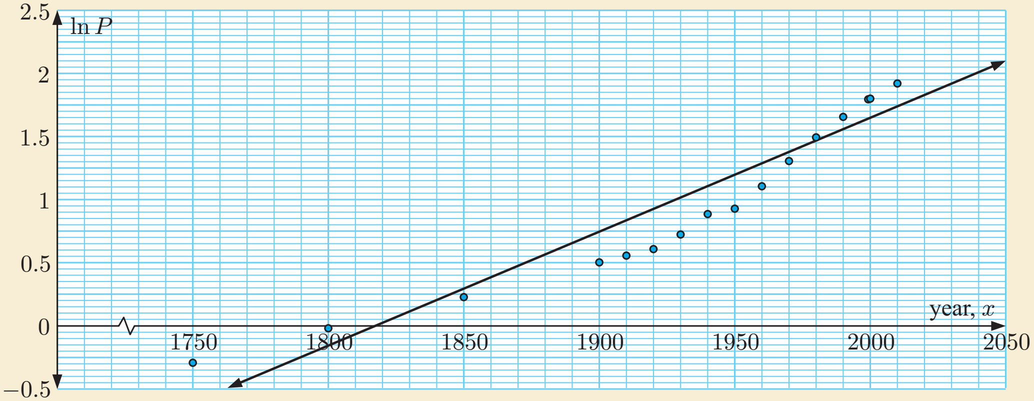
The population data is presented on the graph below:



The "best fit line", $P = 0.0222x - 39.6$, does not fit the data very well. Instead, we try to fit an exponential curve of the form $P = ae^{mx}$.

Taking the natural logarithm of both sides, we have $\ln P = mx + \ln a$, which is the equation of a straight line.

We now plot $\ln P$ against x:



The equation of this "best fit line" is $\ln P = -15.5 + 0.008\,55x$.

Using our model this means that $\ln y = -15.5 + 0.008\,55x$,

$\therefore$ the data can be modelled by $P = e^{-15.5+0.008\,55x}$.

This is shown as a dashed line on the original graph. This is not a perfect fit either, but is a considerable improvement on the original straight line graph.

Example 16

Self Tutor

Consider this table of data connecting x and y:

x	1	2	3	4
y	3.5	10	22.5	44

a Copy and complete the following table:

x^2				
$\frac{y}{x}$				

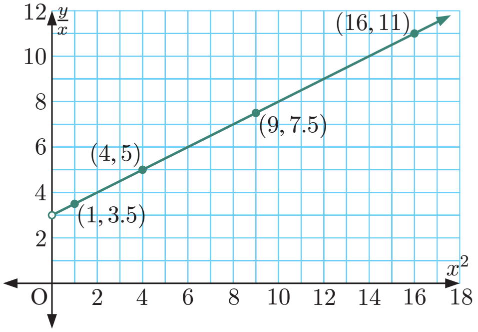
b Plot $\frac{y}{x}$ against x^2, and draw a straight line through the points.

c Find y in terms of x.

a

x^2	1	4	9	16
$\frac{y}{x}$	3.5	5	7.5	11

b



c The graph of $\frac{y}{x}$ against x^2 is linear.

Using the points (4, 5) and (16, 11),

the gradient is $\frac{11-5}{16-4} = \frac{1}{2}$

$$\therefore \text{ the equation is } \quad \frac{y}{x} - 5 = \frac{1}{2}(x^2 - 4)$$

$$\therefore \quad \frac{y}{x} = \frac{1}{2}x^2 + 3$$

$$\therefore \quad y = \tfrac{1}{2}x^3 + 3x$$

$x^2 \geqslant 0$ for all x.
$\frac{y}{x}$ is undefined when $x = 0$.
$\therefore$ the point on the vertical axis is not included.

EXERCISE 7E

1 Consider this table of data connecting x and y:

x	1	2	3	4
y	2	11	26	47

a Copy and complete the following table:

x^2				
y				

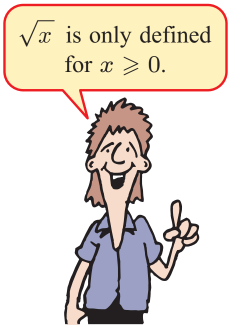
b Plot y against x^2, and draw a straight line through the points.

c Find y in terms of x.

$x^2 \geqslant 0$ for all x.

2 This table shows experimental data values for x and y:

x	1	2	3	4
y	9	9.90	10.97	12



a Copy and complete the following table:

x				
$y\sqrt{x}$				

b Plot $y\sqrt{x}$ against x, and draw a straight line through the points.

c Find y in terms of x.

d Find y when $x = 16$.

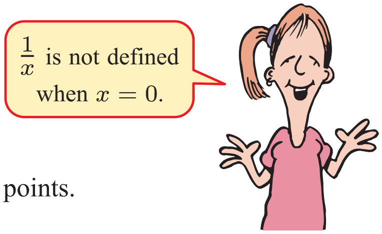
3 This table shows experimental values for x and y.

x	1	2	3	4
y	-1	0	0.11	0.12

It is known that x and y are related by the equation $y = \frac{a}{x} + \frac{b}{x^2}$, where a and b are constants.

a Copy and complete the following table:

$\frac{1}{x}$				
xy				



b Plot xy against $\frac{1}{x}$, and draw a straight line through the points.

c Hence find a and b.

d Find y when $x = 10$.

4 This table shows values of x and y:

x	2	4	6	8
y	5.24	5	5.45	6.12

a Copy and complete the following table:

$x\sqrt{x}$				
$y\sqrt{x}$				

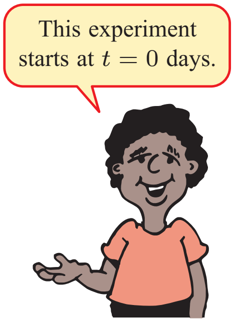
b Plot $y\sqrt{x}$ against $x\sqrt{x}$, and draw a straight line through the points.

c Find y in terms of x.

d Find y when $x = 9$.

5 The mass of bacteria in a culture is measured each day for 5 days.

t (days)	1	2	3	4	5
M (grams)	3.98	6.31	10	15.85	25.12



a Copy and complete the following table:

t					
$\lg M$					

b Plot $\lg M$ against t, and draw a straight line through the points.

c Find M in terms of t.

d Find the original mass of the bacteria.

Example 17

Self Tutor

This table shows experimental data values for x and y:

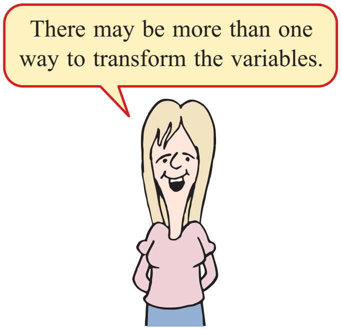
x	1	2	3	4
y	14	10	10	11

By plotting a suitable straight line graph, show that y and x are related by the equation $y = ax + \frac{b}{x}$.

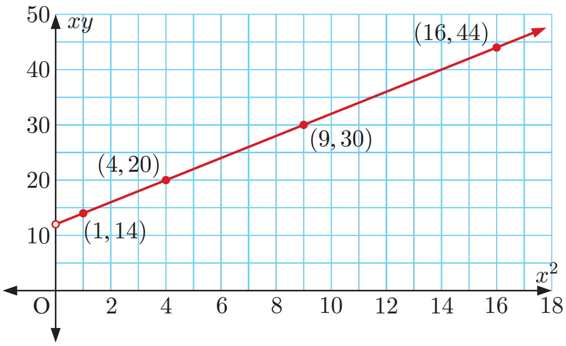
If $y = ax + \frac{b}{x}$, then

$xy = ax^2 + b$

$\therefore$ if y and x are related in this way, then we should observe a linear relationship between xy and x^2.



x^2	1	4	9	16
xy	14	20	30	44



The graph of xy against x^2 is linear.

Using points $(1, 14)$ and $(4, 20)$,

the gradient is $\frac{20-14}{4-1} = 2$.

$$\therefore \text{ the equation is } \quad xy - 14 = 2(x^2 - 1)$$
$$\therefore \quad xy - 14 = 2x^2 - 2$$
$$\therefore \quad xy = 2x^2 + 12$$
$$\therefore \quad y = 2x + \frac{12}{x} \qquad \{a = 2, \; b = 12\}$$

6 This table shows experimental values of x and y:

x	1	2	3	4
y	1	26	99	244

It is known that x and y are related by the equation $y = ax^3 + bx$, where a and b are constants.

- **a** A straight line graph is to be drawn to represent this information. If $\frac{y}{x}$ is plotted on the vertical axis, which variable should be plotted on the horizontal axis?
- **b** Draw the straight line graph.
- **c** Find the values of a and b.
- **d** Find y when $x = 5$.

7 This table shows experimental values of x and y:

x	1	2	3	4
y	4	1.17	0.36	0

By plotting a suitable straight line graph, show that x and y are related by the equation $y = \frac{a}{x} + \frac{b}{\sqrt{x}}$.

8 A stone is dropped from the top of an 80 m high cliff. This table shows the distance the stone has fallen at various times.

Time (t s)	1	1.7	2	2.7
Distance (D m)	4.9	14.16	19.6	35.72

a By plotting a suitable straight line graph, show that t and D are related by the equation $D = a \times t^b$, where a and b are constants.

b How far had the stone fallen after 3 seconds?

c How long did the stone take to hit the water?

Research — Logarithmic scales in science

If your data ranges over many orders of magnitude, it can be difficult to compare or represent on a graph.

For example, the Richter scale for earthquake measurement uses logarithms in base 10. An earthquake measuring 6.0 on the Richter scale has a shaking amplitude $10^{6-4} = 100$ times larger than one that measures 4.0.

Research some other scientific scales that use logarithms to compress very large ranges into manageable values. You may like to consider:

- the decibel scale for the loudness of sound
- the stellar magnitude scale for brightness of stars
- the pH scale for acidity and alkalinity
- counting f-stops for ratios of photographic exposure.

Review set 7A

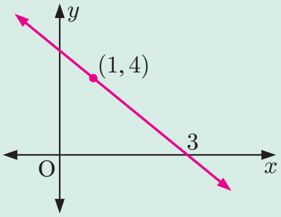
1 Consider the points A$(-1, 6)$ and B$(5, 4)$. Find:

a the distance between A and B

b the midpoint of AB

c the equation of the line through A and B.

2 Determine the equation of the illustrated line:



3 Explain why the vertical straight line in the plane cannot be written in gradient-intercept form $y = mx + c$.

4 Suppose P has coordinates $(-2, -3)$, and Q has coordinates $(1, 3)$. A line perpendicular to PQ, passes through Q.

a Find the equation of the line.

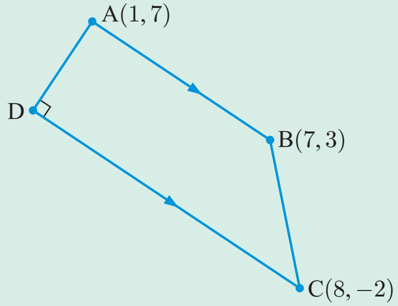
b Find the coordinates of the point where the line cuts the x-axis.

5 Find the point of intersection of the lines $x - 2y = 5$ and $4x + 3y = 9$.

6 ABCD is a trapezium in which AB is parallel to DC, and $\widehat{ADC} = 90°$. Find:

a the coordinates of D

b the area of the trapezium.



7 Find the points where the line $3x + y = 1$ intersects the curve $x^2 + y^2 = 29$.

8 The line $x + y = 5$ meets the curve $x^2 + y^2 + 3xy + 5x = 1$ at P and Q. Find the equation of the perpendicular bisector of PQ.

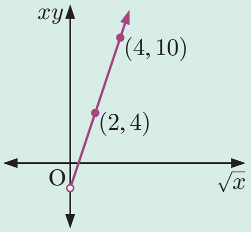
9 Consider two distinct points in the plane (a_1, b_1) and (a_2, b_2) where $a_1 \neq a_2$. Show that the straight line passing through them has equation:

a $y = \dfrac{b_1 - b_2}{a_1 - a_2}x + \dfrac{a_1b_2 - a_2b_1}{a_1 - a_2}$ in gradient-intercept form

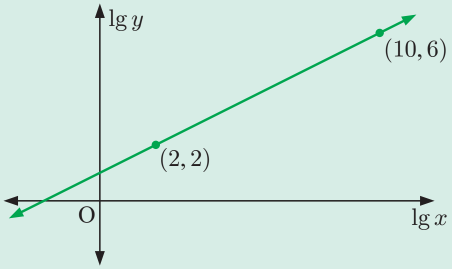
b $(b_1 - b_2)x + (a_2 - a_1)y = a_2b_1 - a_1b_2$ in general form.

10 **a** Write y in terms of x.

b Hence find y when $x = 4$.



11



Consider the graph alongside.

a Write an equation for the line in the form $\lg y = m \lg x + c$.

b Hence write y in terms of x.

12 This table shows experimental values of x and y:

x	1	2	3	4
y	8	7.5	11.33	17.75

a Copy and complete the following table:

x^3				
xy				

b Plot xy against x^3, and draw a straight line through the points.

c Find y in terms of x.

d Hence find y when $x = 7$.

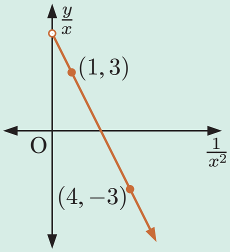
Review set 7B

1 Find the equation of the perpendicular bisector of AB given A$(-2, 3)$ and B$(4, 5)$.

2 The line $x - 2y = 3$ meets the curve $x^2 + 2y^2 - 2xy + 3x = 8$ at P and Q. Find the distance between P and Q.

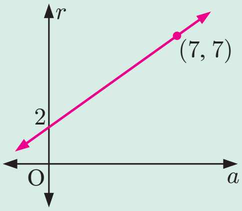
3 **a** Write y in terms of x.

b Hence find y when $x = 8$.

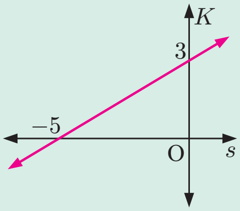


4 Find the equation linking the variables in each graph:

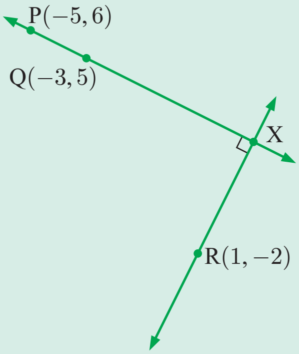
a



b



5 Find the coordinates of X.



6 Find, in general form, the equation of the line passing through $(-5, -7)$ and $(3, -2)$.

7 Consider this table of data connecting x and y:

x	1	2	3	4
y	2.5	5.29	8.13	11

a Copy and complete the following table:

$\sqrt{x}$				
$\frac{y}{\sqrt{x}}$				

b Plot $\frac{y}{\sqrt{x}}$ against $\sqrt{x}$, and draw a straight line through the points.

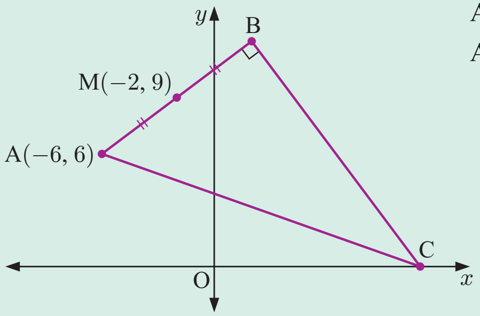
c Hence write y in terms of x.

8 Consider the points $(a, 0)$ and $(0, b)$.

a Find the equation of the straight line through these points, in general form.

b Let θ be the angle between the line and the x-axis. Show that the general form of the equation of the line is $(\sin\theta)x + (\cos\theta)y = d$ where $d = \dfrac{ab}{\sqrt{a^2+b^2}}$ is the shortest distance from the line to the origin.

9



ABC is a triangle in which M is the midpoint of AB, $A\widehat{B}C = 90°$, and C lies on the x-axis.

a Find the coordinates of:

i B **ii** C

b Find the area of the triangle.

10 The line $4x - 3y = 2$ intersects the curve $\dfrac{3}{y} - \dfrac{1}{x} = 1$ at A and B. Find the midpoint of AB.

11 This table shows experimental values of x and y:

x	2	4	6	8
y	21.54	4.64	1	0.21

a By plotting a suitable straight line graph, show that x and y are related by the equation $y = a \times b^x$, where a and b are constants.

b Hence find y when $x = 1$.

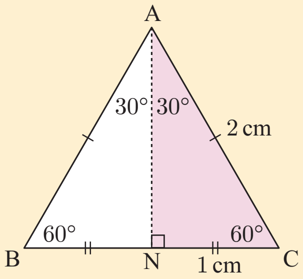
8

The unit circle and radian measure

Contents:

Opening problem

Consider an equilateral triangle with sides 2 cm long. Altitude AN bisects side BC and the vertical angle BAC.



Things to think about:

a Can you use this figure to explain why $\sin 30° = \frac{1}{2}$?

b Use your calculator to find the value of:

  i $\sin 150°$ **ii** $\sin 390°$ **iii** $\sin(-330°)$

c Can you explain each result in **b**, even though the angles are not between 0° and 90°?

A RADIAN MEASURE

DEGREE MEASUREMENT OF ANGLES

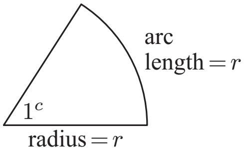
We have seen previously that one full revolution makes an angle of 360°, and the angle on a straight line is 180°.

One **degree**, 1°, is $\frac{1}{360}$th of one full revolution.

This measure of angle is commonly used by surveyors and architects.

RADIAN MEASUREMENT OF ANGLES

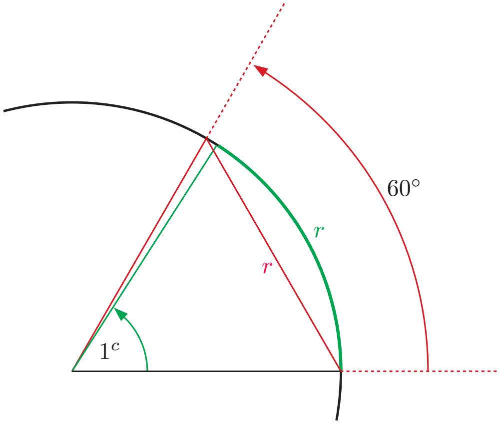
An angle is said to have a measure of one **radian**, 1^c, if it is subtended at the centre of a circle by an arc equal in length to the radius.



The symbol 'c' is used for radian measure but is usually omitted. By contrast, the degree symbol is *always* used when the measure of an angle is given in degrees.

From the diagram below, it can be seen that 1^c is slightly smaller than 60°. In fact, $1^c \approx 57.3°$.

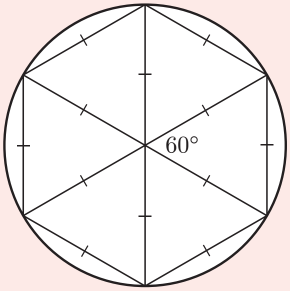
The word 'radian' is an abbreviation of 'radial angle'.



Historical note

There are several theories for why one complete turn was divided into 360 degrees:

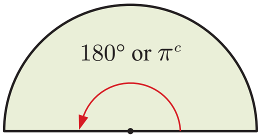
- 360 is approximately the number of days in a year.
- The Babylonians used a counting system in base 60. If they drew 6 equilateral triangles within a circle as shown, and divided each angle into 60 subdivisions, then there were 360 subdivisions in one turn. The division of an hour into 60 minutes, and a minute into 60 seconds, is from this base 60 counting system.
- 360 has 24 divisors, including every integer from 1 to 10 except 7.



The idea of measuring an angle by the length of an arc dates to around 1400 and the Persian mathematician **Al-Kashi**. The concept of a radian is generally credited to **Roger Cotes**, however, who described it as we know it today.

DEGREE-RADIAN CONVERSIONS

If the radius of a circle is r, then an arc of length πr, or half the circumference, will subtend an angle of π radians.



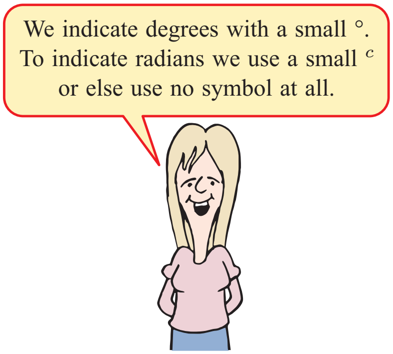
Therefore, $\boxed{\pi \text{ radians} = 180°.}$

So, $1^c = \left(\frac{180}{\pi}\right)^\circ \approx 57.3^\circ$ and $1^\circ = \left(\frac{\pi}{180}\right)^c \approx 0.0175^c$.

To convert from degrees to radians, we multiply by $\frac{\pi}{180}$.

To convert from radians to degrees, we multiply by $\frac{180}{\pi}$.

$$\begin{tikzcd}
\text{Degrees} \arrow[r, bend left, "\times \frac{\pi}{180}"] & \text{Radians} \arrow[l, bend left, "\times \frac{180}{\pi}"]
\end{tikzcd}$$



We indicate degrees with a small °. To indicate radians we use a small ^{c} or else use no symbol at all.

Example 1

Self Tutor

Convert 45° to radians in terms of π.

$$\begin{aligned} 45^\circ &= \left(45 \times \tfrac{\pi}{180}\right) \text{ radians} \\ &= \tfrac{\pi}{4} \text{ radians} \end{aligned}$$

or

$$\begin{aligned} 180^\circ &= \pi \text{ radians} \\ \therefore \left(\tfrac{180}{4}\right)^\circ &= \tfrac{\pi}{4} \text{ radians} \\ \therefore 45^\circ &= \tfrac{\pi}{4} \text{ radians} \end{aligned}$$

Example 2 Self Tutor

Convert 126.5° to radians.

$$\begin{aligned} & 126.5° \\ = & \left(126.5 \times \tfrac{\pi}{180}\right) \text{ radians} \\ \approx & \ 2.21 \text{ radians} \end{aligned}$$

Angles in radians may be expressed either in terms of π or as decimals.

EXERCISE 8A

1 Convert to radians, in terms of π:

a 90° **b** 60° **c** 30° **d** 18° **e** 9°
f 135° **g** 225° **h** 270° **i** 360° **j** 720°
k 315° **l** 540° **m** 36° **n** 80° **o** 230°

2 Convert to radians, correct to 3 significant figures:

a 36.7° **b** 137.2° **c** 317.9° **d** 219.6° **e** 396.7°

Example 3 Self Tutor

Convert the following radian measures to degrees:

a $\frac{5\pi}{6}$ **b** 0.638

a
$$\begin{aligned} & \tfrac{5\pi}{6} \\ = & \left(\tfrac{5\pi}{6} \times \tfrac{180}{\pi}\right)^{\circ} \\ = & \ 150° \end{aligned}$$

b
$$\begin{aligned} & 0.638 \text{ radians} \\ = & \left(0.638 \times \tfrac{180}{\pi}\right)^{\circ} \\ \approx & \ 36.6° \end{aligned}$$

3 Convert the following radian measures to degrees:

a $\frac{\pi}{5}$ **b** $\frac{3\pi}{5}$ **c** $\frac{3\pi}{4}$ **d** $\frac{\pi}{18}$ **e** $\frac{\pi}{9}$
f $\frac{7\pi}{9}$ **g** $\frac{\pi}{10}$ **h** $\frac{3\pi}{20}$ **i** $\frac{7\pi}{6}$ **j** $\frac{\pi}{8}$

4 Convert the following radian measures to degrees. Give your answers correct to 2 decimal places.

a 2 **b** 1.53 **c** 0.867 **d** 3.179 **e** 5.267

5 Copy and complete, giving your answers in terms of π:

a

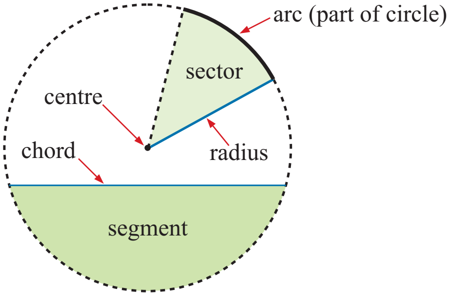
Degrees	0	45	90	135	180	225	270	315	360
Radians									

b

Degrees	0	30	60	90	120	150	180	210	240	270	300	330	360
Radians													

B ARC LENGTH AND SECTOR AREA

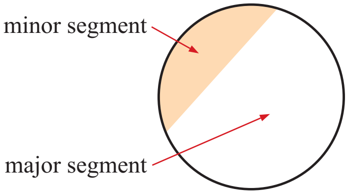
The diagram alongside illustrates terms relating to the parts of a circle.

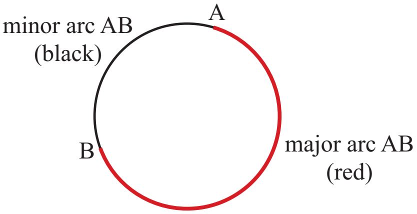


An arc, sector, or segment is described as:

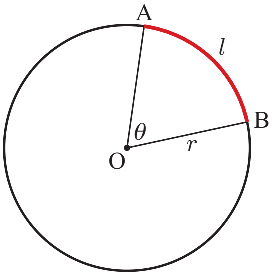
- **minor** if it involves less than half the circle
- **major** if it involves more than half the circle.

For example:





ARC LENGTH



In the diagram, the **arc length** AB is l.

Angle θ is measured in **radians**.

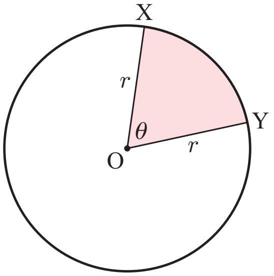
We use a ratio to obtain:
$$\frac{\text{arc length}}{\text{circumference}} = \frac{\theta}{2\pi}$$
$$\therefore \quad \frac{l}{2\pi r} = \frac{\theta}{2\pi}$$
$$\therefore \quad l = \theta r$$

Radians are used in pure mathematics because they make formulae simpler.

For θ in **radians**, arc length $\boldsymbol{l = \theta r}$.

For θ in **degrees**, arc length $\boldsymbol{l = \frac{\theta}{360} \times 2\pi r}$.

AREA OF SECTOR



In the diagram, the area of minor sector XOY is shaded.

θ is measured in **radians**.

We use a ratio to obtain:
$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi}$$
$$\therefore \quad \frac{A}{\pi r^2} = \frac{\theta}{2\pi}$$
$$\therefore \quad A = \tfrac{1}{2}\theta r^2$$

For θ in **radians**, area of sector $\boldsymbol{A = \frac{1}{2}\theta r^2}$.

For θ in **degrees**, area of sector $\boldsymbol{A = \frac{\theta}{360} \times \pi r^2}$.

Example 4

Self Tutor

A sector has radius 12 cm and angle 3 radians. Find its:

a arc length **b** area

a arc length $= \theta r$
$= 3 \times 12$
$= 36$ cm

b area $= \frac{1}{2}\theta r^2$
$= \frac{1}{2} \times 3 \times 12^2$
$= 216$ cm^2

EXERCISE 8B

1 Use radians to find the arc length and area of a sector of a circle of:

a radius 9 cm and angle $\frac{7\pi}{4}$ **b** radius 4.93 cm and angle 4.67 radians.

2 A sector has an angle of 107.9° and an arc length of 5.92 m. Find its:

a radius **b** area.

3 A sector has an angle of 1.19 radians and an area of 20.8 cm^2. Find its:

a radius **b** perimeter.

Example 5

Self Tutor

Find the area of a sector with radius 8.2 cm and arc length 13.3 cm.

For θ in radians, $l = \theta r$

$$\therefore \quad \theta = \frac{l}{r} = \frac{13.3}{8.2}$$

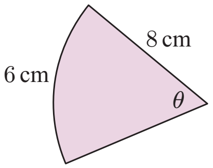
$$\therefore \quad \text{area} = \tfrac{1}{2}\theta r^2$$
$$= \tfrac{1}{2} \times \frac{13.3}{8.2} \times 8.2^2$$
$$\approx 54.5 \text{ cm}^2$$

4 Find, in radians, the angle of a sector of:

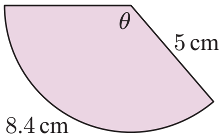
a radius 4.3 m and arc length 2.95 m **b** radius 10 cm and area 30 cm^2.

5 Find θ (in radians) for each of the following, and hence find the area of each figure:

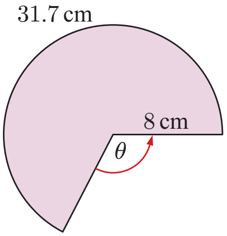
a



b



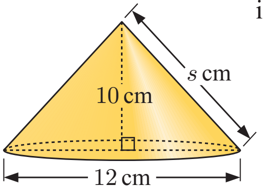
c



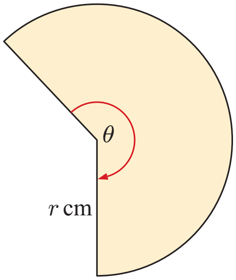
6 Find the arc length and area of a sector of radius 5 cm and angle 2 radians.

7 If a sector has radius $2x$ cm and arc length x cm, show that its area is x^2 cm^2.

8 The cone



is made from this sector:



Find, correct to 3 significant figures:

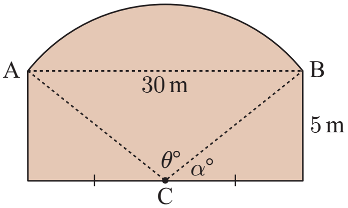
a the slant length s cm

b the value of r

c the arc length of the sector

d the sector angle θ in radians.

9



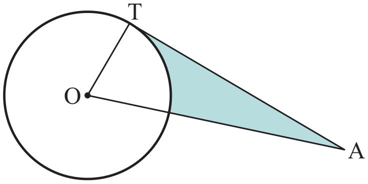
The end wall of a building has the shape illustrated, where the centre of arc AB is C. Find:

a α to 4 significant figures

b θ to 4 significant figures

c the area of the wall.

10



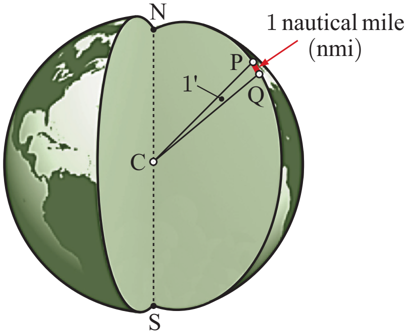
[AT] is a tangent to the given circle. $\text{OA} = 13$ cm and the circle has radius 5 cm. Find the perimeter of the shaded region.

11 A **nautical mile** (nmi) is the distance on the Earth's surface that subtends an angle of 1 minute (or $\frac{1}{60}$ of a degree) of the Great Circle arc measured from the centre of the Earth.

A **knot** is a speed of 1 nautical mile per hour.

a Given that the radius of the Earth is 6370 km, show that 1 nmi $\approx$ 1.853 km.

b Calculate how long it would take a plane to fly from London to Moscow (a distance of 2508 km) if the plane can fly at 480 knots.



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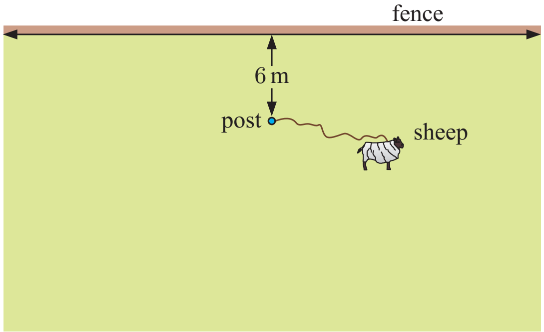
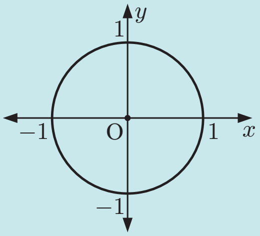


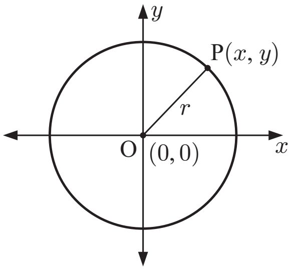
A sheep is tethered to a post which is 6 m from a long fence. The length of rope is 9 m. Find the area which the sheep can feed on.

C THE UNIT CIRCLE AND THE TRIGONOMETRIC RATIOS

The **unit circle** is the circle with centre (0, 0) and radius 1 unit.



CIRCLES WITH CENTRE (0, 0)



Consider a circle with centre (0, 0) and radius r units.
Suppose $P(x, y)$ is any point on this circle.

$$\text{Since } OP = r,$$
$$\sqrt{(x-0)^2 + (y-0)^2} = r \qquad \{\text{distance formula}\}$$
$$\therefore\ x^2 + y^2 = r^2$$

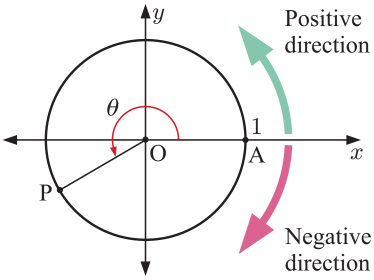
$\boldsymbol{x^2 + y^2 = r^2}$ is the equation of a circle with centre (0, 0) and radius r.

The equation of the **unit circle** is $x^2 + y^2 = 1$.

ANGLE MEASUREMENT

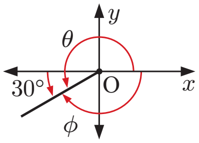
Suppose P lies anywhere on the unit circle, and A is (1, 0).
Let θ be the angle measured from [OA] on the positive x-axis.

θ is **positive** for anticlockwise rotations and **negative** for clockwise rotations.



For example: $\theta = 210° = \frac{7\pi}{6}$

$\phi = -150° = -\frac{5\pi}{6}$



DEFINITION OF SINE AND COSINE

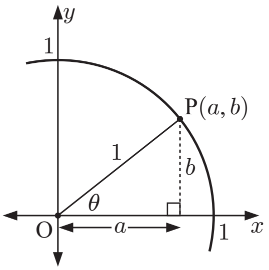
Consider a point $P(a, b)$ which lies on the unit circle in the first quadrant. [OP] makes an angle θ with the x-axis as shown.

Using right angled triangle trigonometry:

$$\cos\theta = \frac{\text{ADJ}}{\text{HYP}} = \frac{a}{1} = a$$

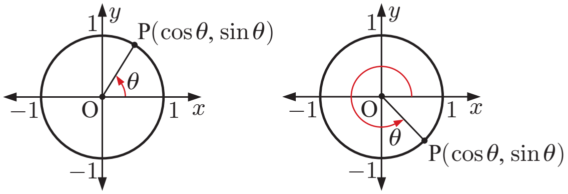
$$\sin\theta = \frac{\text{OPP}}{\text{HYP}} = \frac{b}{1} = b$$

$$\tan\theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{b}{a} = \frac{\sin\theta}{\cos\theta}$$



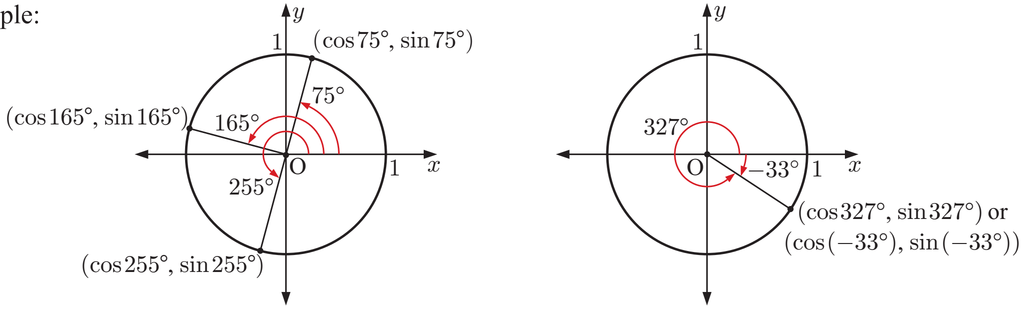
In general, for a point P anywhere on the unit circle:

- **$\cos\theta$** is the x-coordinate of P
- **$\sin\theta$** is the y-coordinate of P



We can hence find the coordinates of any point on the unit circle with given angle θ measured from the positive x-axis.

For example:



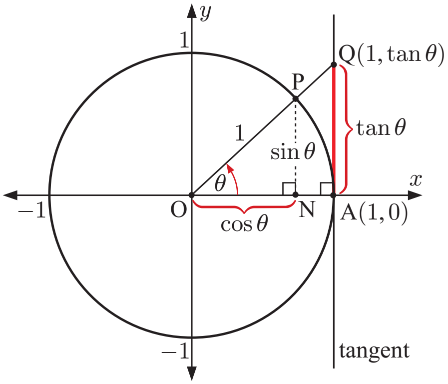
Since the unit circle has equation $x^2 + y^2 = 1$, $(\cos\theta)^2 + (\sin\theta)^2 = 1$ for all θ.

We commonly write this as $\boldsymbol{\cos^2\theta + \sin^2\theta = 1}$.

For all points on the unit circle, $-1 \leqslant x \leqslant 1$ and $-1 \leqslant y \leqslant 1$.

So, $\boldsymbol{-1 \leqslant \cos\theta \leqslant 1}$ and $\boldsymbol{-1 \leqslant \sin\theta \leqslant 1}$ for all θ.

DEFINITION OF TANGENT



Suppose we extend [OP] to meet the tangent from A$(1, 0)$.

We let the intersection between these lines be point Q.

Note that as P moves, so does Q.

The position of Q relative to A is defined as the **tangent function**.

Notice that $\triangle$s ONP and OAQ are equiangular and therefore similar.

Consequently $\frac{\text{AQ}}{\text{OA}} = \frac{\text{NP}}{\text{ON}}$ and hence $\frac{\text{AQ}}{1} = \frac{\sin\theta}{\cos\theta}$.

Under the definition that AQ $= \tan\theta$, $\boldsymbol{\tan\theta = \dfrac{\sin\theta}{\cos\theta}}$.

Discovery 1 The trigonometric ratios

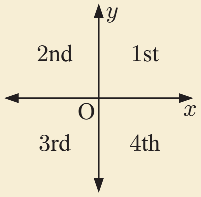
In this Discovery we explore the signs of the trigonometric ratios in each quadrant of the unit circle.

What to do:

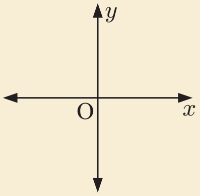
THE UNIT CIRCLE

1 Click on the icon to run the Unit Circle software.
Drag the point P slowly around the circle.
Note the *sign* of each trigonometric ratio in each quadrant.

Quadrant	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	positive		
2			
3			
4			

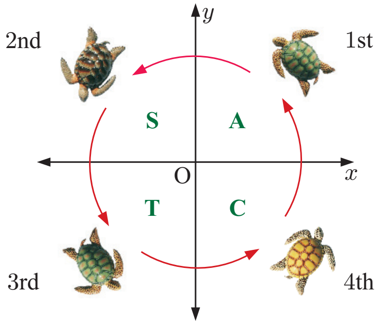


2 Hence note down the trigonometric ratios which are *positive* in each quadrant.



From the **Discovery** you should have found that:

- $\sin\theta$, $\cos\theta$, and $\tan\theta$ are positive in quadrant 1
- only $\sin\theta$ is positive in quadrant 2
- only $\tan\theta$ is positive in quadrant 3
- only $\cos\theta$ is positive in quadrant 4.



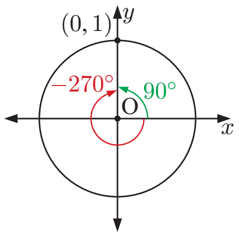
We can use a letter to show which trigonometric ratios are positive in each quadrant. The A stands for *all* of the ratios.

You might like to remember them using

All **S**illy **T**urtles **C**rawl.

Example 6 Self Tutor

Use a unit circle diagram to find the values of $\cos(-270°)$ and $\sin(-270°)$.



$\cos(-270°) = 0$ {the x-coordinate}
$\sin(-270°) = 1$ {the y-coordinate}

PERIODICITY OF TRIGONOMETRIC RATIOS

Since there are 2π radians in a full revolution, if we add any integer multiple of 2π to θ (in radians) then the position of P on the unit circle is unchanged.

For θ in radians and $k \in \mathbb{Z}$,

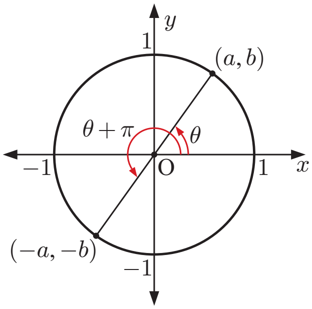
$$\boldsymbol{\cos(\theta + 2k\pi) = \cos\theta} \quad \text{and} \quad \boldsymbol{\sin(\theta + 2k\pi) = \sin\theta}.$$

We notice that for any point $(\cos\theta, \sin\theta)$ on the unit circle, the point directly opposite is $(-\cos\theta, -\sin\theta)$

$$\therefore\ \cos(\theta+\pi) = -\cos\theta$$

$$\sin(\theta+\pi) = -\sin\theta$$

$$\text{and}\quad \tan(\theta+\pi) = \frac{-\sin\theta}{-\cos\theta} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$



For θ in radians and $k \in \mathbb{Z}$, $\boldsymbol{\tan(\theta + k\pi) = \tan\theta}$.

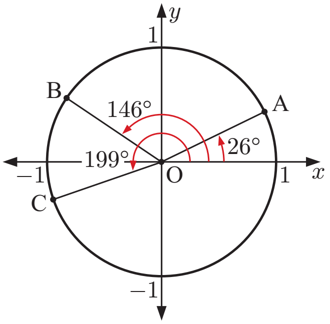
This **periodic** feature is an important property of the trigonometric functions.

EXERCISE 8C

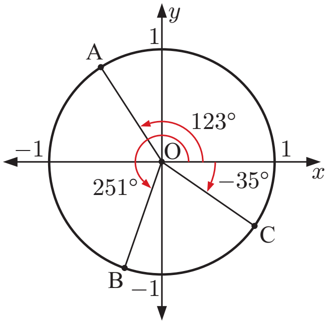
1 For each unit circle illustrated:

- **i** state the exact coordinates of points A, B, and C in terms of sine and cosine
- **ii** use your calculator to give the coordinates of A, B, and C correct to 3 significant figures.

a



b



2 With the aid of a unit circle, complete the following table:

θ (degrees)	0°	90°	180°	270°	360°	450°
θ (radians)						
sine						
cosine						
tangent						

3 **a** Use your calculator to evaluate: **i** $\frac{1}{\sqrt{2}}$ **ii** $\frac{\sqrt{3}}{2}$

b Copy and complete the following table. If necessary, use your calculator to evaluate the trigonometric ratios, then **a** to write them exactly.

θ (degrees)	$30°$	$45°$	$60°$	$135°$	$150°$	$240°$	$315°$
θ (radians)							
sine							
cosine							
tangent							

4 **a** Use your calculator to evaluate:

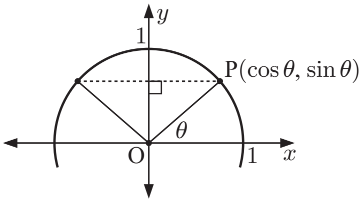
i $\sin 100°$ **ii** $\sin 80°$ **iii** $\sin 120°$ **iv** $\sin 60°$

v $\sin 150°$ **vi** $\sin 30°$ **vii** $\sin 45°$ **viii** $\sin 135°$

b Use the results from **a** to copy and complete:

$\sin(180° - \theta) = ...$

c Justify your answer using the diagram alongside:



d Find the obtuse angle with the same sine as:

i $45°$ **ii** $51°$ **iii** $\frac{\pi}{3}$ **iv** $\frac{\pi}{6}$

5 **a** Use your calculator to evaluate:

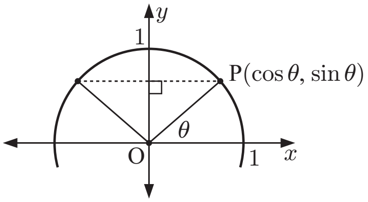
i $\cos 70°$ **ii** $\cos 110°$ **iii** $\cos 60°$ **iv** $\cos 120°$

v $\cos 25°$ **vi** $\cos 155°$ **vii** $\cos 80°$ **viii** $\cos 100°$

b Use the results from **a** to copy and complete:

$\cos(180° - \theta) = ...$

c Justify your answer using the diagram alongside:



d Find the obtuse angle which has the negative cosine of:

i $40°$ **ii** $19°$ **iii** $\frac{\pi}{5}$ **iv** $\frac{2\pi}{5}$

6 Without using your calculator, find:

a $\sin 137°$ if $\sin 43° \approx 0.6820$

b $\sin 59°$ if $\sin 121° \approx 0.8572$

c $\cos 143°$ if $\cos 37° \approx 0.7986$

d $\cos 24°$ if $\cos 156° \approx -0.9135$

e $\sin 115°$ if $\sin 65° \approx 0.9063$

f $\cos 132°$ if $\cos 48° \approx 0.6691$

7 **a** Copy and complete:

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0° < \theta < 90°$	$0 < \theta < \frac{\pi}{2}$	positive	positive	
2					
3					
4					

b In which quadrants are the following true?

i $\cos\theta$ is positive.

ii $\cos\theta$ is negative.

iii $\cos\theta$ and $\sin\theta$ are both negative.

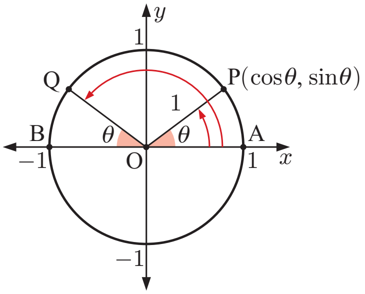
iv $\cos\theta$ is negative and $\sin\theta$ is positive.

8 **a** If $\text{A}\widehat{\text{O}}\text{P} = \text{B}\widehat{\text{O}}\text{Q} = \theta$, what is the measure of $\text{A}\widehat{\text{O}}\text{Q}$?

b Copy and complete:

[OQ] is a reflection of [OP] in the ......
and so Q has coordinates ......

c What trigonometric formulae can be deduced from **a** and **b**?

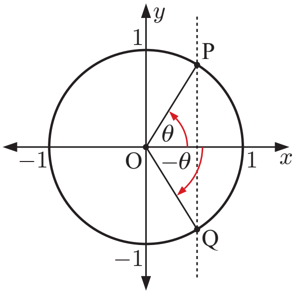


9 **a** Copy and complete:

θ^c	$\sin\theta$	$\sin(-\theta)$	$\cos\theta$	$\cos(-\theta)$
0.75				
1.772				
3.414				
6.25				
−1.17				

b What trigonometric formulae can be deduced from your results in **a**?

c



The coordinates of P in the figure are $(\cos\theta, \sin\theta)$.

i By finding the coordinates of Q in terms of θ in *two different* ways, prove your formulae in **b**.

ii Hence explain why $\cos(2\pi - \theta) = \cos\theta$.

D APPLICATIONS OF THE UNIT CIRCLE

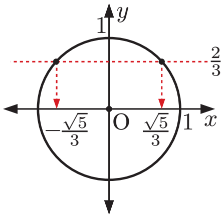
The identity $\cos^2\theta + \sin^2\theta = 1$ is essential for finding trigonometric ratios.

Example 7

Self Tutor

Find the possible values of $\cos\theta$ for $\sin\theta = \frac{2}{3}$. Illustrate your answers.

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore\ \cos^2\theta + \left(\tfrac{2}{3}\right)^2 = 1$$
$$\therefore\ \cos^2\theta = \tfrac{5}{9}$$
$$\therefore\ \cos\theta = \pm\tfrac{\sqrt{5}}{3}$$



EXERCISE 8D.1

1 Find the possible values of $\cos\theta$ for:

a $\sin\theta = \frac{1}{2}$ **b** $\sin\theta = -\frac{1}{3}$ **c** $\sin\theta = 0$ **d** $\sin\theta = -1$

2 Find the possible values of $\sin\theta$ for:

a $\cos\theta = \frac{4}{5}$ **b** $\cos\theta = -\frac{3}{4}$ **c** $\cos\theta = 1$ **d** $\cos\theta = 0$

Example 8

Self Tutor

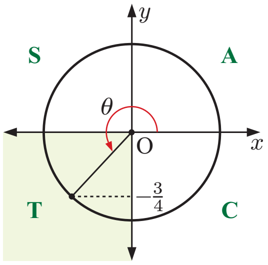
If $\sin\theta = -\frac{3}{4}$ and $\pi < \theta < \frac{3\pi}{2}$, find $\cos\theta$ and $\tan\theta$. Give exact answers.

Now $\cos^2\theta + \sin^2\theta = 1$

$\therefore \cos^2\theta + \frac{9}{16} = 1$

$\therefore \cos^2\theta = \frac{7}{16}$

$\therefore \cos\theta = \pm\frac{\sqrt{7}}{4}$



But $\pi < \theta < \frac{3\pi}{2}$, so θ is a quadrant 3 angle.

$\therefore$ $\cos\theta$ is negative.

$\therefore$ $\cos\theta = -\frac{\sqrt{7}}{4}$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{-\frac{3}{4}}{-\frac{\sqrt{7}}{4}} = \dfrac{3}{\sqrt{7}}$

3 Find the exact value of:

a $\sin\theta$ if $\cos\theta = \frac{2}{3}$ and $0 < \theta < \frac{\pi}{2}$

b $\cos\theta$ if $\sin\theta = \frac{2}{5}$ and $\frac{\pi}{2} < \theta < \pi$

c $\cos\theta$ if $\sin\theta = -\frac{3}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$

d $\sin\theta$ if $\cos\theta = -\frac{5}{13}$ and $\pi < \theta < \frac{3\pi}{2}$.

4 Find the exact value of $\tan\theta$ given that:

a $\sin\theta = \frac{1}{3}$ and $\frac{\pi}{2} < \theta < \pi$

b $\cos\theta = \frac{1}{5}$ and $\frac{3\pi}{2} < \theta < 2\pi$

c $\sin\theta = -\frac{1}{\sqrt{3}}$ and $\pi < \theta < \frac{3\pi}{2}$

d $\cos\theta = -\frac{3}{4}$ and $\frac{\pi}{2} < \theta < \pi$.

Example 9

Self Tutor

If $\tan\theta = -2$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\sin\theta$ and $\cos\theta$. Give exact answers.

$\tan\theta = \dfrac{\sin\theta}{\cos\theta} = -2$

$\therefore \sin\theta = -2\cos\theta$

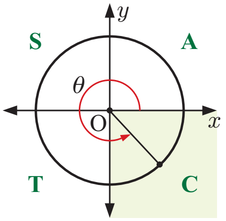
Now $\sin^2\theta + \cos^2\theta = 1$

$\therefore (-2\cos\theta)^2 + \cos^2\theta = 1$

$\therefore 4\cos^2\theta + \cos^2\theta = 1$

$\therefore 5\cos^2\theta = 1$

$\therefore \cos\theta = \pm\frac{1}{\sqrt{5}}$



But $\frac{3\pi}{2} < \theta < 2\pi$, so θ is a quadrant 4 angle.

$\therefore$ $\cos\theta$ is positive and $\sin\theta$ is negative.

$\therefore$ $\cos\theta = \frac{1}{\sqrt{5}}$ and $\sin\theta = -\frac{2}{\sqrt{5}}$.

5 Find exact values for $\sin x$ and $\cos x$ given that:

a $\tan x = \frac{2}{3}$ and $0 < x < \frac{\pi}{2}$

b $\tan x = -\frac{4}{3}$ and $\frac{\pi}{2} < x < \pi$

c $\tan x = \frac{\sqrt{5}}{3}$ and $\pi < x < \frac{3\pi}{2}$

d $\tan x = -\frac{12}{5}$ and $\frac{3\pi}{2} < x < 2\pi$

6 Suppose $\tan\theta = k$ where k is a constant and $\pi < \theta < \frac{3\pi}{2}$. Write expressions for $\sin\theta$ and $\cos\theta$ in terms of k.

FINDING ANGLES WITH PARTICULAR TRIGONOMETRIC RATIOS

From **Exercise 8C** you should have discovered that:

For θ in radians:

- $\sin(\pi - \theta) = \sin\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\cos(2\pi - \theta) = \cos\theta$

We need results such as these, and also the periodicity of the trigonometric ratios, to find angles which have a particular sine, cosine, or tangent.

Example 10

Self Tutor

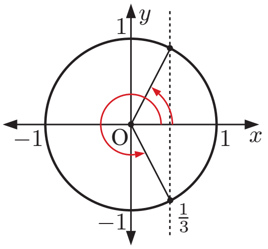
Find the two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\cos\theta = \frac{1}{3}$

b $\sin\theta = \frac{3}{4}$

c $\tan\theta = 2$

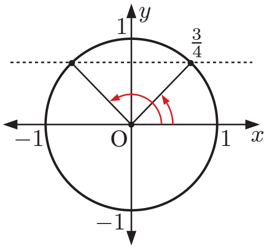
a $\cos^{-1}(\frac{1}{3}) \approx 1.23$



$\therefore\ \theta \approx 1.23$ or $2\pi - 1.23$

$\therefore\ \theta \approx 1.23$ or 5.05

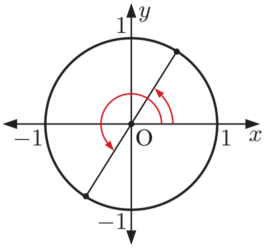
b $\sin^{-1}(\frac{3}{4}) \approx 0.848$



$\therefore\ \theta \approx 0.848$ or $\pi - 0.848$

$\therefore\ \theta \approx 0.848$ or 2.29

c $\tan^{-1}(2) \approx 1.11$



$\therefore\ \theta \approx 1.11$ or $\pi + 1.11$

$\therefore\ \theta \approx 1.11$ or 4.25

If $\cos\theta$, $\sin\theta$, or $\tan\theta$ is positive, your calculator will give θ in the domain $0 < \theta < \frac{\pi}{2}$.

EXERCISE 8D.2

1 Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\tan\theta = 4$

b $\cos\theta = 0.83$

c $\sin\theta = \frac{3}{5}$

d $\cos\theta = 0$

e $\tan\theta = 1.2$

f $\cos\theta = 0.7816$

g $\sin\theta = \frac{1}{11}$

h $\tan\theta = 20.2$

i $\sin\theta = \frac{39}{40}$

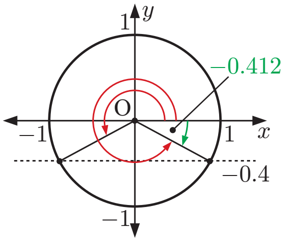
Example 11

Self Tutor

Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\sin\theta = -0.4$ **b** $\cos\theta = -\frac{2}{3}$ **c** $\tan\theta = -\frac{1}{3}$

a $\sin^{-1}(-0.4) \approx -0.412$



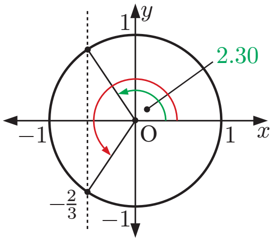
But $0 \leqslant \theta \leqslant 2\pi$

$\therefore \quad \theta \approx \pi + 0.412$ or

$2\pi - 0.412$

$\therefore \quad \theta \approx 3.55$ or 5.87

b $\cos^{-1}(-\frac{2}{3}) \approx 2.30$



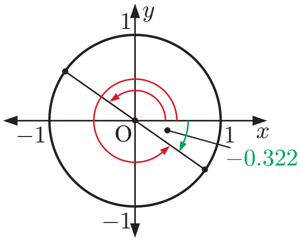
But $0 \leqslant \theta \leqslant 2\pi$

$\therefore \quad \theta \approx 2.30$ or

$2\pi - 2.30$

$\therefore \quad \theta \approx 2.30$ or 3.98

c $\tan^{-1}(-\frac{1}{3}) \approx -0.322$



But $0 \leqslant \theta \leqslant 2\pi$

$\therefore \quad \theta \approx \pi - 0.322$ or

$2\pi - 0.322$

$\therefore \quad \theta \approx 2.82$ or 5.96

If $\sin\theta$ or $\tan\theta$ is negative, your calculator will give θ in the domain $-\frac{\pi}{2} < \theta < 0$.

The green arrow shows the angle that your calculator gives.

2 Find two angles θ on the unit circle, with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\cos\theta = -\frac{1}{4}$ **b** $\sin\theta = 0$ **c** $\tan\theta = -3.1$

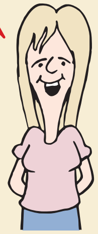
d $\sin\theta = -0.421$ **e** $\tan\theta = -6.67$ **f** $\cos\theta = -\frac{2}{17}$

g $\tan\theta = -\sqrt{5}$ **h** $\cos\theta = -\frac{1}{\sqrt{3}}$ **i** $\sin\theta = -\frac{\sqrt{2}}{\sqrt{5}}$

Discovery 2

Parametric equations

The use of parametric equations is not required for the syllabus.



Usually we write functions in the form $y = f(x)$.

For example: $y = 3x + 7$, $y = x^2 - 6x + 8$, $y = \sin x$

However, sometimes it is useful to express **both** x and y in terms of another variable t, called the **parameter**. In this case we say we have **parametric equations**.

What to do:

PARAMETRIC PLOTTER

1 **a** Use the graphing package to plot $\{(x, y) : x = \cos t,\ y = \sin t,\ 0° \leqslant t \leqslant 360°\}$. Use the same scale on both axes.

b Describe the resulting graph. Is it the graph of a function?

c Evaluate $x^2 + y^2$. Hence determine the equation of this graph in terms of x and y only.

2 Use the graphing package to plot:

a $\{(x,\,y) : x = 2\cos t, \quad y = \sin(2t), \quad 0° \leqslant t \leqslant 360°\}$

b $\{(x,\,y) : x = 2\cos t, \quad y = 2\sin(3t), \quad 0° \leqslant t \leqslant 360°\}$

c $\{(x,\,y) : x = 2\cos t, \quad y = \cos t - \sin t, \quad 0° \leqslant t \leqslant 360°\}$

d $\{(x,\,y) : x = \cos^2 t + \sin 2t, \quad y = \cos t, \quad 0° \leqslant t \leqslant 360°\}$

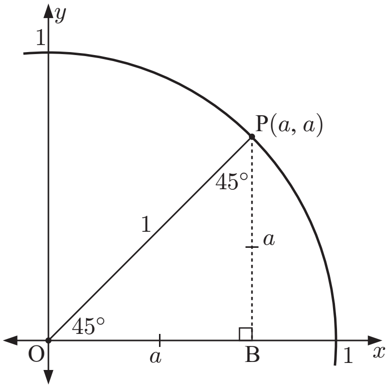
e $\{(x,\,y) : x = \cos^3 t, \quad y = \sin t, \quad 0° \leqslant t \leqslant 360°\}$

E MULTIPLES OF $\frac{\pi}{6}$ AND $\frac{\pi}{4}$

Angles which are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$ occur frequently, so it is important for us to write their trigonometric ratios exactly.

MULTIPLES OF $\frac{\pi}{4}$ OR 45°

Triangle OBP is isosceles as angle OPB also measures 45°.



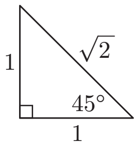
Letting OB = BP = a,

$$a^2 + a^2 = 1^2 \quad \{\text{Pythagoras}\}$$
$$\therefore\ 2a^2 = 1$$
$$\therefore\ a^2 = \tfrac{1}{2}$$
$$\therefore\ a = \tfrac{1}{\sqrt{2}} \quad \{\text{as } a > 0\}$$

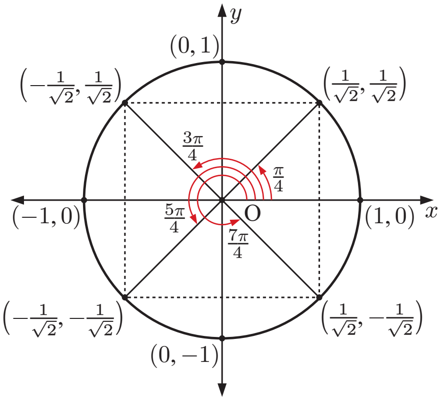
So, P is $\left(\frac{1}{\sqrt{2}},\, \frac{1}{\sqrt{2}}\right)$ where $\frac{1}{\sqrt{2}} \approx 0.707$.

$$\cos\tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}} \quad \text{and} \quad \sin\tfrac{\pi}{4} = \tfrac{1}{\sqrt{2}}$$

You should remember these values. If you forget, draw a right angled isosceles triangle with equal sides of length 1.



For **multiples of $\frac{\pi}{4}$**, we have:



MULTIPLES OF $\frac{\pi}{6}$ OR $30°$

Since OA = OP, triangle OAP is isosceles.

The remaining angles are therefore also $60°$, and so triangle AOP is equilateral.

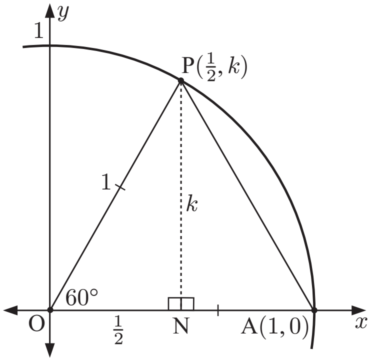
The altitude [PN] bisects base [OA], so ON $= \frac{1}{2}$.

$$\text{If P is } (\tfrac{1}{2}, k), \text{ then } (\tfrac{1}{2})^2 + k^2 = 1 \quad \{\text{Pythagoras}\}$$

$$\therefore \ k^2 = \tfrac{3}{4}$$

$$\therefore \ k = \tfrac{\sqrt{3}}{2} \quad \{\text{as } k > 0\}$$

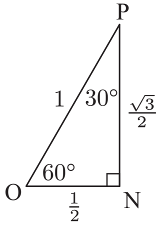
So, P is $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ where $\frac{\sqrt{3}}{2} \approx 0.866$.



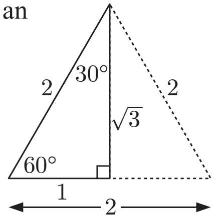
$$\boldsymbol{\cos\tfrac{\pi}{3} = \tfrac{1}{2} \quad \text{and} \quad \sin\tfrac{\pi}{3} = \tfrac{\sqrt{3}}{2}}$$

Now $\widehat{\text{NPO}} = \frac{\pi}{6} = 30°$.

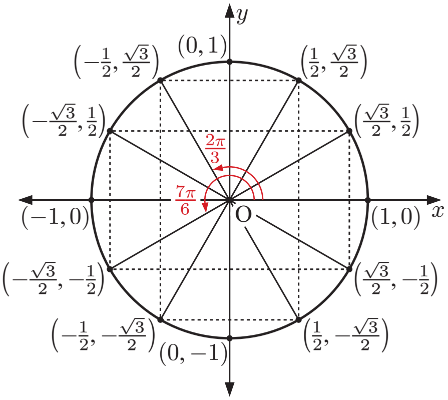
Hence $$\boldsymbol{\cos\tfrac{\pi}{6} = \tfrac{\sqrt{3}}{2} \quad \text{and} \quad \sin\tfrac{\pi}{6} = \tfrac{1}{2}}$$



You should remember these values. If you forget, divide in two an equilateral triangle with side length 2.



For **multiples of** $\frac{\pi}{6}$, we have:



Summary

- For **multiples of** $\frac{\pi}{2}$, the coordinates of the points on the unit circle involve 0 and ± 1.
- For *other* **multiples of** $\frac{\pi}{4}$, the coordinates involve $\pm\frac{1}{\sqrt{2}}$.
- For *other* **multiples of** $\frac{\pi}{6}$, the coordinates involve $\pm\frac{1}{2}$ and $\pm\frac{\sqrt{3}}{2}$.
- The signs of the coordinates are determined by which quadrant the angle is in.

You should be able to use this summary to find the trigonometric ratios for angles which are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$.

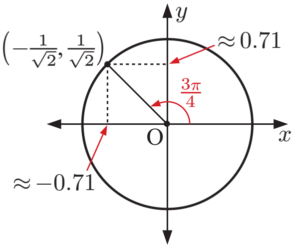
Example 12

Self Tutor

Find the exact values of $\sin\alpha$, $\cos\alpha$, and $\tan\alpha$ for:

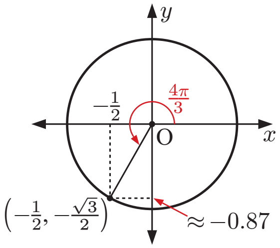
a $\alpha = \frac{3\pi}{4}$ **b** $\alpha = \frac{4\pi}{3}$

a $\frac{3\pi}{4}$ is a multiple of $\frac{\pi}{4}$.
The angle lies in quadrant 2, so only $\sin\frac{3\pi}{4}$ is positive.



$$\sin\left(\tfrac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}$$
$$\cos\left(\tfrac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$
$$\tan\left(\tfrac{3\pi}{4}\right) = -1$$

b $\frac{4\pi}{3}$ is a multiple of $\frac{\pi}{6}$.
The angle lies in quadrant 3, so only $\tan\frac{4\pi}{3}$ is positive.



$$\sin\left(\tfrac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$
$$\cos\left(\tfrac{4\pi}{3}\right) = -\frac{1}{2}$$
$$\tan\left(\tfrac{4\pi}{3}\right) = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$$

EXERCISE 8E

1 Use a unit circle diagram to find exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$, for θ equal to:

a $\frac{\pi}{4}$ **b** $\frac{\pi}{2}$ **c** $\frac{7\pi}{4}$ **d** π **e** $\frac{-3\pi}{4}$

2 Use a unit circle diagram to find exact values for $\sin\beta$, $\cos\beta$, and $\tan\beta$, for β equal to:

a $\frac{\pi}{6}$ **b** $\frac{2\pi}{3}$ **c** $\frac{7\pi}{6}$ **d** $\frac{5\pi}{3}$ **e** $\frac{11\pi}{6}$

3 Find the exact values of:

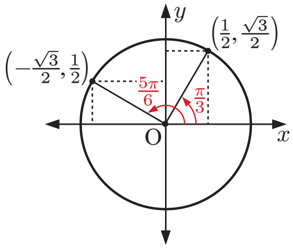
a $\cos 120°$, $\sin 120°$, and $\tan 120°$ **b** $\cos(-45°)$, $\sin(-45°)$, and $\tan(-45°)$

4 **a** Find the exact values of $\cos 270°$ and $\sin 270°$.
b What can you say about $\tan 270°$?

Example 13

Self Tutor

Without using a calculator, show that $8\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{5\pi}{6}\right) = -6$.



$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

$$\therefore\ 8\sin\left(\tfrac{\pi}{3}\right)\cos\left(\tfrac{5\pi}{6}\right) = 8\left(\tfrac{\sqrt{3}}{2}\right)\left(-\tfrac{\sqrt{3}}{2}\right)$$
$$= 2(-3)$$
$$= -6$$

5 Without using a calculator, evaluate:

a $\sin^2 60°$ **b** $\sin 30° \cos 60°$ **c** $4 \sin 60° \cos 30°$

d $1 - \cos^2(\frac{\pi}{6})$ **e** $\sin^2(\frac{2\pi}{3}) - 1$ **f** $\cos^2(\frac{\pi}{4}) - \sin(\frac{7\pi}{6})$

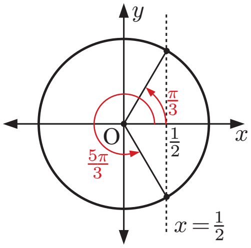
g $\sin(\frac{3\pi}{4}) - \cos(\frac{5\pi}{4})$ **h** $1 - 2\sin^2(\frac{7\pi}{6})$ **i** $\cos^2(\frac{5\pi}{6}) - \sin^2(\frac{5\pi}{6})$

j $\tan^2(\frac{\pi}{3}) - 2\sin^2(\frac{\pi}{4})$ **k** $2\tan(-\frac{5\pi}{4}) - \sin(\frac{3\pi}{2})$ **l** $\dfrac{2\tan 150°}{1 - \tan^2 150°}$

Check your answers using your calculator.

Example 14 Self Tutor

Find all angles $0 \leqslant \theta \leqslant 2\pi$ with a cosine of $\frac{1}{2}$.



Since the cosine is $\frac{1}{2}$, we draw the vertical line $x = \frac{1}{2}$.

Because $\frac{1}{2}$ is involved, we know the required angles are multiples of $\frac{\pi}{6}$.

They are $\frac{\pi}{3}$ and $\frac{5\pi}{3}$.

6 Find all angles between 0° and 360° with:

a a sine of $\frac{1}{2}$ **b** a sine of $\frac{\sqrt{3}}{2}$ **c** a cosine of $\frac{1}{\sqrt{2}}$

d a cosine of $-\frac{1}{2}$ **e** a cosine of $-\frac{1}{\sqrt{2}}$ **f** a sine of $-\frac{\sqrt{3}}{2}$

7 Find all angles between 0 and 2π (inclusive) which have:

a a tangent of 1 **b** a tangent of -1 **c** a tangent of $\sqrt{3}$

d a tangent of 0 **e** a tangent of $\frac{1}{\sqrt{3}}$ **f** a tangent of $-\sqrt{3}$

8 Find all angles between 0 and 4π with:

a a cosine of $\frac{\sqrt{3}}{2}$ **b** a sine of $-\frac{1}{2}$ **c** a sine of -1

9 Find θ if $0 \leqslant \theta \leqslant 2\pi$ and:

a $\cos\theta = \frac{1}{2}$ **b** $\sin\theta = \frac{\sqrt{3}}{2}$ **c** $\cos\theta = -1$ **d** $\sin\theta = 1$

e $\cos\theta = -\frac{1}{\sqrt{2}}$ **f** $\sin^2\theta = 1$ **g** $\cos^2\theta = 1$ **h** $\cos^2\theta = \frac{1}{2}$

i $\tan\theta = -\frac{1}{\sqrt{3}}$ **j** $\tan^2\theta = 3$

10 Find *all* values of θ for which $\tan\theta$ is: **a** zero **b** undefined.

F RECIPROCAL TRIGONOMETRIC RATIOS

We define the reciprocal trigonometric functions cosec θ, secant θ, and cotangent θ as:

$$\mathbf{cosec}\,\boldsymbol{\theta} = \frac{\mathbf{1}}{\mathbf{sin}\,\boldsymbol{\theta}}, \qquad \mathbf{sec}\,\boldsymbol{\theta} = \frac{\mathbf{1}}{\mathbf{cos}\,\boldsymbol{\theta}}, \qquad \text{and} \qquad \mathbf{cot}\,\boldsymbol{\theta} = \frac{\mathbf{1}}{\mathbf{tan}\,\boldsymbol{\theta}} = \frac{\mathbf{cos}\,\boldsymbol{\theta}}{\mathbf{sin}\,\boldsymbol{\theta}}$$

Using these definitions we can derive the identities:

$$\mathbf{tan^2}\,\boldsymbol{\theta} + \mathbf{1} = \mathbf{sec^2}\,\boldsymbol{\theta} \qquad \text{and} \qquad \mathbf{1} + \mathbf{cot^2}\,\boldsymbol{\theta} = \mathbf{cosec^2}\,\boldsymbol{\theta}$$

Proof: Using $\sin^2\theta + \cos^2\theta = 1$,

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \qquad \{\text{dividing each term by } \cos^2\theta\}$$

$$\therefore\ \tan^2\theta + 1 = \sec^2\theta$$

Also using $\sin^2\theta + \cos^2\theta = 1$,

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \qquad \{\text{dividing each term by } \sin^2\theta\}$$

$$\therefore\ 1 + \cot^2\theta = \text{cosec}^2\,\theta$$

EXERCISE 8F

1 Without using a calculator, find:

a $\text{cosec}\left(\frac{\pi}{3}\right)$ **b** $\cot\left(\frac{2\pi}{3}\right)$ **c** $\sec\left(\frac{5\pi}{6}\right)$ **d** $\cot(\pi)$

e $\text{cosec}\left(\frac{4\pi}{3}\right)$ **f** $\sec\left(\frac{7\pi}{4}\right)$

2 Without using a calculator, find $\text{cosec}\,x$, $\sec x$, and $\cot x$ for:

a $\sin x = \frac{3}{5}$, $0 \leqslant x \leqslant \frac{\pi}{2}$ **b** $\cos x = \frac{2}{3}$, $\frac{3\pi}{2} < x < 2\pi$

3 Find the other *five* trigonometric ratios if:

a $\cos\theta = \frac{3}{4}$ and $\frac{3\pi}{2} < \theta < 2\pi$ **b** $\sin x = -\frac{2}{3}$ and $\pi < x < \frac{3\pi}{2}$

c $\sec x = 2\frac{1}{2}$ and $0 < x < \frac{\pi}{2}$ **d** $\text{cosec}\,\theta = 2$ and $\frac{\pi}{2} < \theta < \pi$

e $\tan\beta = \frac{1}{2}$ and $\pi < \beta < \frac{3\pi}{2}$ **f** $\cot\theta = \frac{4}{3}$ and $\pi < \theta < \frac{3\pi}{2}$

4 Find *all* values of θ for which:

a $\text{cosec}\,\theta$ is undefined **b** $\sec\theta$ is undefined

c $\cot\theta$ is zero **d** $\cot\theta$ is undefined.

Review set 8A

1 Convert these to radians in terms of π:

a 120° **b** 225° **c** 150° **d** 540°

2 Find the acute angles that would have the same:

a sine as $\frac{2\pi}{3}$ **b** sine as 165° **c** cosine as 276°.

3 Find:

a $\sin 159°$ if $\sin 21° \approx 0.358$ **b** $\cos 92°$ if $\cos 88° \approx 0.035$

c $\cos 75°$ if $\cos 105° \approx -0.259$ **d** $\sin(-133°)$ if $\sin 47° \approx 0.731$

4 Determine the area of a sector of angle $\frac{5\pi}{12}$ and radius 13 cm.

5 Use the unit circle to find θ such that $\cos\theta = -\sin\theta$, $0 \leqslant \theta \leqslant 2\pi$.

6 Find exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$ for θ equal to:

a $360°$ **b** $\frac{2\pi}{3}$ **c** $-\pi$ **d** $\frac{8\pi}{3}$

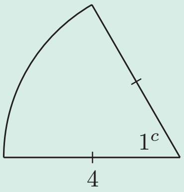
7 If $\cos\theta = \frac{3}{4}$ find the possible values of $\sin\theta$.

8 Evaluate:

a $2\sin(\frac{\pi}{3})\cos(\frac{\pi}{3})$ **b** $\tan^2(\frac{\pi}{4}) - 1$ **c** $\cos^2(\frac{\pi}{6}) - \sin^2(\frac{\pi}{6})$

9 Given $\tan x = -\frac{3}{2}$ and $\frac{3\pi}{2} < x < 2\pi$, find: **a** $\cos x$ **b** $\sin x$.

10



Find the perimeter and area of the sector.

11 Suppose $\cos\theta = \frac{\sqrt{11}}{\sqrt{17}}$ and θ is acute. Find the exact value of $\tan\theta$.

12 Find all angles between 0° and 360° which have:

a a cosine of $-\frac{\sqrt{3}}{2}$ **b** a secant of $\sqrt{2}$ **c** a cotangent of $-\frac{1}{\sqrt{3}}$

13 Find θ for $0 \leqslant \theta \leqslant 2\pi$ if:

a $\cos\theta = -1$ **b** $\sin^2\theta = \frac{3}{4}$

14 If $\sin x = -\frac{1}{4}$ and $\pi < x < \frac{3\pi}{2}$, find the other *five* trigonometric ratios exactly.

Review set 8B

1 Convert these radian measurements to degrees:

a $\frac{2\pi}{5}$ **b** $\frac{5\pi}{4}$ **c** $\frac{7\pi}{9}$ **d** $\frac{11\pi}{6}$

2 Illustrate the regions where $\sin\theta$ and $\cos\theta$ have the same sign.

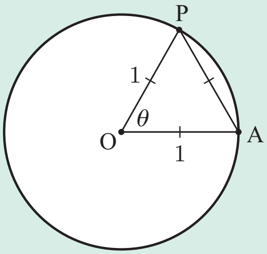
3 Use a unit circle diagram to find:

a $\cos(\frac{3\pi}{2})$ and $\sin(\frac{3\pi}{2})$ **b** $\cos(-\frac{\pi}{2})$ and $\sin(-\frac{\pi}{2})$

4 Suppose $m = \sin p$, where p is acute. Write an expression in terms of m for:

a $\sin(\pi - p)$ **b** $\sin(p + 2\pi)$ **c** $\cos p$ **d** $\tan p$

5



a State the value of θ in:
  i degrees
  ii radians.

b State the arc length AP.

c State the area of the minor *sector* OAP.

6 Show that $\cos(\frac{3\pi}{4}) - \sin(\frac{3\pi}{4}) = -\sqrt{2}$.

7 If $\cos\theta = -\frac{2}{5}$, $\frac{\pi}{2} < \theta < \pi$ find the other *five* trigonometric ratios exactly.

8 Without using a calculator, evaluate:

a $\tan^2 60° - \sin^2 45°$ **b** $\cos^2(\frac{\pi}{4}) + \sin(\frac{\pi}{2})$ **c** $\cos(\frac{5\pi}{3}) - \tan(\frac{5\pi}{4})$

9 Find two angles on the unit circle with $0 \leqslant \theta \leqslant 2\pi$, such that:

a $\cos\theta = \frac{2}{3}$ **b** $\sin\theta = -\frac{1}{4}$ **c** $\tan\theta = 3$

10 Find the perimeter and area of a sector of radius 11 cm and angle 63°.

11 Find the radius and area of a sector of perimeter 36 cm with an angle of $\frac{2\pi}{3}$.

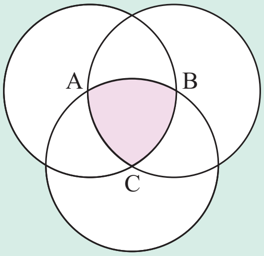
12 Simplify:

a $\sin(\pi - \theta) - \sin\theta$ **b** $\cos\theta\tan\theta$

13 If $\sec\alpha = -3\frac{1}{3}$ and $0 < \alpha < \pi$, find the other *five* trigonometric ratios exactly.

14 Three circles with radius r are drawn as shown, each with its centre on the circumference of the other two circles. A, B, and C are the centres of the three circles.

Prove that an expression for the area of the shaded region is $A = \frac{r^2}{2}(\pi - \sqrt{3})$.



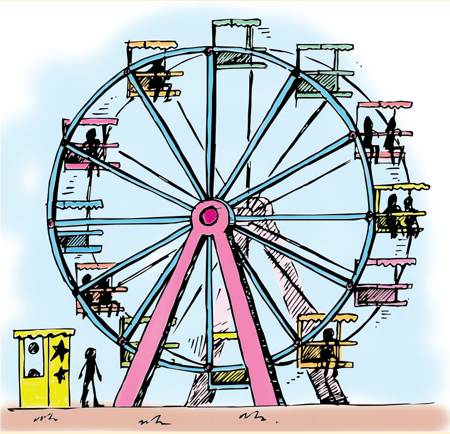
9

Trigonometric functions

Contents:

Opening problem

A Ferris wheel rotates at a constant speed. The wheel's radius is 10 m and the bottom of the wheel is 2 m above ground level. From a point in front of the wheel, Andrew is watching a green light on the perimeter of the wheel. Andrew notices that the green light moves in a circle. He estimates how high the light is above ground level at two second intervals, and draws a scatter diagram of his results.



Things to think about:

a What will Andrew's scatter diagram look like?

b What function can be used to model the data?

c How could this function be used to find:

  i the light's position at any point in time

  ii the times when the light is at its maximum and minimum heights?

d What part of the function indicates the time for one full revolution of the wheel?

Click on the icon to visit a simulation of the Ferris wheel. You will be able to view the light from:

DEMO

- in front of the wheel
- a side-on position
- above the wheel.

You can then observe the graph of the green light's position as the wheel rotates at a constant rate.

A PERIODIC BEHAVIOUR

Periodic phenomena occur all the time in the physical world. Their behaviour repeats again and again over time.

We see periodic behaviour in:

- seasonal variations in our climate
- variations in average maximum and minimum monthly temperatures
- the number of daylight hours at a particular location
- tidal variations in the depth of water in a harbour
- the phases of the moon
- animal populations.

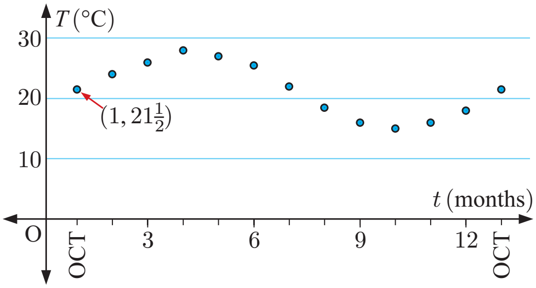
In this chapter we will see how trigonometric functions can be used to model periodic phenomena.

OBSERVING PERIODIC BEHAVIOUR

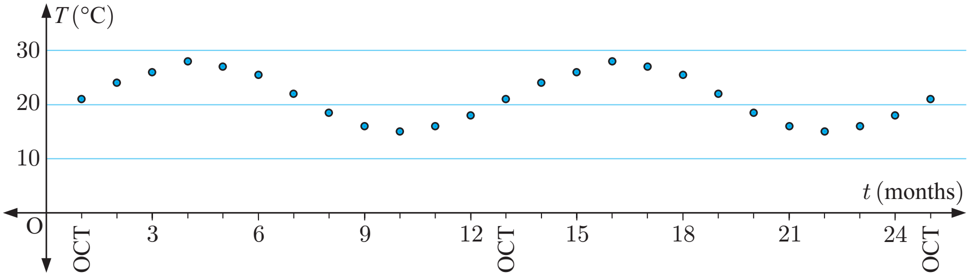
The table below shows the mean monthly maximum temperature for Cape Town, South Africa.

Month	Oct	Nov	Dec	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep
Temperature T (°C)	$21\frac{1}{2}$	24	26	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18

On the scatter diagram alongside we plot the temperature T on the vertical axis. We assign October as $t = 1$ month, November as $t = 2$ months, and so on for the rest of the year.

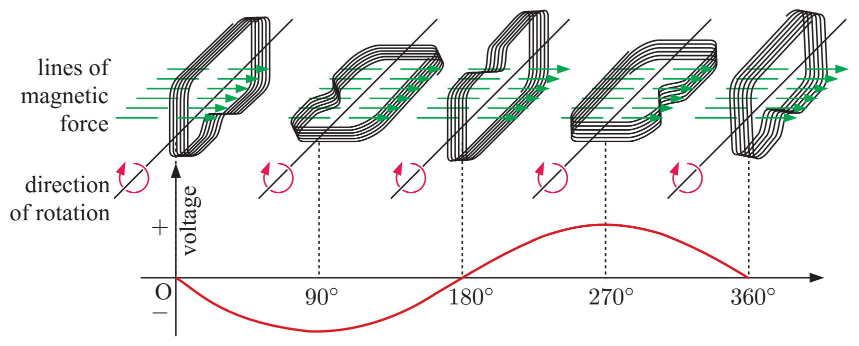


The cycle will approximately repeat itself for each subsequent 12 month period. By the end of the chapter we will be able to establish a **periodic function** which approximately fits this set of points.



Graphs with this basic shape, where the cycle is repeated over and over, are called **sine waves**.

Historical note



In 1831 **Michael Faraday** discovered that an electric current was generated by rotating a coil of wire in a magnetic field. The electric current produced showed a voltage which varied between positive and negative values as the coil rotated through 360°.

GATHERING PERIODIC DATA

Data on a number of periodic phenomena can be found online or in other publications. For example:

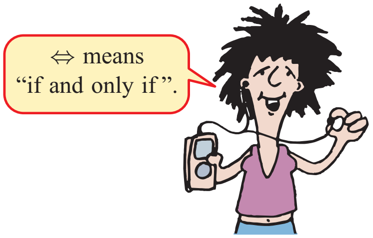
- Maximum and minimum monthly temperatures can be found at www.weatherbase.com
- Tidal details can be obtained from daily newspapers or internet sites such as http://tidesandcurrents.noaa.gov or http://www.bom.gov.au/oceanography

TERMINOLOGY USED TO DESCRIBE PERIODICITY

A **periodic function** is one which repeats itself over and over in a horizontal direction, in intervals of the same length.

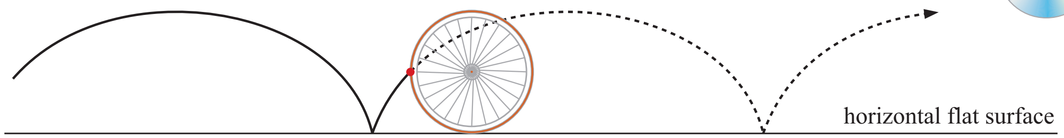
The **period** of a periodic function is the length of one repetition or cycle.

$f(x)$ is a periodic function with period $p \Leftrightarrow \boldsymbol{f(x+p) = f(x)}$ for all x, and p is the smallest positive value for this to be true.



A **cycloid** is an example of a periodic function. It is the curve traced out by a point on a circle as the circle rolls across a flat surface in a straight line.

DEMO



Use a **graphing package** to examine the function $f(x) = x - [x]$ where $[x]$ is "the largest integer less than or equal to x".
Is $f(x)$ periodic? What is its period?

GRAPHING PACKAGE

WAVES

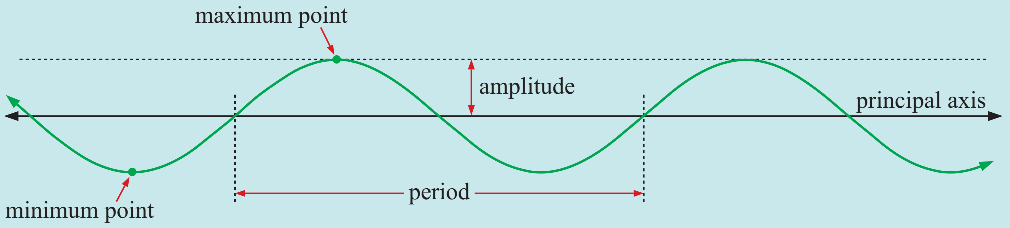
In this course we are mainly concerned with periodic phenomena which show a wave pattern.

The wave oscillates about a horizontal line called the **principal axis** or **mean line** which has equation $\boldsymbol{y = \dfrac{\textbf{max} + \textbf{min}}{2}}$.

A **maximum point** occurs at the top of a crest, and a **minimum point** at the bottom of a trough.

The **amplitude** is the distance between a maximum (or minimum) point and the principal axis.

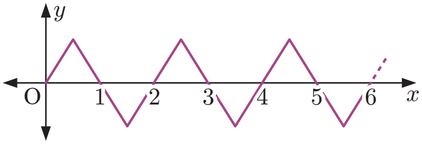
$$\textbf{amplitude} = \frac{\textbf{max} - \textbf{min}}{2}$$



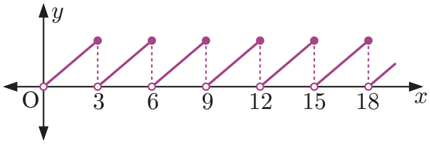
EXERCISE 9A

1 Which of these graphs show periodic behaviour?

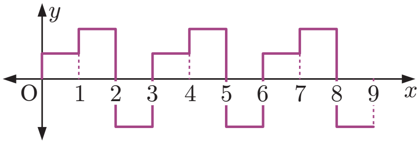
a



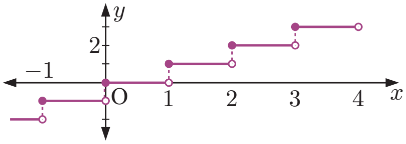
b



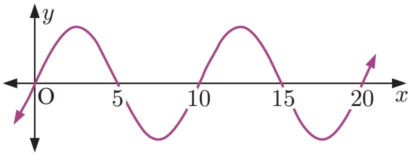
c



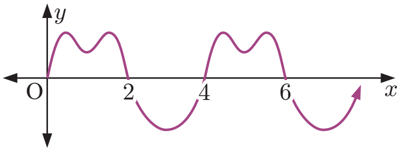
d



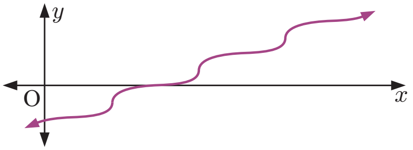
e



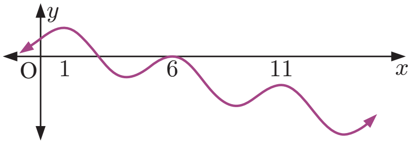
f



g



h



2 The table below shows the height above the ground of a point on a bicycle wheel as it is rolled along a flat surface.

Distance travelled (cm)	0	20	40	60	80	100	120	140	160	180	200
Height above ground (cm)	0	6	23	42	57	64	59	43	23	7	1

Distance travelled (cm)	220	240	260	280	300	320	340	360	380	400
Height above ground (cm)	5	27	40	55	63	60	44	24	9	3

a Plot the graph of height against distance.

b Is it reasonable to fit a curve to this data, or should we leave it as discrete points?

c Is the data periodic? If so, estimate:

- **i** the equation of the principal axis
- **ii** the maximum value
- **iii** the period
- **iv** the amplitude.

3 Draw a scatter diagram for each set of data below. Is there evidence to suggest the data is periodic?

a

x	0	1	2	3	4	5	6	7	8	9	10	11	12
y	0	1	1.4	1	0	−1	−1.4	−1	0	1	1.4	1	0

b

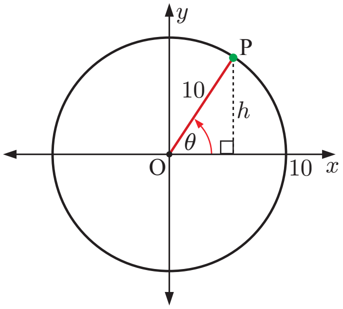
x	0	2	3	4	5	6	7	8	9	10	12
y	0	4.7	3.4	1.7	2.1	5.2	8.9	10.9	10.2	8.4	10.4

B THE SINE FUNCTION

In previous studies of trigonometry we have only considered static situations where an angle is fixed. However, when an object moves around a circle, the situation is dynamic. The angle θ between the radius [OP] and the positive x-axis continually changes with time.

Consider again the **Opening Problem** in which a Ferris wheel of radius 10 m revolves at constant speed. We let P represent the green light on the wheel.

The height of P relative to the x-axis can be determined using right angled triangle trigonometry:

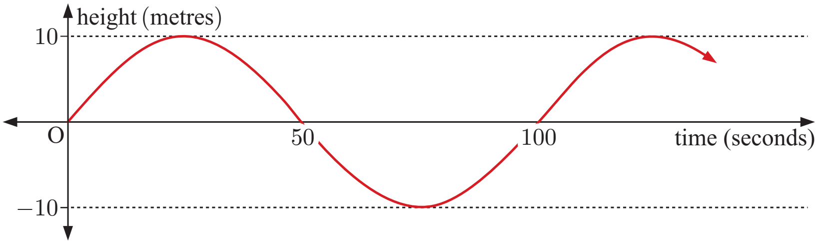


$$\sin\theta = \frac{h}{10}, \quad \text{so} \quad h = 10\sin\theta.$$

As time goes by, θ changes and so does h.

So, we can write h as a function of θ, or alternatively we can write h as a function of time t.

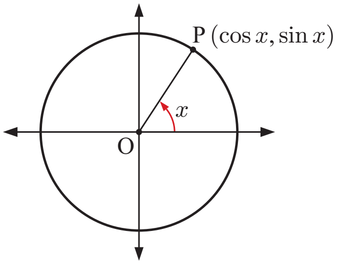
For example, suppose the Ferris wheel observed by Andrew takes 100 seconds for a full revolution. The graph below shows the height of the light above or below the principal axis against the time in seconds.



We observe that the amplitude is 10 metres and the period is 100 seconds.

THE BASIC SINE CURVE $y = \sin x$

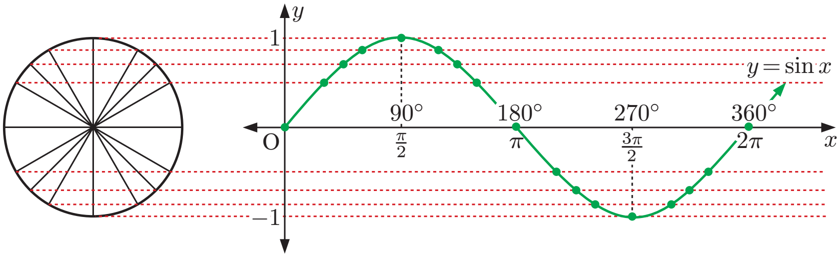
Suppose point P moves around the unit circle so the angle [OP] makes with the positive horizontal axis is x. In this case P has coordinates $(\cos x, \sin x)$.



If we project the values of $\sin x$ from the unit circle to a set of axes alongside, we can obtain the graph of $y = \sin x$.

Note carefully that x on the unit circle diagram is an *angle*, and becomes the horizontal coordinate of the sine function.

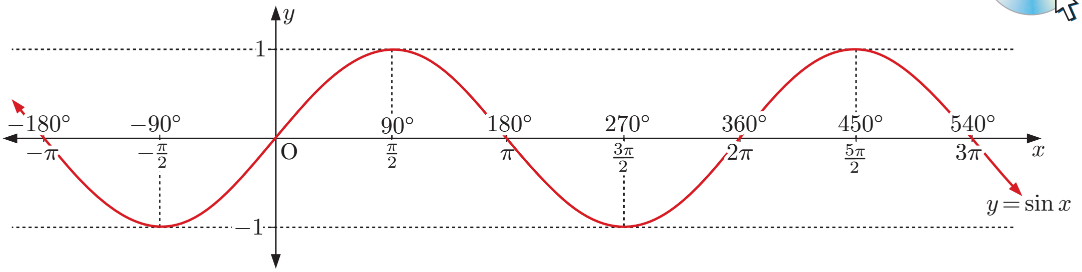
Unless indicated otherwise, you should assume that x is measured in radians. Degrees are only included on this graph for the sake of completeness.



SINE FUNCTION

Click on the icon to generate the sine function for yourself.

You should observe that the sine function can be continued beyond $0 \leqslant x \leqslant 2\pi$ in either direction.



The unit circle repeats itself after one full revolution, so the **period** of $y = \sin x$ is 2π.

The *maximum* value is 1 and the *minimum* is -1, as $-1 \leqslant y \leqslant 1$ on the unit circle.

The **amplitude** of $y = \sin x$ is 1.

TRANSFORMATIONS OF THE SINE CURVE

In the **Discoveries** that follow, we will consider different transformations of the sine curve $y = \sin x$. We will hence be able to generate the curve for the general sine function $y = a\sin bx + c$, $a > 0$, $b > 0$.

Discovery 1 The family $y = a\sin x$, $a > 0$

DYNAMIC SINE FUNCTION

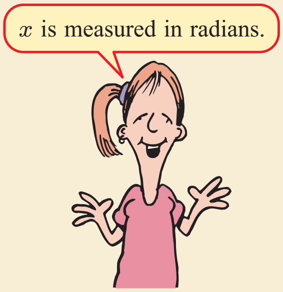
Click on the icon to explore the family $y = a\sin x$, $a > 0$.

What to do:

1 Use the slider to vary the value of a. Observe the changes to the graph of the function.

2 Use the software to help complete the table:

a	*Function*	*Maximum*	*Minimum*	*Period*	*Amplitude*
1	$y = \sin x$	1	-1	2π	1
2	$y = 2\sin x$				
3	$y = 3\sin x$				
0.5	$y = 0.5\sin x$				
a	$y = a\sin x$				



3 How does a affect the function $y = a\sin x$?

Discovery 2 The family $y = \sin bx$, $b > 0$

DYNAMIC SINE FUNCTION

Click on the icon to explore the family $y = \sin bx$, $b > 0$.

What to do:

1 Use the slider to vary the value of b. Observe the changes to the graph of the function.

2 Use the software to help complete the table:

b	*Function*	*Maximum*	*Minimum*	*Period*	*Amplitude*
1	$y = \sin x$	1	-1	2π	1
2	$y = \sin 2x$				
3	$y = \sin 3x$				
$\frac{1}{2}$	$y = \sin(\frac{1}{2}x)$				
b	$y = \sin bx$				

3 How does b affect the function $y = \sin bx$?

Discovery 3 *The family* $y = \sin x + c$

Click on the icon to explore the family $y = \sin x + c$.

DYNAMIC SINE FUNCTION

What to do:

1 Use the slider to vary the value of c. Observe the changes to the graph of the function.

2 Use the software to help complete the table:

d	*Function*	*Maximum*	*Minimum*	*Period*	*Amplitude*
0	$y = \sin x$	1	-1	2π	1
3	$y = \sin x + 3$				
-2	$y = \sin x - 2$				
d	$y = \sin x + c$				

3 How does c affect the function $y = \sin x + c$?

THE GENERAL SINE FUNCTION

The **general sine function** is

$$\boldsymbol{y = \underset{\substack{\uparrow \\ \text{affects} \\ \textbf{amplitude}}}{a} \sin \underset{\substack{\uparrow \\ \text{affects} \\ \textbf{period}}}{b}x + \underset{\substack{\uparrow \\ \text{affects} \\ \textbf{vertical translation}}}{c}} \quad \text{where} \quad a > 0, \; b > 0.$$

The **principal axis** of the general sine function is $\boldsymbol{y = c}$.

The **period** of the general sine function is $\boldsymbol{\frac{2\pi}{b}}$.

The **amplitude** of the general sine function is $\boldsymbol{a}$.

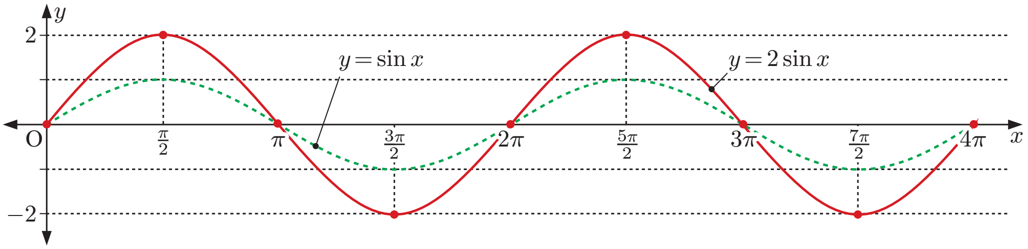
Example 1

Self Tutor

Without using technology, sketch the following graphs for $0 \leqslant x \leqslant 4\pi$:

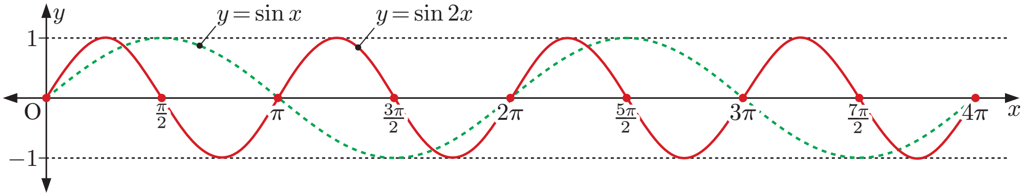
a $y = 2\sin x$ **b** $y = \sin 2x$ **c** $y = \sin x - 1$

a The amplitude is 2 and the period is 2π.



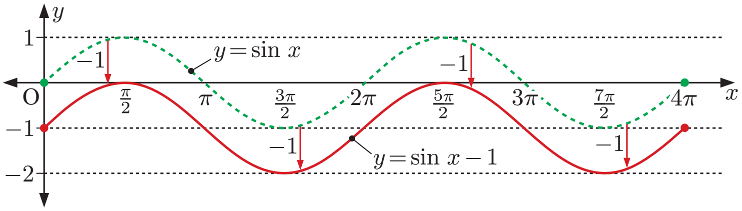
b The period is $\frac{2\pi}{2} = \pi$.

$\therefore$ the maximum values are π units apart.



Since sin $2x$ has half the period of sin x, the first maximum is at $\frac{\pi}{4}$ not $\frac{\pi}{2}$.

c This is a vertical translation of $y = \sin x$ downwards by 1 unit. The principal axis is now $y = -1$.



EXERCISE 9B

GRAPHING PACKAGE

1 Without using technology, sketch the following graphs for $0 \leqslant x \leqslant 4\pi$:

a $y = 3\sin x$ **b** $y = 4\sin x$ **c** $y = \sin 3x$

d $y = \sin 4x$ **e** $y = \sin x + 2$ **f** $y = \sin x - 3$

Check your answers using technology.

2 Find the value of a given that the function $y = a\sin x$, $a > 0$, has amplitude:

a 2 **b** 5 **c** 11

3 Find the value of b given that the function $y = \sin bx$, $b > 0$, has period:

a $\frac{2\pi}{3}$ **b** $\frac{2\pi}{5}$ **c** $\frac{\pi}{3}$ **d** $\frac{\pi}{2}$

4 Find the value of c given that the function $y = \sin x + c$ has principal axis:

a $y = 3$ **b** $y = -1$ **c** $y = 5$

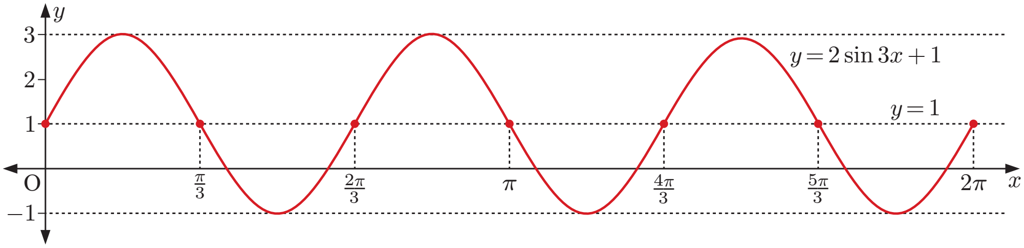
Example 2

Self Tutor

Without using technology, sketch $y = 2\sin 3x + 1$ for $0 \leqslant x \leqslant 2\pi$.

We start with $y = \sin x$. We then:

- double the amplitude to produce $y = 2\sin x$, then
- divide the period by 3 to produce $y = 2\sin 3x$, then
- translate the graph 1 unit upwards to produce $y = 2\sin 3x + 1$, so the principal axis is now $y = 1$.



5 Without using technology, sketch the following graphs for $0 \leqslant x \leqslant 2\pi$:

a $y = 3\sin x - 1$ **b** $y = 2\sin 3x$ **c** $y = \sin 2x + 3$

d $y = 3\sin 2x - 1$ **e** $y = 5\sin 2x + 3$ **f** $y = 4\sin 3x - 2$

Check your answers using technology.

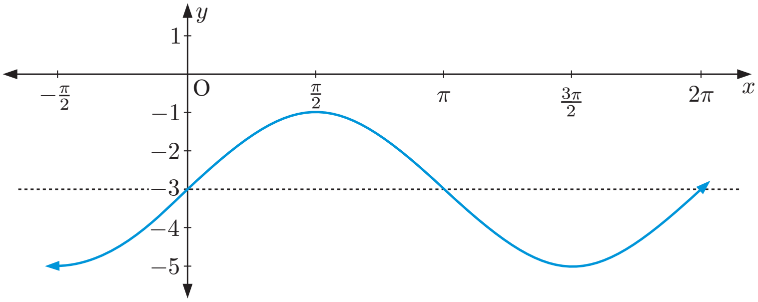
6 Find a, b, and c given that the function $y = a\sin bx + c$, $a > 0$, $b > 0$, has:

a amplitude 3, period 2π, and principal axis $y = 0$

b amplitude 2, period $\frac{2\pi}{5}$, and principal axis $y = 6$

c amplitude 5, period $\frac{2\pi}{3}$, and principal axis $y = -2$.

7 Find m and n given the following graph of the function $y = m\sin x + n$.



8 On the same set of axes, sketch for $0 \leqslant x \leqslant 2\pi$:

a $y = \sin x$ and $y = |\sin x|$ **b** $y = 3\sin 2x$ and $y = |3\sin 2x|$

Discovery 4 Modelling using sine functions

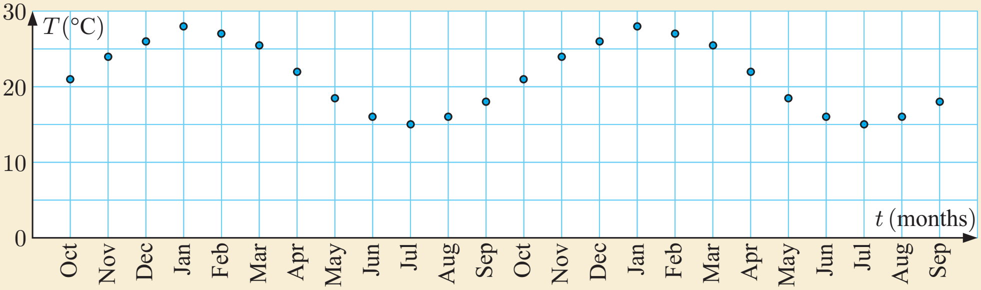
When patterns of variation can be identified and quantified using a formula or equation, predictions may be made about behaviour in the future. Examples of this include tidal movement which can be predicted many months ahead, and the date of a future full moon.

What to do:

1 Consider again the mean monthly maximum temperature for Cape Town:

Month	Oct	Nov	Dec	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep
Temperature T (°C)	$21\frac{1}{2}$	24	26	28	27	$25\frac{1}{2}$	22	$18\frac{1}{2}$	16	15	16	18

The graph over a two year period is shown below:



We attempt to model this data using the general sine function $y = a\sin bx + c$,
or in this case $T = a\sin bt + c$.

a State the period of the function. Hence show that $b = \frac{\pi}{6}$.

b Use the amplitude to show that $a \approx 6.5$.

c Use the principal axis to show that $c \approx 21.5$.

d Superimpose the model $T \approx 6.5\sin(\frac{\pi}{6}t) + 21.5$ on the original data to confirm its accuracy.

2 Some of the largest tides in the world are observed in Canada's Bay of Fundy. The difference between high and low tides is 14 metres, and the average time difference between high tides is about 12.4 hours.

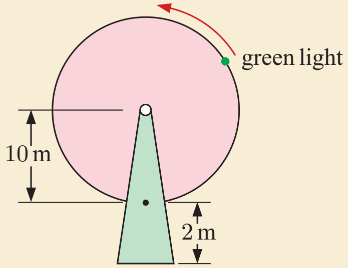
Suppose the mean tide occurs at midnight.

a Find a sine model for the height of the tide H in terms of the time t.

b Sketch the graph of the model over one period.

3 Revisit the **Opening Problem** on page **226**.

The wheel takes 100 seconds to complete one revolution. Find the sine model which gives the height of the light above the ground at any point in time. Assume that at time $t = 0$, the light is at its mean position.



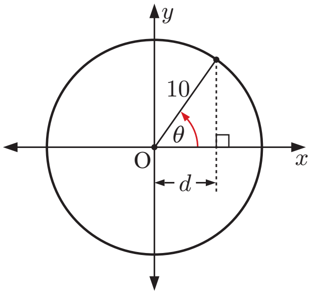
C THE COSINE FUNCTION

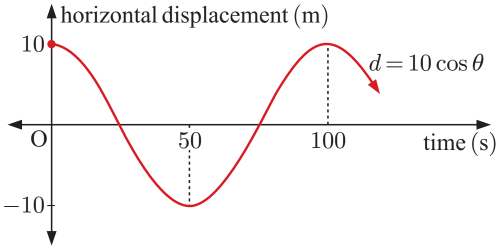
We return to the Ferris wheel and now view the movement of the green light from above.

Now $\cos\theta = \dfrac{d}{10}$ so $d = 10\cos\theta$.

The graph being generated over time is therefore a **cosine function**.

DEMO

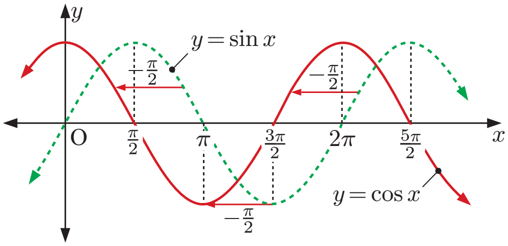




GRAPHING PACKAGE

Use the graphing package to graph $y = \cos x$ and $y = \sin x$ on the same set of axes.

Like the sine curve $y = \sin x$, the cosine curve $y = \cos x$ has a **period** of 2π, an **amplitude** of 1, and its **range** is $-1 \leqslant y \leqslant 1$.



You should observe that $y = \cos x$ and $y = \sin x$ are identical in shape, but the cosine function is $\frac{\pi}{2}$ units left of the sine function.

Use the graphing package to graph $y = \cos x$ and $y = \sin\left(x + \frac{\pi}{2}\right)$ on the same set of axes.

You should observe that $\boldsymbol{\cos x = \sin\left(x + \frac{\pi}{2}\right)}$.

THE GENERAL COSINE FUNCTION

The **general cosine function** is $y = a\cos bx + c$ where $a > 0$, $b > 0$.

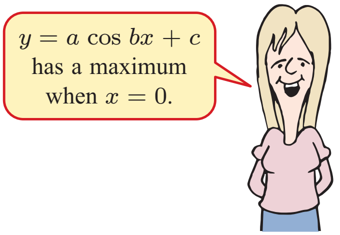
DYNAMIC COSINE FUNCTION

Since the cosine function is a horizontal translation of the sine function, the constants a, b, and c have the same effects as for the general sine function. Click on the icon to check this.

The **principal axis** of the general cosine function is $\boldsymbol{y = c}$.

The **period** of the general cosine function is $\boldsymbol{\dfrac{2\pi}{b}}$.

The **amplitude** of the general cosine function is $\boldsymbol{a}$.



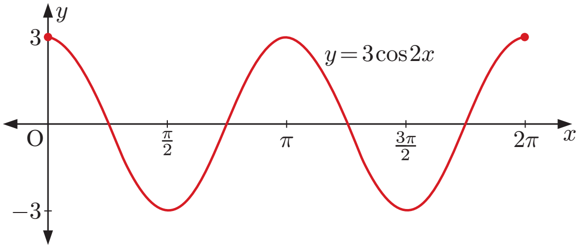
Example 3

Self Tutor

Without using technology, sketch the graph of $y = 3\cos 2x$ for $0 \leqslant x \leqslant 2\pi$.

$a = 3$, so the amplitude is 3.

$b = 2$, so the period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$.



EXERCISE 9C

1 Without using technology, sketch the following graphs for $0 \leqslant x \leqslant 2\pi$:

a $y = 3\cos x$
b $y = 5\cos x$
c $y = \cos 2x$
d $y = \cos 3x$
e $y = \cos x + 2$
f $y = \cos x - 1$
g $y = 2\cos 2x$
h $y = \cos 3x + 1$
i $y = 4\cos x + 10$
j $y = 2\cos 3x + 4$
k $y = 4\cos 2x - 2$
l $y = 3\cos 2x + 5$

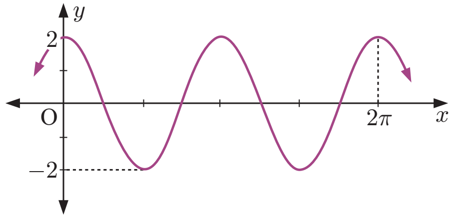
2 Find a, b, and c given that the function $y = a\cos bx + c$, $a > 0$, $b > 0$, has:

a amplitude 4, period $\frac{2\pi}{3}$, and principal axis $y = -1$

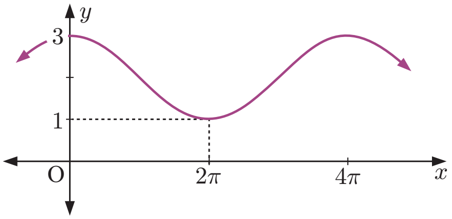
b amplitude 3, period $\frac{2\pi}{5}$, and principal axis $y = 3$.

3 Find the cosine function shown in the graph:

a



b



4 The function $y = a\cos bx + c$, $a > 0$, $b > 0$, has amplitude 5, period 2π, and principal axis $y = 1$.

a Find the values of a, b, and c.

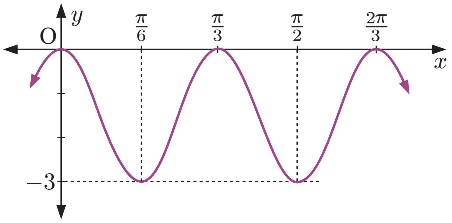
b Sketch the function for $0 \leqslant x \leqslant 2\pi$.

5 The graph shown has the form $y = a\cos bx + c$ where $a > 0$, $b > 0$.

a Find the values of a, b, and c.

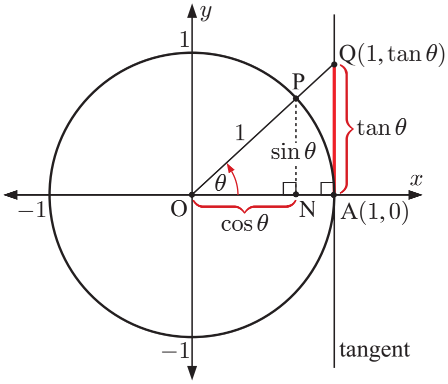
b Sketch the reflection of the function in the x-axis.

c Write down the equation of the reflection in **b**.



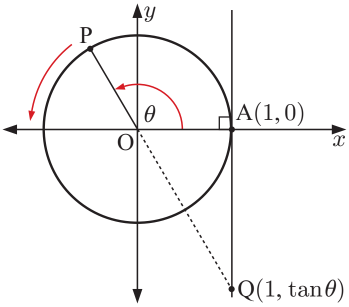
D THE TANGENT FUNCTION

We have seen that if $P(\cos\theta,\, \sin\theta)$ is a point which is free to move around the unit circle, and if [OP] is extended to meet the tangent at $A(1,\, 0)$, the intersection between these lines occurs at $Q(1,\, \tan\theta)$.



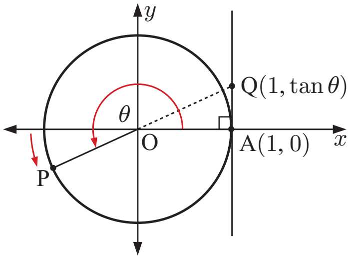
This enables us to define the **tangent function**

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}.}$$

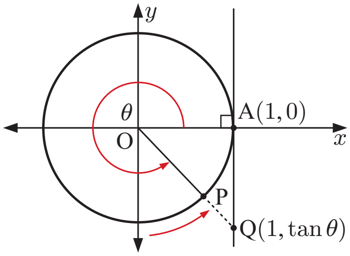


For θ in quadrant 2, $\sin\theta$ is positive and $\cos\theta$ is negative and so $\tan\theta = \frac{\sin\theta}{\cos\theta}$ is negative.

As before, [OP] is extended to meet the tangent at A at $Q(1,\, \tan\theta)$. We see that Q is below the x-axis.



For θ in quadrant 3, $\sin\theta$ and $\cos\theta$ are both negative and so $\tan\theta$ is positive. This is clearly demonstrated as Q is back above the x-axis.



For θ in quadrant 4, $\sin\theta$ is negative and $\cos\theta$ is positive. $\tan\theta$ is again negative. We see that Q is below the x-axis.

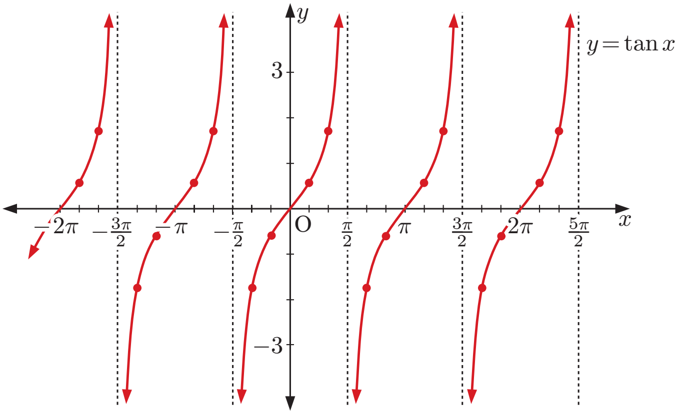
Discussion

What happens to $\tan\theta$ when P is at $(0,\, 1)$ and $(0,\, -1)$?

THE GRAPH OF $y = \tan x$

Since $\tan x = \dfrac{\sin x}{\cos x}$, $\tan x$ will be undefined whenever $\cos x = 0$.

The zeros of the function $y = \cos x$ correspond to vertical asymptotes of the function $y = \tan x$.



DEMO

We observe that $y = \tan x$ has:

- **period** π
- **range** $y \in \mathbb{R}$
- **vertical asymptotes** $x = \frac{\pi}{2} + k\pi$ for all $k \in \mathbb{Z}$.

TANGENT FUNCTION

Click on the icon to explore how the tangent function is produced from the unit circle.

THE GENERAL TANGENT FUNCTION

The **general tangent function** is $y = a \tan bx + c$, $a > 0$, $b > 0$.

- The **principal axis** is $y = c$.
- The **period** of this function is $\dfrac{\pi}{b}$.
- The **amplitude** of this function is undefined.

DYNAMIC TANGENT FUNCTION

Click on the icon to explore the properties of this function.

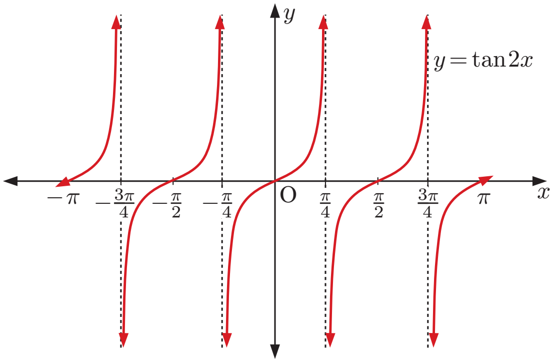
Example 4

Self Tutor

Without using technology, sketch the graph of $y = \tan 2x$ for $-\pi \leqslant x \leqslant \pi$.

Since $b = 2$, the period is $\frac{\pi}{2}$.

The vertical asymptotes are
$x = \pm\frac{\pi}{4}$, $x = \pm\frac{3\pi}{4}$,
and the x-axis intercepts are at
$0, \pm\frac{\pi}{2}, \pm\pi$.



Discussion

- Discuss how to find the x-intercepts of $y = \tan x$.
- How can we simplify $\tan(x - \pi)$?
- How many solutions does the equation $\tan x = 2$ have?

EXERCISE 9D

1 *Sketch* the following functions for $-\pi \leqslant x \leqslant \pi$:

GRAPHING PACKAGE

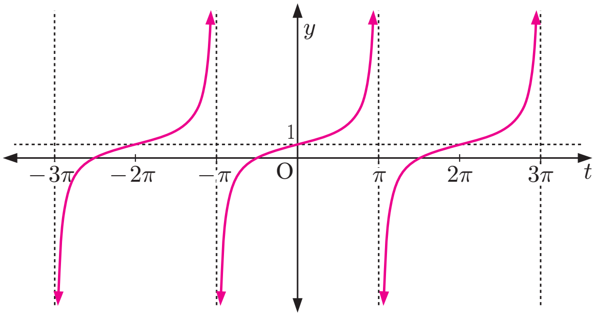
a $y = 2\tan x$
b $y = \tan 3x$
c $y = \tan x + 2$
d $y = 3\tan 2x$
e $y = 2\tan x - 1$
f $y = 2\tan 3x + 2$

Use technology to check your answers.

2 Find b and c given that the function $y = \tan bx + c$, $b > 0$, has:

a period $\frac{2\pi}{3}$ and principal axis $y = 2$
b period $\frac{\pi}{2}$ and principal axis $y = -3$.

3 Find p and q given the following graph of the function $y = \tan pt + q$.



Activity

Click on the icon to run a card game for trigonometric functions.

CARD GAME

E TRIGONOMETRIC EQUATIONS

Linear equations such as $2x + 3 = 11$ have exactly one solution. Quadratic equations of the form $ax^2 + bx + c = 0$, $a \neq 0$ have at most two real solutions.

Trigonometric equations generally have infinitely many solutions unless a restricted domain such as $0 \leqslant x \leqslant 3\pi$ is given.

For example, suppose that Andrew in the **Opening Problem** wants to know when the green light will be 16 metres above the ground. To find out, he will need to solve a trigonometric equation. If the wheel keeps rotating, the equation would have infinitely many solutions. Andrew may therefore specify that he is interested in the *first* time the green light is 16 metres above the ground.

If a periodic function $f(x)$ has period p then the domain $0 \leqslant x < p$ is called the **principal domain**. By solving an equation on the principal domain, all the other solutions can be found using the periodic behaviour.

If $x = a$ is a solution, then $x = a + kp$ will also be a solution for all $k \in \mathbb{Z}$.

For example, $\sin x$ has period 2π, so it is normal to consider the domain $0 \leqslant x < 2\pi$.

Discussion

What would you choose as the principal domain for:

- $y = \cos x$
- $y = \sin(2x)$
- $y = \tan x$?

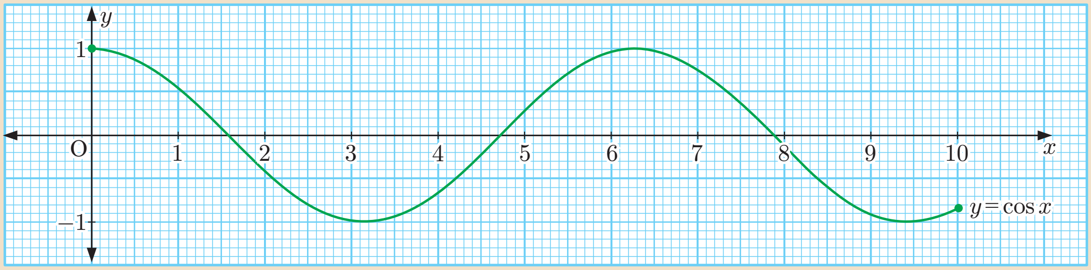
GRAPHICAL SOLUTION OF TRIGONOMETRIC EQUATIONS

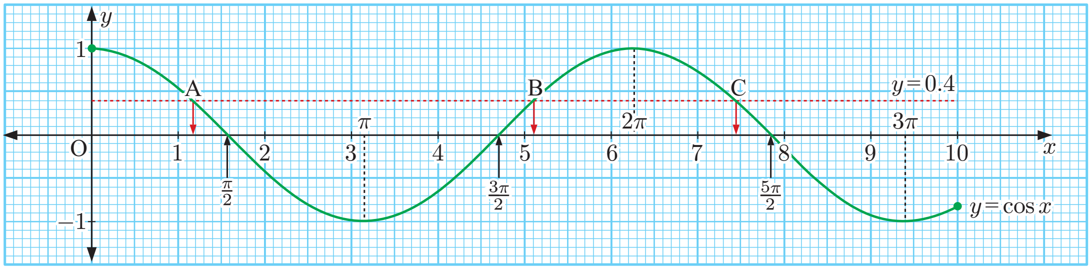
Sometimes simple trigonometric graphs are available on grid paper. In such cases we can estimate solutions straight from the graph.

Example 5

Self Tutor

Solve $\cos x = 0.4$ for $0 \leqslant x \leqslant 10$ radians using the graph of $y = \cos x$.





$y = 0.4$ meets $y = \cos x$ at A, B, and C. Hence $x \approx 1.2$, 5.1, or 7.4.

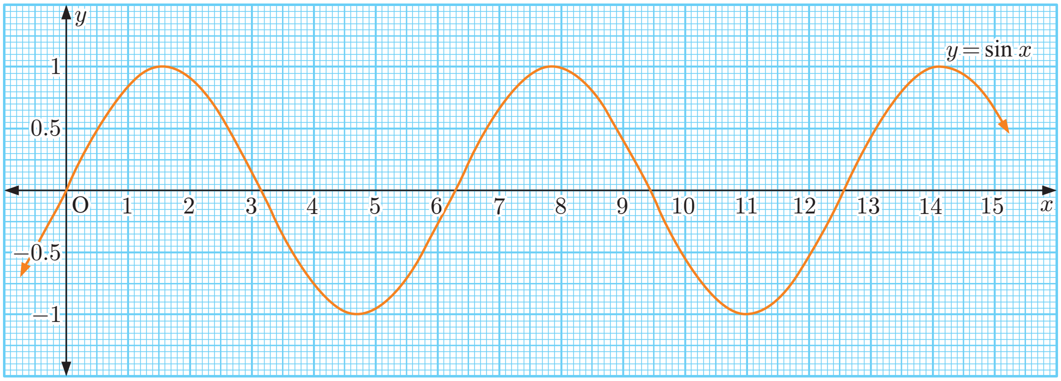
The solutions of $\cos x = 0.4$ for $0 \leqslant x \leqslant 10$ radians are 1.2, 5.1, and 7.4.

Trigonometric equations may also be solved using the graphing package.

GRAPHING PACKAGE

EXERCISE 9E.1

1

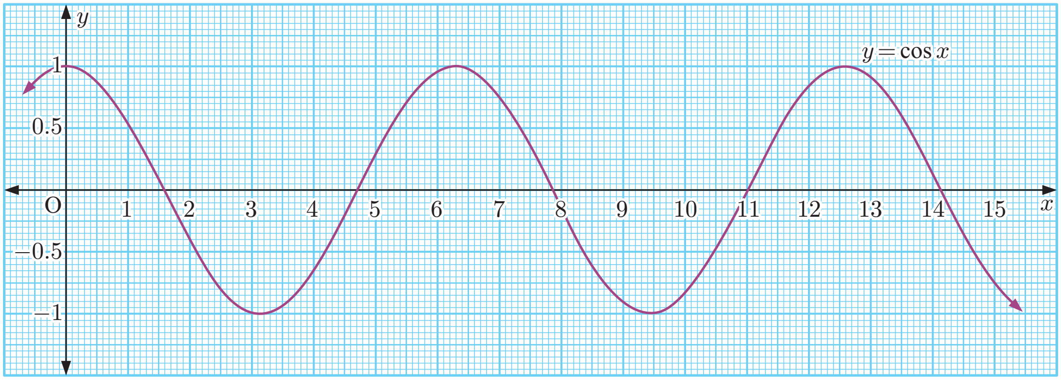


Use the graph of $y = \sin x$ to find, correct to 1 decimal place, the solutions of:

a $\sin x = 0.3$ for $0 \leqslant x \leqslant 15$

b $\sin x = -0.4$ for $5 \leqslant x \leqslant 15$.

2

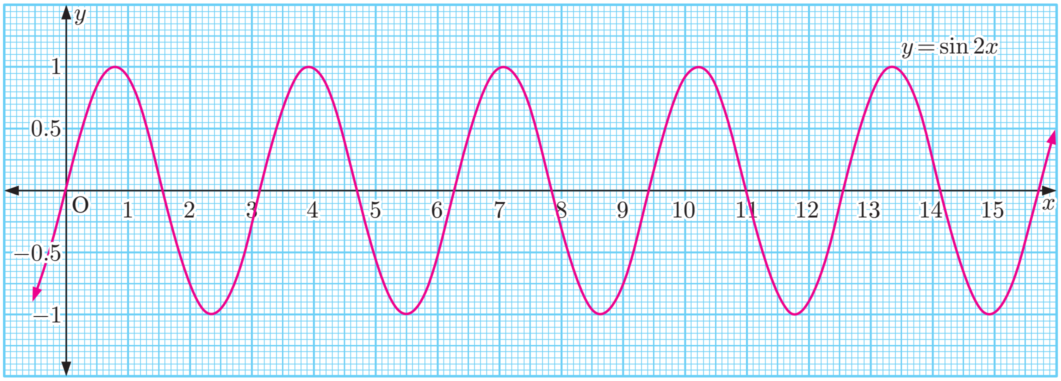


Use the graph of $y = \cos x$ to find, correct to 1 decimal place, the solutions of:

a $\cos x = 0.6$ for $0 \leqslant x \leqslant 10$

b $\cos x = -0.3$ for $4 \leqslant x \leqslant 12$.

3

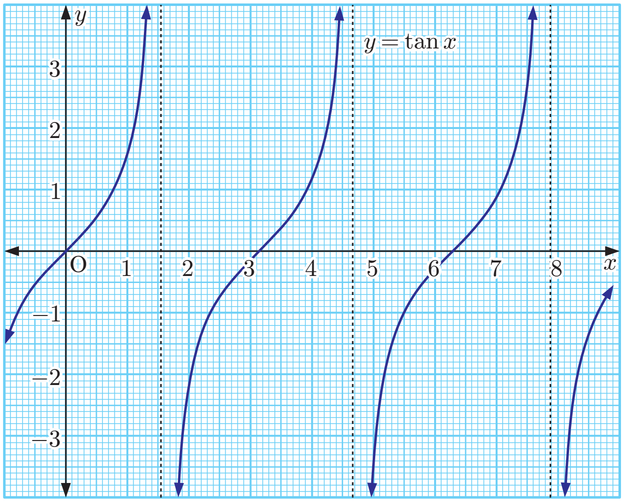


Use the graph of $y = \sin 2x$ to find, correct to 1 decimal place, the solutions of:

a $\sin 2x = 0.7$ for $0 \leqslant x \leqslant 16$

b $\sin 2x = -0.3$ for $0 \leqslant x \leqslant 16$.

4



The graph of $y = \tan x$ is illustrated.

GRAPHING PACKAGE

a Use the graph to estimate: **i** $\tan 1$ **ii** $\tan 2.3$

Check your answers using a calculator.

b Find, correct to 1 decimal place, the solutions of:

i $\tan x = 2$ for $0 \leqslant x \leqslant 8$ **ii** $\tan x = -1.4$ for $2 \leqslant x \leqslant 7$.

5 Use the graphing package to solve for x on the domain $0 < x < 4\pi$:

a $\sin x = 0.431$ **b** $\cos x = -0.814$ **c** $3\tan x - 2 = 0$

6 Use the graphing package to solve for x on the domain $-5 \leqslant x \leqslant 5$:

a $5\cos x - 4 = 0$ **b** $2\tan x + 13 = 0$ **c** $8\sin x + 3 = 0$

7 **a** Use the graphing package to solve $\sin^2 x + \sin x - 2 = 0$ for $0 \leqslant x \leqslant 2\pi$.

b Solve for m: $m^2 + m - 2 = 0$.

c Hence explain your answer in **a**.

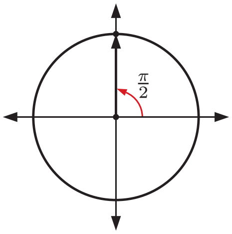
SOLVING TRIGONOMETRIC EQUATIONS USING ALGEBRA

Using a graph we get approximate decimal or **numerical** solutions to trigonometric equations.

Sometimes exact solutions are needed in terms of π, and these arise when the solutions are multiples of $\frac{\pi}{6}$ or $\frac{\pi}{4}$. Exact solutions obtained using algebra are called **analytical** solutions.

We use the periodicity of the trigonometric functions to give us all solutions in the required domain.

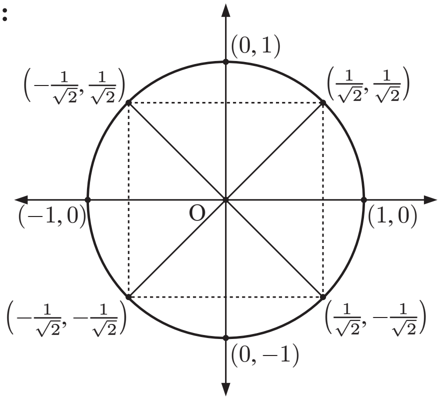
For example, consider $\sin x = 1$. We know from the unit circle that a solution is $x = \frac{\pi}{2}$. However, since the period of $\sin x$ is 2π, there are infinitely many solutions spaced 2π apart.

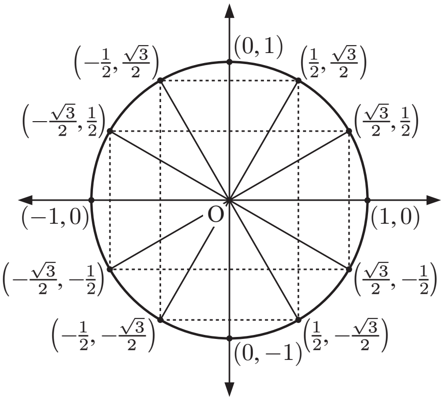


Hence $x = \frac{\pi}{2} + k2\pi$ is a solution for any $k \in \mathbb{Z}$.

In this course we will be solving equations on a fixed domain. This means there will be a finite number of solutions.

Reminder:





Example 6

Self Tutor

Solve for x: $2\sin x - 1 = 0, \quad 0 \leqslant x \leqslant \pi$

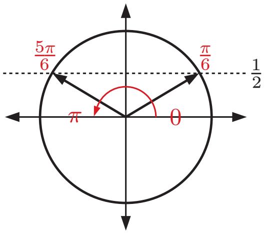
$2\sin x - 1 = 0$

$\therefore \quad \sin x = \frac{1}{2}$

There are two points on the unit circle with sine $\frac{1}{2}$.

They correspond to angles $\frac{\pi}{6}$ and $\frac{5\pi}{6}$.

These are the only solutions on the domain $0 \leqslant x \leqslant \pi$, so $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$.



Since the tangent function is periodic with period π we see that $\tan(x + \pi) = \tan x$ for all values of x. This means that equal tan values are π units apart.

Example 7

Self Tutor

Solve $\tan x + \sqrt{3} = 0$ for $0 < x < 4\pi$.

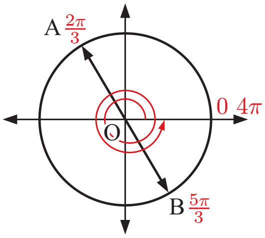
$\tan x + \sqrt{3} = 0$

$\therefore \quad \tan x = -\sqrt{3}$

There are two points on the unit circle with tangent $-\sqrt{3}$.

They correspond to angles $\frac{2\pi}{3}$ and $\frac{5\pi}{3}$.

For the domain $0 < x < 4\pi$ we have 4 solutions: $x = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}$, or $\frac{11\pi}{3}$.



Start at angle 0 and work around to 4π, noting down the angle every time you reach points A and B.

EXERCISE 9E.2

1 Solve for x on the domain $0 \leqslant x \leqslant 4\pi$:

a $2\cos x - 1 = 0$ **b** $\sqrt{2}\sin x = 1$ **c** $\tan x = 1$

2 Solve for x on the domain $-2\pi \leqslant x \leqslant 2\pi$:

a $2\sin x - \sqrt{3} = 0$ **b** $\sqrt{2}\cos x + 1 = 0$ **c** $\tan x = -1$

Example 8

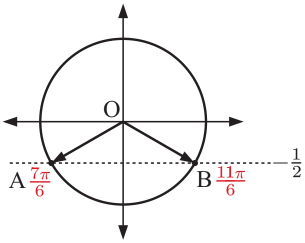
Self Tutor

Solve exactly for $0 \leqslant x \leqslant 3\pi$: **a** $\sin x = -\frac{1}{2}$ **b** $\sin 2x = -\frac{1}{2}$

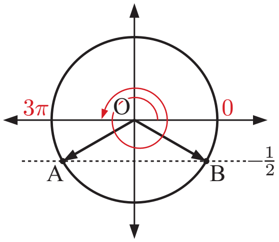
The equations both have the form $\sin\theta = -\frac{1}{2}$.

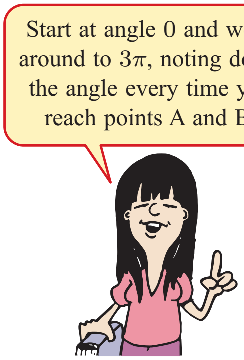
There are two points on the unit circle with sine $-\frac{1}{2}$.

They correspond to angles $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.



a In this case θ is simply x, so we have the domain $0 \leqslant x \leqslant 3\pi$.
The only solutions for this domain are $x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$.




Start at angle 0 and work around to 3π, noting down the angle every time you reach points A and B.

b In this case θ is $2x$.
If $0 \leqslant x \leqslant 3\pi$ then $0 \leqslant 2x \leqslant 6\pi$.

$\therefore \quad 2x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}, \frac{31\pi}{6}, \text{ or } \frac{35\pi}{6}$

$\therefore \quad x = \frac{7\pi}{12}, \frac{11\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}, \frac{31\pi}{12}, \text{ or } \frac{35\pi}{12}$

3 Solve exactly for $0 \leqslant x \leqslant 3\pi$: **a** $\cos x = \frac{1}{2}$ **b** $\cos 2x = \frac{1}{2}$

4 Solve exactly for $0 \leqslant x \leqslant 2\pi$: **a** $\sin x = -\frac{1}{\sqrt{2}}$ **b** $\sin 3x = -\frac{1}{\sqrt{2}}$

5 Find the exact solutions of:

- **a** $\cos x = -\frac{1}{2}, \ 0 \leqslant x \leqslant 5\pi$
- **b** $2\sin x - 1 = 0, \ -360° \leqslant x \leqslant 360°$
- **c** $2\cos x + \sqrt{3} = 0, \ 0 \leqslant x \leqslant 3\pi$
- **d** $3\cos 2x + 3 = 0, \ 0 \leqslant x \leqslant 3\pi$
- **e** $4\cos 3x + 2 = 0, \ -\pi \leqslant x \leqslant \pi$

Example 9

Self Tutor

Solve $\tan 2x + 1 = 2$ for $-\pi \leqslant x \leqslant \pi$.

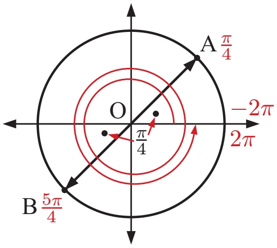
$\tan 2x = 1$

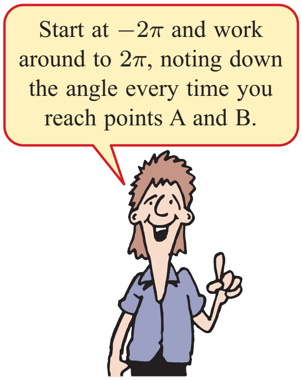
There are two points on the unit circle which have tangent 1.

Since $-\pi \leqslant x \leqslant \pi$,
$-2\pi \leqslant 2x \leqslant 2\pi$

So, $2x = -\frac{7\pi}{4}, \ -\frac{3\pi}{4}, \ \frac{\pi}{4}, \text{ or } \frac{5\pi}{4}$

$\therefore \quad x = -\frac{7\pi}{8}, \ -\frac{3\pi}{8}, \ \frac{\pi}{8}, \text{ or } \frac{5\pi}{8}$




Start at -2π and work around to 2π, noting down the angle every time you reach points A and B.

6 Solve $\tan x = \sqrt{3}$ for $0 \leqslant x \leqslant 2\pi$. Hence solve the following equations for $0 \leqslant x \leqslant 2\pi$:

a $\tan 4x = \sqrt{3}$ **b** $\tan^2 x = 3$

7 Solve $\sqrt{3}\tan 3x = 1$ for $0 \leqslant x \leqslant \pi$.

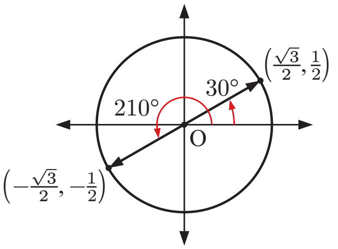
8 Solve for $-\pi \leqslant x \leqslant \pi$:

a $\sec x = -2$ **b** $\sqrt{3}\operatorname{cosec} 2x = 2$ **c** $\cot x = 0$

Example 10

Self Tutor

Find the exact solutions of $\sqrt{3}\sin x = \cos x$ for $0° \leqslant x \leqslant 360°$.



$$\sqrt{3}\sin x = \cos x$$

$$\therefore \quad \frac{\sin x}{\cos x} = \frac{1}{\sqrt{3}} \quad \{\text{dividing both sides by } \sqrt{3}\cos x\}$$

$$\therefore \quad \tan x = \frac{1}{\sqrt{3}}$$

$$\therefore \quad x = 30° \text{ or } 210°$$

9 Solve for $0 \leqslant x \leqslant 2\pi$:

a $\sin x - \cos x = 0$ **b** $\sin x = -\cos x$

c $\sin 3x = \cos 3x$ **d** $\sin 2x = \sqrt{3}\cos 2x$

Check your answers using the graphing package.

GRAPHING PACKAGE

10 Solve for $0 \leqslant x \leqslant \pi$: $\sin x = \operatorname{cosec} x$

F TRIGONOMETRIC RELATIONSHIPS

There are a vast number of trigonometric relationships. However, we only need to remember a few because we can obtain the rest by rearrangement or substitution.

SIMPLIFYING TRIGONOMETRIC EXPRESSIONS

For any given angle θ, $\sin\theta$ and $\cos\theta$ are real numbers. $\tan\theta$ is also real whenever it is defined. The algebra of trigonometry is therefore identical to the algebra of real numbers.

An expression like $2\sin\theta + 3\sin\theta$ compares with $2x + 3x$, so $2\sin\theta + 3\sin\theta = 5\sin\theta$.

To simplify more complicated trigonometric expressions, we often use the identities:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$\sin^2\theta + \cos^2\theta = 1$ is a special form of Pythagoras' theorem

We can also use rearrangements of these formulae, such as:

$$\sin^2\theta = 1 - \cos^2\theta \qquad \tan^2\theta = \sec^2\theta - 1$$
$$\cos^2\theta = 1 - \sin^2\theta \qquad \cot^2\theta = \operatorname{cosec}^2\theta - 1$$

Example 11

Self Tutor

Simplify:

a $3\cos\theta + 4\cos\theta$

b $\tan\alpha - 3\tan\alpha$

a $3\cos\theta + 4\cos\theta = 7\cos\theta$
{compare with $3x + 4x = 7x$}

b $\tan\alpha - 3\tan\alpha = -2\tan\alpha$
{compare with $x - 3x = -2x$}

Example 12

Self Tutor

Simplify:

a $2 - 2\sin^2\theta$

b $\cos^2\theta\sin\theta + \sin^3\theta$

a
$$\begin{aligned} &2 - 2\sin^2\theta \\ &= 2(1 - \sin^2\theta) \\ &= 2\cos^2\theta \\ &\quad \{\cos^2\theta + \sin^2\theta = 1\} \end{aligned}$$

b
$$\begin{aligned} &\cos^2\theta\sin\theta + \sin^3\theta \\ &= \sin\theta(\cos^2\theta + \sin^2\theta) \\ &= \sin\theta \times 1 \\ &= \sin\theta \end{aligned}$$

EXERCISE 9F.1

1 Simplify:

a $\sin\theta + \sin\theta$

b $2\cos\theta + \cos\theta$

c $3\sin\theta - \sin\theta$

d $3\sin\theta - 2\sin\theta$

e $\tan\theta - 3\tan\theta$

f $2\cos^2\theta - 5\cos^2\theta$

Example 13

Self Tutor

Expand and simplify: $(\cos\theta - \sin\theta)^2$

$$\begin{aligned} &(\cos\theta - \sin\theta)^2 \\ &= \cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta && \{\text{using } (a-b)^2 = a^2 - 2ab + b^2\} \\ &= \cos^2\theta + \sin^2\theta - 2\cos\theta\sin\theta \\ &= 1 - 2\cos\theta\sin\theta \end{aligned}$$

2 Simplify:

a $3\sin^2\theta + 3\cos^2\theta$

b $-2\sin^2\theta - 2\cos^2\theta$

c $-\cos^2\theta - \sin^2\theta$

d $3 - 3\sin^2\theta$

e $4 - 4\cos^2\theta$

f $\cos^3\theta + \cos\theta\sin^2\theta$

g $\cos^2\theta - 1$

h $\sin^2\theta - 1$

i $2\cos^2\theta - 2$

j $\dfrac{1 - \sin^2\theta}{\cos^2\theta}$

k $\dfrac{1 - \cos^2\theta}{\sin\theta}$

l $\dfrac{\cos^2\theta - 1}{-\sin\theta}$

3 Simplify:

a $3\tan x - \dfrac{\sin x}{\cos x}$ **b** $\dfrac{\sin^2 x}{\cos^2 x}$ **c** $\tan x \cos x$

d $\dfrac{\sin x}{\tan x}$ **e** $3\sin x + 2\cos x \tan x$ **f** $\dfrac{2\tan x}{\sin x}$

g $\tan x \cot x$ **h** $\sin x \operatorname{cosec} x$ **i** $\sec x \cot x$

j $\sin x \cot x$ **k** $\dfrac{\cot x}{\operatorname{cosec} x}$ **l** $\dfrac{2\sin x \cot x + 3\cos x}{\cot x}$

4 Expand and simplify if possible:

a $(1+\sin\theta)^2$ **b** $(\sin\alpha - 2)^2$ **c** $(\tan\alpha - 1)^2$

d $(\sin\alpha + \cos\alpha)^2$ **e** $(\sin\beta - \cos\beta)^2$ **f** $-(2-\cos\alpha)^2$

5 Simplify:

a $1 - \sec^2\beta$ **b** $\dfrac{\tan^2\theta(\cot^2\theta + 1)}{\tan^2\theta + 1}$

c $\cos^2\alpha(\sec^2\alpha - 1)$ **d** $(\sin x + \tan x)(\sin x - \tan x)$

e $(2\sin\theta + 3\cos\theta)^2 + (3\sin\theta - 2\cos\theta)^2$ **f** $(1+\operatorname{cosec}\theta)(\sin\theta - \sin^2\theta)$

g $\sec A - \sin A \tan A - \cos A$

FACTORISING TRIGONOMETRIC EXPRESSIONS

Example 14

Self Tutor

Factorise:

a $\cos^2\alpha - \sin^2\alpha$ **b** $\tan^2\theta - 3\tan\theta + 2$

a $\cos^2\alpha - \sin^2\alpha$

$= (\cos\alpha + \sin\alpha)(\cos\alpha - \sin\alpha)$ {compare with $a^2 - b^2 = (a+b)(a-b)$}

b $\tan^2\theta - 3\tan\theta + 2$

$= (\tan\theta - 2)(\tan\theta - 1)$ {compare with $x^2 - 3x + 2 = (x-2)(x-1)$}

EXERCISE 9F.2

1 Factorise:

a $1 - \sin^2\theta$ **b** $\sin^2\alpha - \cos^2\alpha$

c $\tan^2\alpha - 1$ **d** $2\sin^2\beta - \sin\beta$

e $2\cos\phi + 3\cos^2\phi$ **f** $3\sin^2\theta - 6\sin\theta$

g $\tan^2\theta + 5\tan\theta + 6$ **h** $2\cos^2\theta + 7\cos\theta + 3$

i $6\cos^2\alpha - \cos\alpha - 1$ **j** $3\tan^2\alpha - 2\tan\alpha$

k $\sec^2\beta - \operatorname{cosec}^2\beta$ **l** $2\cot^2 x - 3\cot x + 1$

m $2\sin^2 x + 7\sin x\cos x + 3\cos^2 x$

Example 15

Self Tutor

Simplify:

a $\dfrac{2-2\cos^2\theta}{1+\cos\theta}$ **b** $\dfrac{\cos\theta-\sin\theta}{\cos^2\theta-\sin^2\theta}$

a
$$\begin{aligned}&\frac{2-2\cos^2\theta}{1+\cos\theta}\\ &=\frac{2(1-\cos^2\theta)}{1+\cos\theta}\\ &=\frac{2\cancel{(1+\cos\theta)}^{1}(1-\cos\theta)}{\cancel{(1+\cos\theta)}_{1}}\\ &=2(1-\cos\theta)\end{aligned}$$

b
$$\begin{aligned}&\frac{\cos\theta-\sin\theta}{\cos^2\theta-\sin^2\theta}\\ &=\frac{\cancel{(\cos\theta-\sin\theta)}^{1}}{(\cos\theta+\sin\theta)\cancel{(\cos\theta-\sin\theta)}_{1}}\\ &=\frac{1}{\cos\theta+\sin\theta}\end{aligned}$$

2 Simplify:

a $\dfrac{1-\sin^2\alpha}{1-\sin\alpha}$ **b** $\dfrac{\tan^2\beta-1}{\tan\beta+1}$ **c** $\dfrac{\cos^2\phi-\sin^2\phi}{\cos\phi+\sin\phi}$

d $\dfrac{\cos^2\phi-\sin^2\phi}{\cos\phi-\sin\phi}$ **e** $\dfrac{\sin\alpha+\cos\alpha}{\sin^2\alpha-\cos^2\alpha}$ **f** $\dfrac{3-3\sin^2\theta}{6\cos\theta}$

g $1-\dfrac{\cos^2\theta}{1+\sin\theta}$ **h** $\dfrac{1+\cot\theta}{\operatorname{cosec}\theta}-\dfrac{\sec\theta}{\tan\theta+\cot\theta}$ **i** $\dfrac{\tan^2\theta}{\sec\theta-1}$

3 Show that:

a $(\cos\theta+\sin\theta)^2+(\cos\theta-\sin\theta)^2=2$ **b** $(2\sin\theta+3\cos\theta)^2+(3\sin\theta-2\cos\theta)^2=13$

c $(1-\cos\theta)\left(1+\dfrac{1}{\cos\theta}\right)=\tan\theta\sin\theta$ **d** $\left(1+\dfrac{1}{\sin\theta}\right)(\sin\theta-\sin^2\theta)=\cos^2\theta$

e $\sec A-\cos A=\tan A\sin A$ **f** $\dfrac{\cos\theta}{1-\sin\theta}=\sec\theta+\tan\theta$

g $\dfrac{\cos\alpha}{1-\tan\alpha}+\dfrac{\sin\alpha}{1-\cot\alpha}=\sin\alpha+\cos\alpha$ **h** $\dfrac{\sin\theta}{1+\cos\theta}+\dfrac{1+\cos\theta}{\sin\theta}=2\operatorname{cosec}\theta$

i $\dfrac{\sin\theta}{1-\cos\theta}-\dfrac{\sin\theta}{1+\cos\theta}=2\cot\theta$ **j** $\dfrac{1}{1-\sin\theta}+\dfrac{1}{1+\sin\theta}=2\sec^2\theta$

GRAPHING PACKAGE

Use a graphing package to check these simplifications by graphing each function on the same set of axes.

Discovery 5 Double angle formulae

What to do:

1 Copy and complete, using angles of your choice as well:

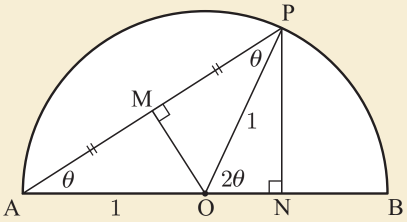
θ	$\sin 2\theta$	$2\sin\theta$	$2\sin\theta\cos\theta$	$\cos 2\theta$	$2\cos\theta$	$\cos^2\theta-\sin^2\theta$
0.631						
57.81°						
−3.697						

2 Write down any discoveries from your table of values in **1**.

3 In the diagram alongside, the semi-circle has radius 1 unit, and $\widehat{PAB} = \theta$.

$\widehat{APO} = \theta$ {△AOP is isosceles}

$\widehat{PON} = 2\theta$ {exterior angle of a triangle}



a Find in terms of θ, the lengths of:

i OM **ii** AM **iii** ON **iv** PN

b Use △ANP and the lengths in **a** to show that:

i $\cos\theta = \dfrac{\sin 2\theta}{2\sin\theta}$ **ii** $\cos\theta = \dfrac{1+\cos 2\theta}{2\cos\theta}$

c Hence deduce that:

i $\sin 2\theta = 2\sin\theta\cos\theta$ **ii** $\cos 2\theta = 2\cos^2\theta - 1$

4 Starting with $\cos 2\theta = 2\cos^2\theta - 1$, show that:

a $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ **b** $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

The double angle formulae are not required for the syllabus but are very useful.

G TRIGONOMETRIC EQUATIONS IN QUADRATIC FORM

Sometimes we may be given trigonometric equations in quadratic form.

For example, $2\sin^2 x + \sin x = 0$ and $2\cos^2 x + \cos x - 1 = 0$ are quadratic equations where the variables are $\sin x$ and $\cos x$ respectively.

These equations can be factorised by quadratic factorisation and then solved for x.

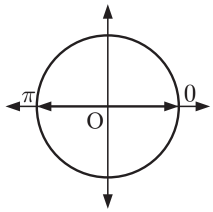
Example 16

Self Tutor

Solve for $0 \leqslant x \leqslant 2\pi$:

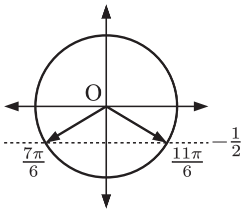
a $2\sin^2 x + \sin x = 0$ **b** $2\cos^2 x + \cos x - 1 = 0$

a
$$2\sin^2 x + \sin x = 0$$
$$\therefore \sin x(2\sin x + 1) = 0$$
$$\therefore \sin x = 0 \text{ or } -\tfrac{1}{2}$$



$\sin x = 0$ when

$x = 0, \pi, \text{ or } 2\pi$

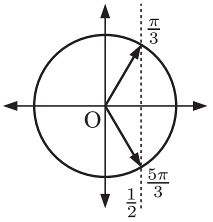


$\sin x = -\frac{1}{2}$ when

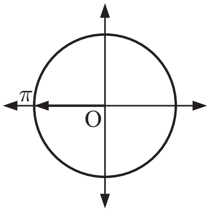
$x = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$

The solutions are: $x = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, \text{ or } 2\pi$.

b $$2\cos^2 x + \cos x - 1 = 0$$
$$\therefore\quad (2\cos x - 1)(\cos x + 1) = 0$$
$$\therefore\quad \cos x = \tfrac{1}{2} \text{ or } -1$$



$\cos x = \frac{1}{2}$ when
$x = \frac{\pi}{3}$ or $\frac{5\pi}{3}$



$\cos x = -1$ when
$x = \pi$

The solutions are: $x = \frac{\pi}{3}$, π, or $\frac{5\pi}{3}$.

EXERCISE 9G

1 Solve for $0 \leqslant x \leqslant 2\pi$:

- **a** $2\sin^2 x + \sin x = 0$
- **b** $2\cos^2 x = \cos x$
- **c** $2\cos^2 x + \cos x - 1 = 0$
- **d** $2\sin^2 x + 3\sin x + 1 = 0$
- **e** $\sin^2 x = 2 - \cos x$
- **f** $\cos x + \sec x = 2$

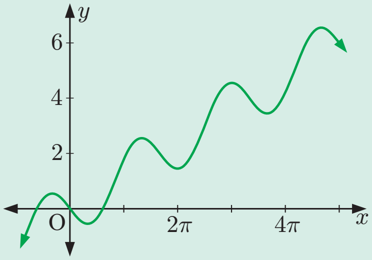
2 Solve for $0 \leqslant x \leqslant 2\pi$:

- **a** $\sin^2 x + \cos x = -1$
- **b** $2\cos^2 x = 3\sin x$

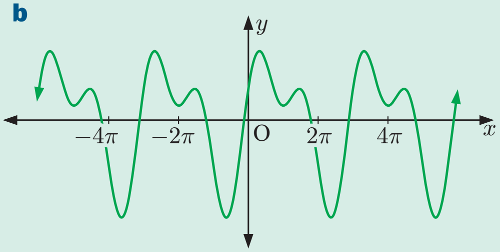
Review set 9A

1 Which of the following graphs displays periodic behaviour?

a



b



2 Draw each of the following graphs for $0 \leqslant x \leqslant 2\pi$:

- **a** $y = 5\sin x$
- **b** $y = \cos 3x - 1$
- **c** $y = \tan 2x + 4$

3 State the minimum and maximum values of:

- **a** $1 + \sin x$
- **b** $2\cos 3x$
- **c** $y = 3\sin 2x$
- **d** $y = \cos 4x - 1$

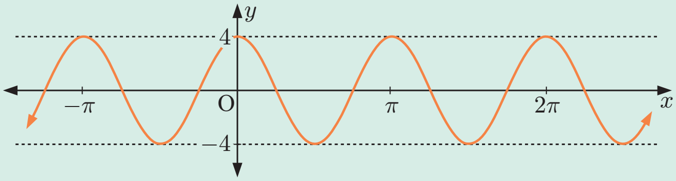
4 State the period of:

- **a** $y = 4\sin x$
- **b** $y = 2\cos 4x$
- **c** $y = 4\cos 2x + 4$
- **d** $y = 2\tan 3x$

5 Complete the table:

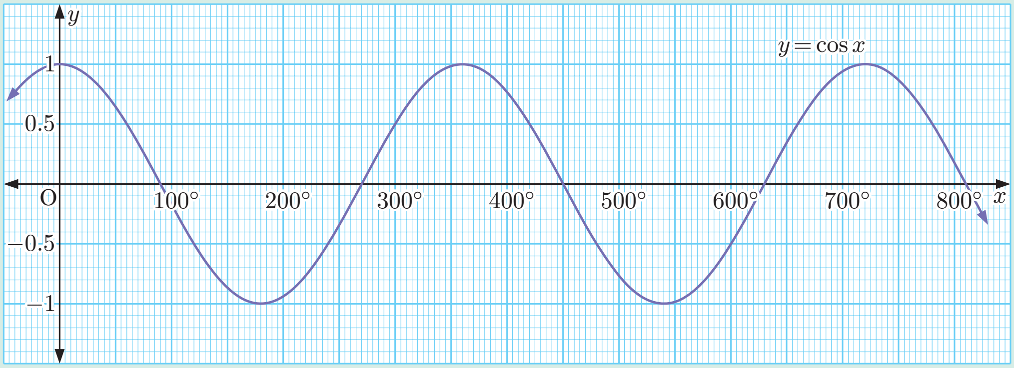
Function	*Period*	*Amplitude*	*Domain*	*Range*
$y = 3\sin 2x + 1$				
$y = \tan 2x$				
$y = 2\cos 3x - 3$				

6 Find the cosine function represented in the graph.



7 On the same set of axes, graph $y = 2\cos x$ and $y = |2\cos x|$ for $0 \leqslant x \leqslant 2\pi$.

8



Use the graph of $y = \cos x$ to find the solutions of:

- **a** $\cos x = -0.4, \quad 0 \leqslant x \leqslant 800°$
- **b** $\cos x = 0.9, \quad 0 \leqslant x \leqslant 600°$

9 Solve in terms of π:

- **a** $2\sin x = -1$ for $0 \leqslant x \leqslant 4\pi$
- **b** $\sqrt{2}\sin x - 1 = 0$ for $-2\pi \leqslant x \leqslant 2\pi$
- **c** $2\sin 3x + \sqrt{3} = 0$ for $0 \leqslant x \leqslant 2\pi$
- **d** $\sqrt{2}\cos x - 1 = 0$ for $0 \leqslant x \leqslant 4\pi$

10 Simplify:

- **a** $\dfrac{1 - \cos^2\theta}{1 + \cos\theta}$
- **b** $\dfrac{\sin\alpha - \cos\alpha}{\sin^2\alpha - \cos^2\alpha}$
- **c** $\dfrac{4\sin^2\alpha - 4}{8\cos\alpha}$
- **d** $\dfrac{\cot^2\theta}{\operatorname{cosec}\theta - 1}$

11 Show that $\dfrac{\cos\theta - \sec\theta}{\tan\theta}$ simplifies to $-\sin\theta$.

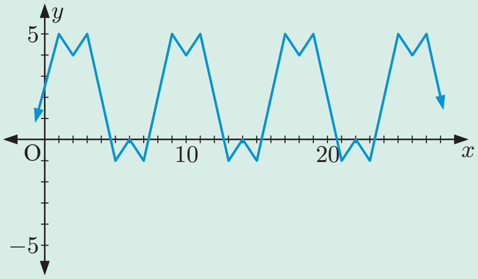
12 Find exact solutions for $-\pi \leqslant x \leqslant \pi$:

- **a** $\tan 2x = -\sqrt{3}$
- **b** $\tan^2 x - 3 = 0$

Review set 9B

1 Consider the graph alongside.

- **a** Explain why this graph shows periodic behaviour.
- **b** State:
  - **i** the period
  - **ii** the maximum value
  - **iii** the minimum value



2 Find b given that the function $y = \sin bx$, $b > 0$ has period:

a $\frac{\pi}{3}$ **b** $\frac{\pi}{12}$

3 State the minimum and maximum values of:

a $y = 5\sin x - 3$ **b** $y = 3\cos x + 1$ **c** $y = 4\cos 2x + 9$

4 On the same set of axes, for the domain $0 \leqslant x \leqslant 2\pi$, sketch:

a $y = \cos x$ and $y = \cos x - 3$ **b** $y = \tan x$ and $y = 2\tan x$

c $y = \cos x$ and $y = \cos 2x + 1$ **d** $y = \sin x$ and $y = 3\sin x + 1$

5 The function $y = a\sin bx + c$, $a > 0$, $b > 0$, has amplitude 2, period $\frac{\pi}{3}$, and principal axis $y = -2$.

a Find the values of a, b, and c. **b** Sketch the function for $0 \leqslant x \leqslant \pi$.

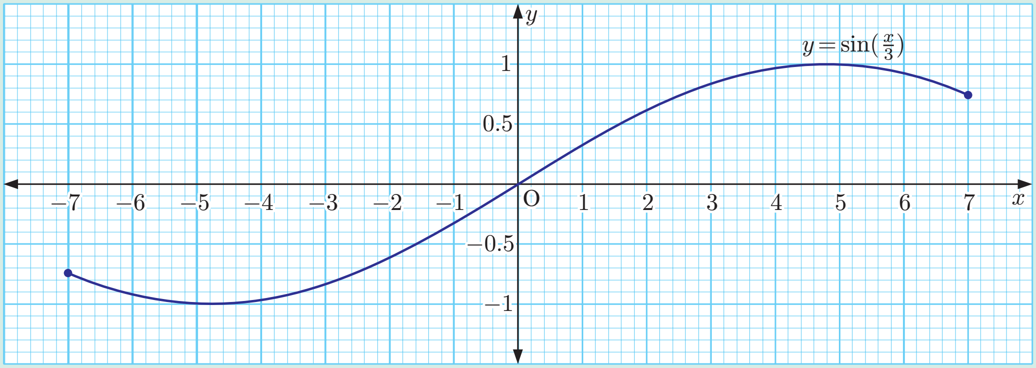
6 Consider the function $y = 2\tan x$.

a State a function which has the same shape, but has principal axis $y = 2$.

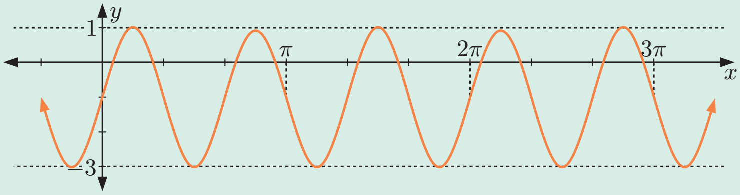
b Draw $y = 2\tan x$ and your function from **a** on the same set of axes, for $-2\pi \leqslant x \leqslant 2\pi$.

7 Consider $y = \sin(\frac{x}{3})$ on the domain $-7 \leqslant x \leqslant 7$. Use the graph to solve, correct to 1 decimal place:

a $\sin(\frac{x}{3}) = -0.9$ **b** $\sin(\frac{x}{3}) = \frac{1}{4}$



8 Find m and n given the following graph of the function $y = 2\sin mx + n$:



9 Solve for $0 \leqslant x \leqslant 2\pi$:

a $\sin^2 x - \sin x - 2 = 0$ **b** $4\sin^2 x = 1$

10 Simplify:

a $\cos^3\theta + \sin^2\theta\cos\theta$ **b** $\dfrac{\cos^2\theta - 1}{\sin\theta}$

c $5 - 5\sin^2\theta$ **d** $\dfrac{\sin^2\theta - 1}{\cos\theta}$

11 Expand and simplify if possible:

a $(2\sin\alpha - 1)^2$

b $(\cos\alpha - \sin\alpha)^2$

12 Show that:

a $\dfrac{\cos\theta}{1+\sin\theta} + \dfrac{1+\sin\theta}{\cos\theta} = 2\sec\theta$

b $(1+\sec\theta)\left(\cos\theta - \cos^2\theta\right) = \sin^2\theta$

10

Counting and the binomial expansion

Contents:

Opening problem

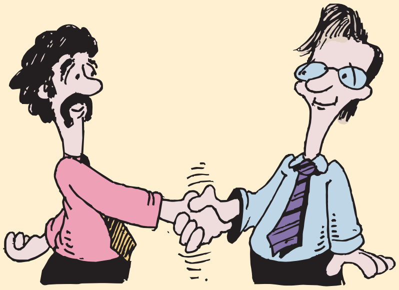
At a mathematics teachers' conference there are 273 delegates present. The organising committee consists of 10 people.

Things to think about:

a If each committee member shakes hands with every other committee member, how many handshakes take place?
Can a 10-sided convex polygon be used to solve this problem?

b If all 273 delegates shake hands with all other delegates, how many handshakes take place now?

c If the organising committee lines up on stage to face the delegates in the audience, in how many different orders can they line up?



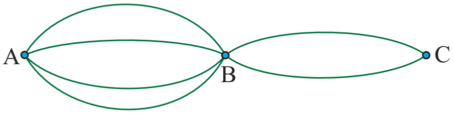
The **Opening Problem** is an example of a **counting** problem.

The following exercises will help us to solve counting problems without having to list and count the possibilities one by one. To do this we will examine:

- the product principle
- counting permutations
- counting combinations.

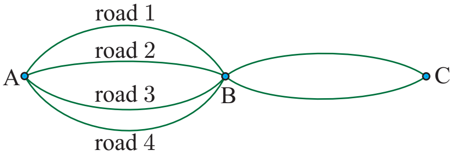
A THE PRODUCT PRINCIPLE

Suppose there are three towns A, B, and C. Four different roads could be taken from A to B, and two different roads from B to C.



How many different pathways are there from A to C going through B?

If we take road 1, there are two alternative roads to complete our trip.



Similarly, if we take road 2, there are two alternative roads to complete our trip.

The same is true for roads 3 and 4.

So, there are $2+2+2+2=4\times 2$ different pathways from A to C going through B.

Notice that the 4 corresponds to the number of roads from A to B and the 2 corresponds to the number of roads from B to C.

THE PRODUCT PRINCIPLE

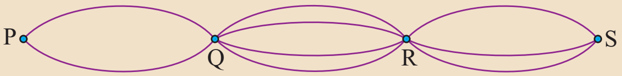
If there are m different ways of performing an operation, and for each of these there are n different ways of performing a second **independent** operation, then there are mn different ways of performing the two operations in succession.

The product principle can be extended to three or more successive independent operations.

Example 1

Self Tutor

P, Q, R, and S represent where Pauline, Quentin, Reiko, and Sam live. There are two different paths from P to Q, four different paths from Q to R, and 3 different paths from R to S.

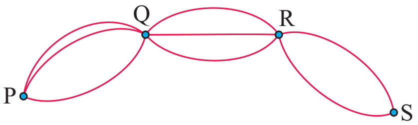


How many different pathways could Pauline take to visit Sam if she stops to see Quentin and then Reiko on the way?

The total number of different pathways $= 2 \times 4 \times 3 = 24$ {product principle}

EXERCISE 10A

1 The illustration shows the different map routes for a bus service which goes from P to S through both Q and R. How many different routes are possible?

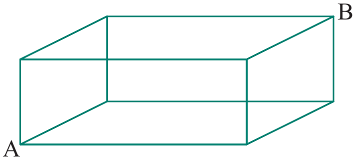


2 

In how many ways can the vertices of a rectangle be labelled with the letters A, B, C, and D:

- **a** in clockwise alphabetical order
- **b** in alphabetical order
- **c** in random order?

3 The wire frame shown forms the outline of a box. An ant crawls along the wire from A to B. How many different paths of shortest length lead from A to B?



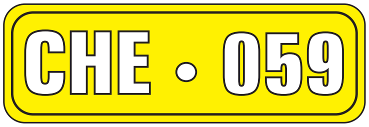
4 A table tennis competition has 7 teams. In how many different ways can the top two positions be filled in order of premiership points obtained?

5 A football competition is organised between 8 teams. In how many ways can the top 4 places be filled in order of premiership points obtained?

6 How many 3-digit numbers can be formed using the digits 2, 3, 4, 5, and 6:

- **a** as often as desired
- **b** at most once each?

7 How many different alpha-numeric plates for motor car registration can be made if the first 3 places are English alphabet letters and the remaining places are 3 digits from 0 to 9?

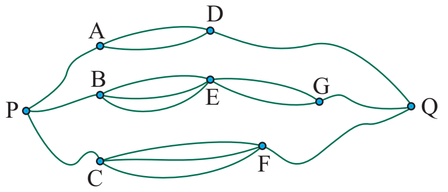


8 In how many ways can:

- **a** 2 postcards be mailed into 2 mail boxes
- **b** 2 postcards be mailed into 3 mail boxes
- **c** 4 postcards be mailed into 3 mail boxes?

B COUNTING PATHS

Consider the road system illustrated which shows the roads from P to Q.



From A to Q there are 2 paths.
From B to Q there are $3 \times 2 = 6$ paths.
From C to Q there are 3 paths.

$\therefore$ from P to Q there are $2 + 6 + 3 = 11$ paths.

Notice that:

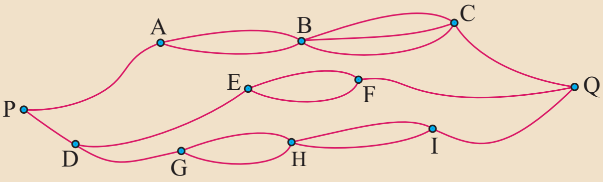
- When going from B to G, we go from B to E **and** then from E to G. We **multiply** the possibilities.
- When going from P to Q, we must first go from P to A **or** P to B **or** P to C. We **add** the possibilities from each of these first steps.

The word **and** suggests *multiplying* the possibilities.
The word **or** suggests *adding* the possibilities.

Example 2

Self Tutor

How many different paths lead from P to Q?



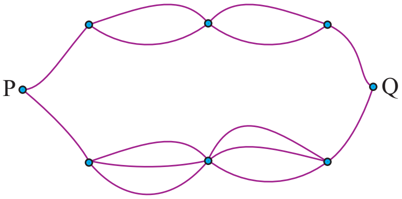
From P to A to B to C to Q there are $2 \times 3 = 6$ paths
or from P to D to E to F to Q there are 2 paths
or from P to D to G to H to I to Q there are $2 \times 2 = 4$ paths.

In total there are $6 + 2 + 4 = 12$ different paths.

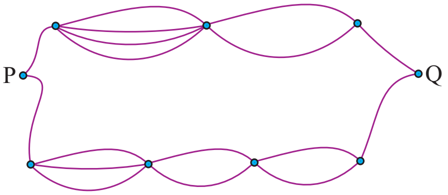
EXERCISE 10B

1 How many different paths lead from P to Q?

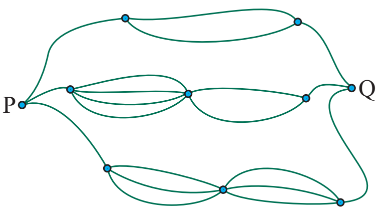
a



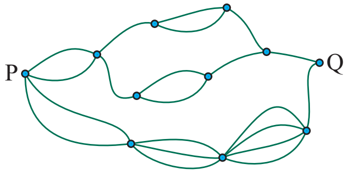
b



c

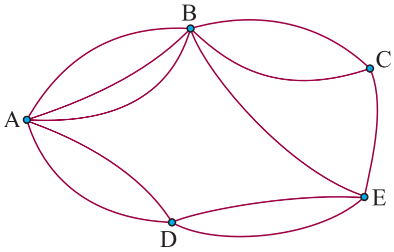


d



2 Katie is going on a long journey to visit her family. She lives in city A and is travelling to city E. Unfortunately there are no direct trains. However, she has the choice of several trains which stop in different cities along the way. These are illustrated in the diagram.

How many different train journeys does Katie have to choose from?



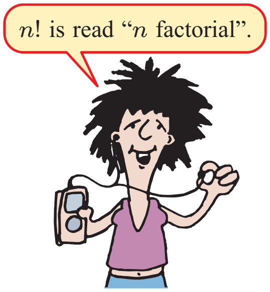
C FACTORIAL NOTATION

In problems involving counting, products of consecutive positive integers such as $8 \times 7 \times 6$ and $6 \times 5 \times 4 \times 3 \times 2 \times 1$ are common.

For convenience, we introduce **factorial numbers** to represent the products of consecutive positive integers.

For $n \geqslant 1$, $n!$ is the product of the first n positive integers.

$$\boldsymbol{n! = n(n-1)(n-2)(n-3)\ldots. \times 3 \times 2 \times 1}$$



For example, the product $6 \times 5 \times 4 \times 3 \times 2 \times 1$ can be written as 6!.

Notice that $8 \times 7 \times 6$ can be written using factorial numbers only as

$$8 \times 7 \times 6 = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{8!}{5!}$$

An alternative **recursive definition** of factorial numbers is $\boldsymbol{n! = n \times (n-1)!}$ for $n \geqslant 1$

which can be extended to $n! = n(n-1)(n-2)!$ and so on.

Using the factorial rule with $n = 1$, we have $1! = 1 \times 0!$

Therefore, for completeness we define $\boldsymbol{0! = 1}$

Example 3

Self Tutor

Simplify: **a** $4!$ **b** $\dfrac{5!}{3!}$ **c** $\dfrac{7!}{4! \times 3!}$

a $4! = 4 \times 3 \times 2 \times 1 = 24$

b $\dfrac{5!}{3!} = \dfrac{5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = 5 \times 4 = 20$

c $\dfrac{7!}{4! \times 3!} = \dfrac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1} = 35$

If your problem involves factorials of large numbers then it is important to cancel as many factors as possible before using a calculator to evaluate the rest.

For example, if you have $\frac{300!}{297!}$ in your problem, you will find you cannot calculate 300! on your calculator. However, we can see that

$$\begin{aligned}\frac{300!}{297!} = \frac{300 \times 299 \times 298 \times 297!}{297!} &= 300 \times 299 \times 298 \\ &= 26\,730\,600.\end{aligned}$$

EXERCISE 10C.1

1 Find $n!$ for $n = 0, 1, 2, 3, ..., 10$.

2 Simplify without using a calculator:

a $\frac{6!}{5!}$ **b** $\frac{6!}{4!}$ **c** $\frac{6!}{7!}$ **d** $\frac{4!}{6!}$ **e** $\frac{100!}{99!}$ **f** $\frac{7!}{5! \times 2!}$

3 Simplify:

a $\frac{n!}{(n-1)!}$ **b** $\frac{(n+2)!}{n!}$ **c** $\frac{(n+1)!}{(n-1)!}$

Example 4 Self Tutor

Express in factorial form:

a $10 \times 9 \times 8 \times 7$ **b** $\frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1}$

a $10 \times 9 \times 8 \times 7 = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{10!}{6!}$

b $\frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{10!}{4! \times 6!}$

4 Express in factorial form:

a $7 \times 6 \times 5$ **b** 10×9 **c** $11 \times 10 \times 9 \times 8 \times 7$

d $\frac{13 \times 12 \times 11}{3 \times 2 \times 1}$ **e** $\frac{1}{6 \times 5 \times 4}$ **f** $\frac{4 \times 3 \times 2 \times 1}{20 \times 19 \times 18 \times 17}$

Example 5 Self Tutor

Write as a product by factorising:

a $8! + 6!$ **b** $10! - 9! + 8!$

a
$$\begin{aligned}&8! + 6! \\ &= 8 \times 7 \times 6! + 6! \\ &= 6!(8 \times 7 + 1) \\ &= 6! \times 57\end{aligned}$$

b
$$\begin{aligned}&10! - 9! + 8! \\ &= 10 \times 9 \times 8! - 9 \times 8! + 8! \\ &= 8!(90 - 9 + 1) \\ &= 8! \times 82\end{aligned}$$

5 Write as a product by factorising:

a $5! + 4!$ **b** $11! - 10!$ **c** $7! + 9!$ **d** $12! - 10!$

e $9! + 8! + 7!$ **f** $7! - 6! + 8!$ **g** $12! - 2 \times 11!$ **h** $3 \times 9! + 5 \times 8!$

Example 6

Self Tutor

Simplify $\frac{7! - 6!}{6}$ by factorising.

$$\begin{aligned} & \frac{7! - 6!}{6} \\ &= \frac{7 \times 6! - 6!}{6} \\ &= \frac{6!\cancel{(7-1)}^{1}}{\cancel{6}_1} \\ &= 6! \end{aligned}$$

6 Simplify by factorising:

a $\frac{12! - 11!}{11}$ **b** $\frac{10! + 9!}{11}$ **c** $\frac{10! - 8!}{89}$ **d** $\frac{10! - 9!}{9!}$

e $\frac{6! + 5! - 4!}{4!}$ **f** $\frac{n! + (n-1)!}{(n-1)!}$ **g** $\frac{n! - (n-1)!}{n-1}$ **h** $\frac{(n+2)! + (n+1)!}{n+3}$

THE BINOMIAL COEFFICIENT

The **binomial coefficient** is defined by

$$\binom{n}{r} = \underbrace{\frac{n(n-1)(n-2)\ldots.(n-r+2)(n-r+1)}{r(r-1)(r-2)\ldots.2 \times 1}}_{\text{factor form}} = \underbrace{\frac{n!}{r!(n-r)!}}_{\text{factorial form}}$$

The binomial coefficient is sometimes written ${}^nC_r$ or C_r^n.

Example 7

Self Tutor

Use the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ to evaluate: **a** $\binom{5}{2}$ **b** $\binom{11}{7}$

a

$$\begin{aligned} \binom{5}{2} &= \frac{5!}{2!(5-2)!} \\ &= \frac{5!}{2! \times 3!} \\ &= \frac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 3 \times 2 \times 1} \\ &= 10 \end{aligned}$$

b

$$\begin{aligned} \binom{11}{7} &= \frac{11!}{7!(11-7)!} \\ &= \frac{11!}{7! \times 4!} \\ &= \frac{11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 4 \times 3 \times 2 \times 1} \\ &= \frac{7920}{24} \\ &= 330 \end{aligned}$$

EXERCISE 10C.2

1 Use the formula $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ to evaluate:

a $\binom{3}{1}$ **b** $\binom{4}{2}$ **c** $\binom{7}{3}$ **d** $\binom{10}{4}$

Check your answers using technology.

2 **a** Use the formula $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ to evaluate:

i $\binom{8}{2}$ **ii** $\binom{8}{6}$

b Show that $\binom{n}{r} = \binom{n}{n-r}$ for all $n \in \mathbb{Z}^+$, $r = 0, 1, 2, ..., n$.

3 Find k if $\binom{9}{k} = 4\binom{7}{k-1}$.

D PERMUTATIONS

A **permutation** of a group of symbols is *any arrangement* of those symbols in a definite *order*.

For example, BAC is a permutation on the symbols A, B, and C in which all three of them are used. We say the symbols are "taken 3 at a time".

The set of all the different permutations on the symbols A, B, and C taken 3 at a time, is {ABC, ACB, BAC, BCA, CAB, CBA}.

Example 8

Self Tutor

List the set of all permutations on the symbols P, Q, and R taken:

a 1 at a time **b** 2 at a time **c** 3 at a time.

a {P, Q, R}

b {PQ, QP, RP, PR, QR, RQ}

c {PQR, PRQ, QPR, QRP, RPQ, RQP}

Example 9

Self Tutor

List all permutations on the symbols W, X, Y, and Z taken 4 at a time.

WXYZ	WXZY	WYXZ	WYZX	WZXY	WZYX
XWYZ	XWZY	XYWZ	XYZW	XZYW	XZWY
YWXZ	YWZX	YXWZ	YXZW	YZWX	YZXW
ZWXY	ZWYX	ZXWY	ZXYW	ZYWX	ZYXW

There are 24 of them.

For large numbers of symbols, listing the complete set of permutations is absurd. However, we can still count them by considering the number of options we have for filling each position.

Suppose we want to find the number of different permutations on the symbols A, B, C, D, E, F, and G, taken 3 at a time.

There are 3 positions to fill:

1st	2nd	3rd

In the 1st position, any of the 7 symbols could be used, so we have 7 options.

7		
1st	2nd	3rd

This leaves any of 6 symbols to go in the 2nd position, and this leaves any of 5 symbols to go in the 3rd position.

7	6	5
1st	2nd	3rd

$$\begin{aligned}\text{So, the total number of permutations} &= 7 \times 6 \times 5 \quad \{\text{product principle}\}\\ &= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}\\ &= \frac{7!}{4!} \quad \text{or} \quad \frac{7!}{(7-3)!}\end{aligned}$$

The number of **permutations** on n distinct symbols taken r at a time is:

$$\underbrace{n \times (n-1) \times (n-2) \times \ldots. \times (n-r+1)}_{r \text{ of these}} = \frac{n!}{(n-r)!}$$

If we are finding permutations on the complete set of n symbols, as in **Example 9**, then $r = n$, and the number of permutations is $n!$.

Example 10

Self Tutor

A chess association runs a tournament with 16 teams. In how many different ways could the top 5 positions be filled on the competition ladder?

Any of the 16 teams could fill the 'top' position.
Any of the remaining 15 teams could fill the 2nd position.
Any of the remaining 14 teams could fill the 3rd position.
⋮
Any of the remaining 12 teams could fill the 5th position.

16	15	14	13	12
1st	2nd	3rd	4th	5th

$$\begin{aligned}\text{The total number of permutations} &= 16 \times 15 \times 14 \times 13 \times 12\\ &= \frac{16!}{11!}\\ &= 524\,160\end{aligned}$$

So the top 5 positions could be filled in 524 160 ways.

Example 11

Self Tutor

The alphabet blocks A, B, C, D, and E are placed in a row in front of you.

a How many different permutations could you have?

b How many permutations end in C?

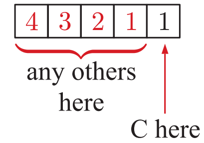
c How many permutations have the form $\boxed{\ldots}\boxed{A}\boxed{\ldots}\boxed{B}\boxed{\ldots}$?

d How many begin and end with a vowel (A or E)?

a There are 5 letters taken 5 at a time.

$\therefore$ the total number of permutations $= 5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$.

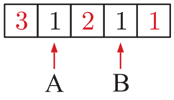
b



C must be in the last position. The other 4 letters could go into the remaining 4 places in 4! ways.

$\therefore$ the number of permutations $= 1 \times 4! = 24$.

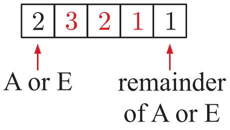
c



A goes into 1 place. B goes into 1 place. The remaining 3 letters go into the remaining 3 places in 3! ways.

$\therefore$ the number of permutations $= 1 \times 1 \times 3! = 6$.

d



A or E could go into the 1st position, so there are two options. The other one must go into the last position.

The remaining 3 letters could go into the 3 remaining places in 3! ways.

$\therefore$ the number of permutations $= 2 \times 1 \times 3! = 12$.

EXERCISE 10D

1 List the set of all permutations on the symbols W, X, Y, and Z taken:

a 1 at a time **b** two at a time **c** three at a time.

2 List the set of all permutations on the symbols A, B, C, D, and E taken:

a 2 at a time **b** 3 at a time.

3 In how many ways can:

a 5 different books be arranged on a shelf

b 3 different paintings be chosen from a collection of 8, and hung in a row

c a signal consisting of 4 coloured flags in a row be made if there are 10 different flags to choose from?

4 A captain and vice-captain are to be selected from a team of 11 cricketers. In how many ways can this be done?

5 Suppose you have 4 different coloured flags. How many different signals could you make using:

a 2 flags in a row **b** 3 flags in a row **c** 2 or 3 flags in a row?

6 Nine boxes are each labelled with a different whole number from 1 to 9. Five people are allowed to take one box each. In how many different ways can this be done if:

a there are no restrictions

b the first three people decide that they will take even numbered boxes?

7 **a** How many different permutations on the letters A, B, C, D, E, and F are there if each letter can be used once only?

b How many of these permutations:

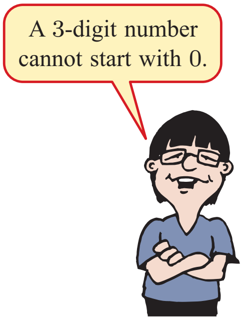
- **i** end in ED
- **ii** begin with F and end with A
- **iii** begin and end with a vowel (A or E)?

8 How many 3-digit numbers can be constructed from the digits 1, 2, 3, 4, 5, 6, and 7 if each digit may be used:

- **a** as often as desired
- **b** only once
- **c** once only and the number must be odd?

9 3-digit numbers are constructed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 using each digit at most once. How many such numbers:

- **a** can be constructed
- **b** end in 5
- **c** end in 0
- **d** are divisible by 5?



10 Arrangements containing 5 different letters from the word TRIANGLE are to be made. How many possible arrangements are there if:

- **a** there are no restrictions
- **b** the arrangement must start with R and end with A or E
- **c** the arrangement must include the letter G?

Example 12

Self Tutor

There are 6 different books arranged in a row on a shelf. In how many ways can two of the books, A and B, be together?

Method 1: We could have any of the following locations for A and B

$$\left.\begin{array}{cccccc} \text{A} & \text{B} & \times & \times & \times & \times \\ \text{B} & \text{A} & \times & \times & \times & \times \\ \times & \text{A} & \text{B} & \times & \times & \times \\ \times & \text{B} & \text{A} & \times & \times & \times \\ \times & \times & \text{A} & \text{B} & \times & \times \\ \times & \times & \text{B} & \text{A} & \times & \times \\ \times & \times & \times & \text{A} & \text{B} & \times \\ \times & \times & \times & \text{B} & \text{A} & \times \\ \times & \times & \times & \times & \text{A} & \text{B} \\ \times & \times & \times & \times & \text{B} & \text{A} \end{array}\right\} \text{10 of these}$$

If we consider any one of these, the remaining 4 books could be placed in 4! different orderings.

$\therefore$ total number of ways
$= 10 \times 4! = 240.$

Method 2: A and B can be put together in 2! ways (AB or BA).

Now consider this pairing as one book (effectively tying a string around them) which together with the other 4 books can be ordered in 5! different ways.

$\therefore$ the total number of ways $= 2! \times 5! = 240.$

11 In how many ways can 5 different books be arranged on a shelf if:

- **a** there are no restrictions
- **b** books X and Y must be together
- **c** books X and Y must not be together?

12 10 students sit in a row of 10 chairs. In how many ways can this be done if:

- **a** there are no restrictions
- **b** students A, B, and C insist on sitting together?

13 3 boys and 3 girls are to sit in a row. How many ways can this be done if:

- **a** there are no restrictions
- **b** there is a girl at each end
- **c** boys and girls must alternate
- **d** all the boys sit together?

14 How many three-digit numbers can be made using the digits 0, 1, 3, 5, and 8 at most once each, if:

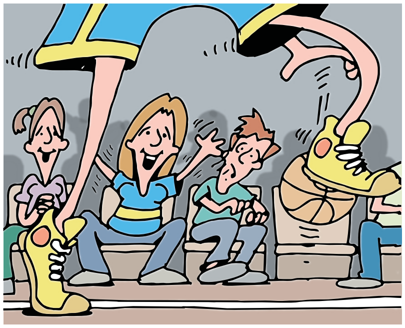
- **a** there are no restrictions
- **b** the numbers must be less than 500
- **c** the numbers must be even and greater than 300?

15 Consider the letters of the word MONDAY. How many permutations of four different letters can be chosen if:

- **a** there are no restrictions
- **b** at least one vowel (A or O) must be used
- **c** the two vowels are not together?

16 Alice has booked ten adjacent front-row seats for a basketball game for herself and nine friends.

- **a** How many different arrangements are possible if there are no restrictions?
- **b** Due to a severe snowstorm, only five of Alice's friends are able to join her for the game. In how many different ways can they be seated in the 10 seats if:
  - **i** there are no restrictions
  - **ii** any two of Alice's friends are to sit next to her?

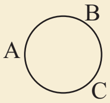


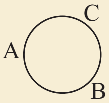
Discovery 1 *Permutations in a circle*

There are 6 permutations on the symbols A, B, and C **in a line**. These are:

ABC ACB BAC BCA CAB CBA.

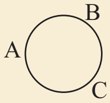
However **in a circle** there are only 2 different permutations on these 3 symbols. They are the only possibilities with different right-hand and left-hand neighbours.

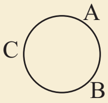


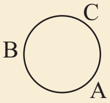


Permutations in a circle are not required for the syllabus.

In contrast, these three diagrams show the same cyclic permutation:







What to do:

1 Draw diagrams showing different cyclic permutations for:

- **a** one symbol: A
- **b** two symbols: A and B
- **c** three symbols: A, B, and C
- **d** four symbols: A, B, C, and D

2 Copy and complete:

Number of symbols	*Permutations in a line*	*Permutations in a circle*
1		
2		
3	$6 = 3!$	$2 = 2!$
4		

3 If there are n symbols to be arranged around a circle, how many different cyclic permutations are possible?

E COMBINATIONS

A **combination** is a selection of objects *without* regard to order.

For example, the possible teams of 3 people that can be selected from A, B, C, D, and E are:

ABC ABD ABE ACD ACE ADE
BCD BCE BDE
CDE

There are 10 combinations in total.

Now given the five people A, B, C, D, and E, we know that there are $5 \times 4 \times 3 = 60$ permutations for taking three of them at a time. So why is this 6 times larger than the number of combinations?

The answer is that for the combinations, order is not important. Selecting A, B, and C for the team is the same as selecting B, C, and A. For each of the 10 possible combinations, there are $3! = 6$ ways of ordering the members of the team.

In general, when choosing r objects from n objects,

$$\begin{aligned}\text{number of combinations} &= \text{number of permutations} \div r! \\ &= \frac{n!}{(n-r)!} \div r! \\ &= \frac{n!}{r!(n-r)!}\end{aligned}$$

This is the binomial coefficient we encountered in **Section C**.

The number of **combinations** on n distinct symbols taken r at a time is $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$.

Example 13

Self Tutor

How many different teams of 4 can be selected from a squad of 7 if:

a there are no restrictions **b** the teams must include the captain?

a There are 7 players up for selection and we want any 4 of them.

There are $\binom{7}{4} = 35$ possible combinations.

b The captain must be included *and* we need any 3 of the other 6.

There are $\binom{1}{1} \times \binom{6}{3} = 20$ possible combinations.

Example 14

Self Tutor

A committee of 4 is chosen from 7 men and 6 women. How many different committees can be chosen if:

a there are no restrictions

b there must be 2 of each sex

c there must be at least one of each sex?

a There are $7 + 6 = 13$ people up for selection and we want any 4 of them.

There are $\binom{13}{4} = 715$ possible committees.

b The 2 men can be chosen out of 7 in $\binom{7}{2}$ ways.

The 2 women can be chosen out of 6 in $\binom{6}{2}$ ways.

$\therefore$ there are $\binom{7}{2} \times \binom{6}{2} = 315$ possible committees.

c The total number of committees

= the number with 3 men and 1 woman + the number with 2 men and 2 women + the number with 1 man and 3 women

$= \binom{7}{3} \times \binom{6}{1} + \binom{7}{2} \times \binom{6}{2} + \binom{7}{1} \times \binom{6}{3}$

$= 665$

or The total number of committees

$= \binom{13}{4}$ − the number with all men − the number with all women

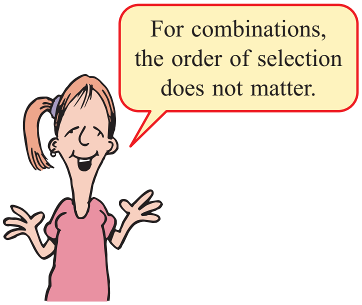
$= \binom{13}{4} - \binom{7}{4} \times \binom{6}{0} - \binom{7}{0} \times \binom{6}{4}$

$= 665$

EXERCISE 10E

1 Determine whether the following are examples of combinations or permutations:

- **a** making a 3-digit number using the digits 1, 2, 3, 4, and 5 at most once each
- **b** selecting a committee of 3 people from a list of 5
- **c** selecting the chairperson and treasurer from a committee of 8 people
- **d** selecting 2 pieces of fruit to take to school from a bowl of 10 pieces.



2 List the different teams of 4 that can be chosen from a squad of 6 (named A, B, C, D, E, and F).

Check that the formula $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$ gives the total number of teams.

3 How many different teams of 11 can be chosen from a squad of 17?

4 Candidates for an examination are required to answer 5 questions out of 9.

- **a** In how many ways can the questions be chosen if there are no restrictions?
- **b** If question 1 was made compulsory, how many selections would be possible?

5
- **a** How many different committees of 3 can be selected from 13 candidates?
- **b** How many of these committees consist of the president and 2 others?

6 **a** How many different teams of 5 can be selected from a squad of 12?

b How many of these teams contain:

i the captain and vice-captain **ii** exactly one of the captain or the vice-captain?

7 A team of 9 is selected from a squad of 15. 3 particular players *must* be included, and another must be excluded because of injury. In how many ways can the team be chosen?

8 In how many ways can 4 people be selected from 10 if:

a one particular person *must* be selected

b two particular people are excluded from every selection

c one particular person is always included and two particular people are always excluded?

9 A committee of 5 is chosen from 10 men and 6 women. Determine the number of ways of selecting the committee if:

a there are no restrictions **b** it must contain 3 men and 2 women

c it must contain all men **d** it must contain at least 3 men

e it must contain at least one of each sex.

10 A committee of 8 is chosen from 9 boys and 6 girls. In how many ways can this be done if:

a there are no restrictions **b** there must be 5 boys and 3 girls

c all the girls are selected **d** there are more boys than girls?

11 A music class consists of 5 piano players, 7 guitarists, and 4 violinists. A band of 1 piano player, 3 guitarists, and 2 violinists must be chosen to play at a school concert. In how many different ways can the band be chosen?

12 A committee of 5 is chosen from 6 doctors, 3 dentists, and 7 others.
Determine the number of ways of selecting the committee if it is to contain:

a exactly 2 doctors and 1 dentist **b** exactly 2 doctors

c at least one person from either of the two given professions.

13 How many diagonals does a 20-sided convex polygon have?

14 There are 12 distinct points A, B, C, D, ..., L on a circle. Lines are drawn between each pair of points.

a How many lines: **i** are there in total **ii** pass through B?

b How many triangles: **i** are determined by the lines **ii** have one vertex B?

15 How many 4-digit numbers can be constructed for which the digits are in ascending order from left to right? You cannot start a number with 0.

16 **a** Give an example which demonstrates that:

$$\binom{5}{0}\times\binom{6}{4}+\binom{5}{1}\times\binom{6}{3}+\binom{5}{2}\times\binom{6}{2}+\binom{5}{3}\times\binom{6}{1}+\binom{5}{4}\times\binom{6}{0}=\binom{11}{4}.$$

b Copy and complete:

$$\binom{m}{0}\times\binom{n}{r}+\binom{m}{1}\times\binom{n}{r-1}+\binom{m}{2}\times\binom{n}{r-2}+\ ....\ +\binom{m}{r-1}\times\binom{n}{1}+\binom{m}{r}\times\binom{n}{0}=\ ....$$

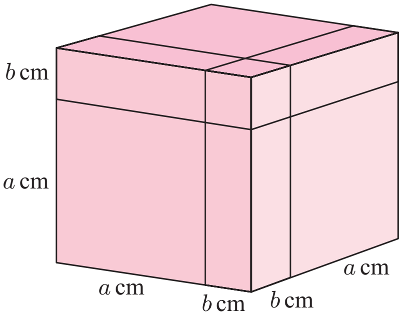
17 In how many ways can 12 people be divided into:

a two equal groups **b** three equal groups?

18 Answer the **Opening Problem** on page **256**.

F BINOMIAL EXPANSIONS

Consider the cube alongside, which has sides of length $(a+b)$ cm.



The cube has been subdivided into 8 blocks by making 3 cuts parallel to the cube's surfaces as shown.

We know that the total volume of the cube is $(a+b)^3$ cm^3. However, we can also find an expression for the cube's volume by adding the volumes of the 8 individual blocks.

We have:

	1 block	$a \times a \times a$
	3 blocks	$a \times a \times b$
	3 blocks	$a \times b \times b$
	1 block	$b \times b \times b$

ANIMATION

$\therefore$ the cube's volume $= a^3 + 3a^2b + 3ab^2 + b^3$

$\therefore \quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

The sum $\boldsymbol{a+b}$ is called a **binomial** as it contains two terms.

Any expression of the form $\boldsymbol{(a+b)^n}$ is called a **power of a binomial**.

All binomials raised to a power can be expanded using the same general principles. In this chapter, therefore, we consider the expansion of the general expression $(a+b)^n$ where $n \in \mathbb{N}$.

Consider the following algebraic expansions:

$$(a+b)^1 = a+b$$
$$(a+b)^2 = a^2+2ab+b^2$$

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2\\ &= (a+b)(a^2+2ab+b^2)\\ &= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned}$$

The **binomial expansion** of $\boldsymbol{(a+b)^2}$ is $\boldsymbol{a^2+2ab+b^2}$.

The **binomial expansion** of $\boldsymbol{(a+b)^3}$ is $\boldsymbol{a^3+3a^2b+3ab^2+b^3}$.

Discovery 2 — The binomial expansion

What to do:

1 Expand $(a+b)^4$ in the same way as for $(a+b)^3$ above.
Hence expand $(a+b)^5$ and $(a+b)^6$.

2 The cubic expansion $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$ contains 4 terms. Observe that their coefficients are: 1 3 3 1

- **a** What happens to the powers of a and b in each term of the expansion of $(a+b)^3$?
- **b** Does the pattern in **a** continue for the expansions of $(a+b)^4$, $(a+b)^5$, and $(a+b)^6$?

c Write down the triangle of coefficients to row 6:

$$\begin{array}{lccccccc} n=1 & & & 1 & & 1 & & \\ n=2 & & 1 & & 2 & & 1 & \\ n=3 & 1 & & 3 & & 3 & & 1 \longleftarrow \text{row } 3 \\ & & & & \vdots & & & \end{array}$$

3 The triangle of coefficients in **c** above is called **Pascal's triangle**. Investigate:

a the predictability of each row from the previous one

b a formula for finding the sum of the numbers in the nth row of Pascal's triangle.

4 **a** Use your results from **3** to predict the elements of the 7th row of Pascal's triangle.

b Hence write down the binomial expansion of $(a+b)^7$.

c Check your result algebraically by using $(a+b)^7 = (a+b)(a+b)^6$ and your results from **1**.

From the **Discovery** we obtained
$$\begin{aligned}(a+b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\ &= a^4 + 4a^3b^1 + 6a^2b^2 + 4a^1b^3 + b^4\end{aligned}$$

Notice that:

- As we look from left to right across the expansion, the powers of a decrease by 1, while the powers of b increase by 1.
- The sum of the powers of a and b in each term of the expansion is 4.
- The number of terms in the expansion is $4+1=5$.
- The coefficients of the terms are row 4 of Pascal's triangle.

For the expansion of $(a+b)^n$ where $n \in \mathbb{N}$:

- As we look from left to right across the expansion, the powers of a *decrease* by 1, while the powers of b *increase* by 1.
- The sum of the powers of a and b in each term of the expansion is n.
- The number of terms in the expansion is $n+1$.
- The coefficients of the terms are row n of Pascal's triangle.

In the following examples we see how the general binomial expansion $(a+b)^n$ may be put to use.

Example 15

Self Tutor

Using $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, find the binomial expansion of:

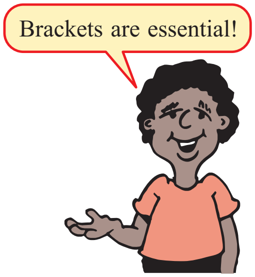
a $(2x+3)^3$ **b** $(x-5)^3$

a In the expansion of $(a+b)^3$ we substitute $a=(2x)$ and $b=(3)$.

$$\begin{aligned}\therefore\ (2x+3)^3 &= (2x)^3 + 3(2x)^2(3) + 3(2x)^1(3)^2 + (3)^3 \\ &= 8x^3 + 36x^2 + 54x + 27\end{aligned}$$

b We substitute $a=(x)$ and $b=(-5)$

$$\begin{aligned}\therefore\ (x-5)^3 &= (x)^3 + 3(x)^2(-5) + 3(x)(-5)^2 + (-5)^3 \\ &= x^3 - 15x^2 + 75x - 125\end{aligned}$$



Example 16

Self Tutor

Find the:

a 5th row of Pascal's triangle

b binomial expansion of $\left(x-\frac{2}{x}\right)^5$.

a

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

1 ← the 0th row, for $(a+b)^0$

1 1 ← the 1st row, for $(a+b)^1$

1 5 10 10 5 1 ← the 5th row, for $(a+b)^5$

b Using the coefficients obtained in **a**, $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Letting $a=(x)$ and $b=\left(\frac{-2}{x}\right)$, we find

$$\left(x-\frac{2}{x}\right)^5 = (x)^5 + 5(x)^4\left(\frac{-2}{x}\right) + 10(x)^3\left(\frac{-2}{x}\right)^2 + 10(x)^2\left(\frac{-2}{x}\right)^3 + 5(x)\left(\frac{-2}{x}\right)^4 + \left(\frac{-2}{x}\right)^5$$

$$= x^5 - 10x^3 + 40x - \frac{80}{x} + \frac{80}{x^3} - \frac{32}{x^5}$$

EXERCISE 10F

1 Use the binomial expansion of $(a+b)^3$ to expand and simplify:

- **a** $(p+q)^3$
- **b** $(x+1)^3$
- **c** $(x-3)^3$
- **d** $(2+x)^3$
- **e** $(3x-1)^3$
- **f** $(2x+5)^3$
- **g** $(2a-b)^3$
- **h** $\left(3x-\frac{1}{3}\right)^3$
- **i** $\left(2x+\frac{1}{x}\right)^3$

2 Use $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ to expand and simplify:

- **a** $(1+x)^4$
- **b** $(p-q)^4$
- **c** $(x-2)^4$
- **d** $(3-x)^4$
- **e** $(1+2x)^4$
- **f** $(2x-3)^4$
- **g** $(2x+b)^4$
- **h** $\left(x+\frac{1}{x}\right)^4$
- **i** $\left(2x-\frac{1}{x}\right)^4$

3 Expand and simplify:

- **a** $(x+2)^5$
- **b** $(x-2y)^5$
- **c** $(1+2x)^5$
- **d** $\left(x-\frac{1}{x}\right)^5$

4 Expand and simplify $(2+x)^5 + (2-x)^5$.

5 **a** Write down the 6th row of Pascal's triangle.

b Find the binomial expansion of:

- **i** $(x+2)^6$
- **ii** $(2x-1)^6$
- **iii** $\left(x+\frac{1}{x}\right)^6$

6 Expand and simplify:

- **a** $(1+\sqrt{2})^3$
- **b** $(\sqrt{5}+2)^4$
- **c** $(2-\sqrt{2})^5$

7 Find $\dfrac{(2+\sqrt{3})^3}{4+\sqrt{3}}$, giving your answer in the form $\dfrac{a+b\sqrt{3}}{c}$, where $a,\ b,\ c \in \mathbb{Z}$.

8 **a** Expand $(2+x)^6$. **b** Hence find the value of $(2.01)^6$.

9 The first two terms in a binomial expansion are: $(a+b)^3 = 8 + 12e^x + ...$

a Find a and b. **b** Hence determine the remaining two terms of the expansion.

10 Expand and simplify $(2x+3)(x+1)^4$.

11 Find the coefficient of:

a a^3b^2 in the expansion of $(3a+b)^5$ **b** a^3b^3 in the expansion of $(2a+3b)^6$.

G THE BINOMIAL THEOREM

Historical note Binomial Theorem

The Binomial Theorem is one of the most important results in mathematics.

Multiplying out binomial terms is a basic process which dates back to the beginning of algebra. Mathematicians had noticed relationships between the coefficients for many centuries, and Pascal's triangle was certainly widely used long before Pascal.

Isaac Newton discovered the Binomial Theorem in 1665, but he did not publish his results until much later. Newton was the first person to give a formula for the binomial coefficients. He did this because he wanted to go further. Newton's ground-breaking result included a generalisation of the Binomial Theorem to the case of $(a+b)^n$ where n is a rational number, such as $\frac{1}{2}$. This results in a sum with an infinite number of terms, called an infinite series. In doing this, Newton was the first person to confidently use the exponential notation that we recognise today for both negative and fractional powers.

In the previous Section we saw how the coefficients of the binomial expansion $(a+b)^n$ can be found in the nth row of Pascal's triangle. These coefficients are in fact the **binomial coefficients** $\binom{n}{r}$ for $r = 0,\ 1,\ 2,\ ....,\ n$.

$$\begin{array}{ccccccccc} & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array} \qquad \begin{array}{ccccccccc} & & & \binom{1}{0} & & \binom{1}{1} & & & \\ & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\ & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\ \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} \end{array}$$

The **Binomial Theorem** states that

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + ... + \binom{n}{r} a^{n-r}b^r + ... + b^n$$

where $\binom{n}{r}$ is the **binomial coefficient** of $a^{n-r}b^r$ and $r = 0,\ 1,\ 2,\ 3,\ ....,\ n$.

The **general term** or $(r+1)$th term in the binomial expansion is $T_{r+1} = \binom{n}{r} a^{n-r}b^r$.

Using sigma notation we write $(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r}b^r$.

Example 17

Self Tutor

Write down the first three and last two terms of the expansion of $\left(2x+\frac{1}{x}\right)^{12}$.

Do not simplify your answer.

$$\left(2x+\frac{1}{x}\right)^{12} = (2x)^{12} + \binom{12}{1}(2x)^{11}\left(\frac{1}{x}\right)^{1} + \binom{12}{2}(2x)^{10}\left(\frac{1}{x}\right)^{2} + \ldots.$$

$$\ldots. + \binom{12}{11}(2x)^{1}\left(\frac{1}{x}\right)^{11} + \left(\frac{1}{x}\right)^{12}$$

Example 18

Self Tutor

Find the 7th term of $\left(3x-\frac{4}{x^2}\right)^{14}$. Do not simplify your answer.

$a = (3x)$, $b = \left(\frac{-4}{x^2}\right)$, and $n = 14$

Given the general term $T_{r+1} = \binom{n}{r}a^{n-r}b^r$, we let $r = 6$

$$\therefore\ T_7 = \binom{14}{6}(3x)^8\left(\frac{-4}{x^2}\right)^6$$

Example 19

Self Tutor

In the expansion of $\left(x^2+\frac{4}{x}\right)^{12}$, find:

a the coefficient of x^6

b the constant term.

$a = (x^2)$, $b = \left(\frac{4}{x}\right)$, and $n = 12$

$$\therefore \text{ the general term } T_{r+1} = \binom{12}{r}(x^2)^{12-r}\left(\frac{4}{x}\right)^r$$

$$= \binom{12}{r}x^{24-2r} \times \frac{4^r}{x^r}$$

$$= \binom{12}{r}4^r x^{24-3r}$$

a If $24 - 3r = 6$

then $3r = 18$

$\therefore\ r = 6$

$\therefore\ T_7 = \binom{12}{6}4^6x^6$

$\therefore$ the coefficient of x^6 is $\binom{12}{6}4^6$ or 3 784 704.

b If $24 - 3r = 0$

then $3r = 24$

$\therefore\ r = 8$

$\therefore\ T_9 = \binom{12}{8}4^8x^0$

$\therefore$ the constant term is $\binom{12}{8}4^8$ or 32 440 320.

EXERCISE 10G

1 Write down the first three and last two terms of the following binomial expansions. Do not simplify your answers.

a $(1+2x)^{11}$ **b** $\left(3x+\frac{2}{x}\right)^{15}$ **c** $\left(2x-\frac{3}{x}\right)^{20}$

2 Without simplifying, write down:

a the 6th term of $(2x+5)^{15}$

b the 4th term of $\left(x^2+y\right)^9$

c the 10th term of $\left(x-\frac{2}{x}\right)^{17}$

d the 9th term of $\left(2x^2-\frac{1}{x}\right)^{21}$.

3 In the expansion of $(2x+3)^{12}$, find:

a the coefficient of x^8 **b** the coefficient of x^5.

4 In the expansion of $(1-3x)^{10}$, find:

a the coefficient of x^3 **b** the coefficient of x^7.

5 In the expansion of $\left(x^2+\frac{2}{x}\right)^9$, find:

a the coefficient of x^{12} **b** the constant term **c** the coefficient of x^{-6}.

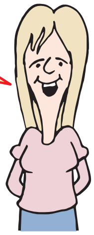
6 Consider the expansion of $(x+b)^7$.

a Write down the general term of the expansion.

b Find b given that the coefficient of x^4 is -280.

7 Find the term independent of x in the expansion of:

a $\left(x+\frac{2}{x^2}\right)^{15}$ **b** $\left(x-\frac{3}{x^2}\right)^9$.



The "term independent of x" is the constant term.

8 Find the coefficient of:

a x^{10} in the expansion of $(3+2x^2)^{10}$

b x^3 in the expansion of $\left(2x^2-\frac{3}{x}\right)^6$

c x^6y^3 in the expansion of $\left(2x^2-3y\right)^6$

d x^{12} in the expansion of $\left(2x^2-\frac{1}{x}\right)^{12}$.

9 In the expansion of $(k+x)^8$, the coefficient of x^5 is 10 times the coefficient of x^6. Find the value of k.

10 The coefficient of x^5 in the expansion of $(ax-2)^7$ is twice the coefficient of x^5 in the expansion of $(a+x)^9$. Find the value of a.

11 In the expansion of $\left(ax+\frac{b}{x}\right)^6$, the constant term is 20 000, and the coefficient of x^4 is equal to the coefficient of x^2.

a Show that $ab=10$ and $b=\frac{2a}{5}$.

b Find a and b given that they are both positive.

Example 20

Self Tutor

Find the coefficient of x^5 in the expansion of $(x+3)(2x-1)^6$.

$$\begin{aligned}&(x+3)(2x-1)^6\\ &= (x+3)[(2x)^6 + \binom{6}{1}(2x)^5(-1) + \binom{6}{2}(2x)^4(-1)^2 + ...]\\ &= (x+3)(2^6x^6 - \binom{6}{1}2^5x^5 + \binom{6}{2}2^4x^4 - ....)\end{aligned}$$

(2)

(1)

So, the terms containing x^5 are $\binom{6}{2}2^4x^5$ from (1)

and $-3\binom{6}{1}2^5x^5$ from (2)

$\therefore$ the coefficient of x^5 is $\binom{6}{2}2^4 - 3\binom{6}{1}2^5 = -336$

12 Find the coefficient of x^5 in the expansion of $(x+2)(x^2+1)^8$.

13 Find the term containing x^6 in the expansion of $(2-x)(3x+1)^9$.

14 Find the coefficient of x^4 in the expansion of:

a $(3-2x)^7$ **b** $(1+3x)(3-2x)^7$

15 Find:

a the coefficient of x^7 in the expansion of $(x^2-3)(2x-5)^8$

b the term independent of x in the expansion of $(1-x^2)\left(x+\frac{2}{x}\right)^6$.

16 When the expansion of $(a+bx)(1-x)^6$ is written in ascending powers of x, the first three terms are $3-20x+cx^2$. Find the values of a, b, and c.

Example 21

Self Tutor

Consider the expansion of $(1+3x)^n$, where $n \in \mathbb{Z}^+$.

If the coefficient of x^2 is 90, find the value of n.

$(1+3x)^n$ has general term $T_{r+1} = \binom{n}{r}1^{n-r}(3x)^r$

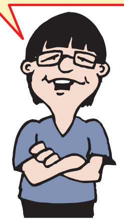
$= \binom{n}{r}3^rx^r$

$\therefore$ $T_3 = \binom{n}{2}3^2x^2$ is the x^2 term.

Since the coefficient of x^2 is 90, $\binom{n}{2}\times 9 = 90$

$$\begin{aligned}\therefore\ \frac{n(n-1)}{2} &= 10\\ \therefore\ n^2 - n &= 20\\ \therefore\ n^2 - n - 20 &= 0\\ \therefore\ (n-5)(n+4) &= 0\\ \therefore\ n &= 5 \qquad \{n>0\}\end{aligned}$$

$\binom{n}{2} = \frac{n(n-1)}{2}$ for all integers $n \geqslant 2$.



17 The coefficient of x^2 in the expansion of $(1+2x)^n$ is 112. Find n.

18 The coefficient of x^2 in the expansion of $\left(1-\frac{x}{3}\right)^n$ is $\frac{5}{3}$. Find n.

19 The third term of $(1+x)^n$ is $36x^2$. Find the fourth term.

20 Suppose $(1+kx)^n = 1 - 12x + 60x^2 - ....$. Find the values of k and n.

Review set 10A

1 Simplify: **a** $\dfrac{n!}{(n-2)!}$ **b** $\dfrac{n!+(n+1)!}{n!}$

2 Eight people enter a room and each person shakes hands with every other person. How many hand shakes are made?

3 The letters P, Q, R, S, and T are to be arranged in a row. How many of the possible arrangements:

- **a** end with T
- **b** begin with P and end with T?

4
- **a** How many three digit numbers can be formed using the digits 0 to 9?
- **b** How many of these numbers are divisible by 5?

5 The first two terms in a binomial expansion are: $(a+b)^4 = e^{4x} - 4e^{2x} + ...$

- **a** Find a and b.
- **b** Copy and complete the expansion.

6 Expand and simplify $(\sqrt{3}+2)^5$, giving your answer in the form $a+b\sqrt{3}$, $a, b \in \mathbb{Z}$.

7 Find the constant term in the expansion of $\left(3x^2+\frac{1}{x}\right)^8$.

8 Find c given that the expansion $(1+cx)(1+x)^4$ includes the term $22x^3$.

9 Steven and nine of his classmates are in a school committee. The committee must select a president, vice-president, and secretary. In how many ways can this be done if:

- **a** there are no restrictions
- **b** Steven must be the president
- **c** Steven cannot hold any of the key positions?

10 Find the coefficient of x^3 in the expansion of $(x+5)^6$.

11 A team of five is chosen from six men and four women.

- **a** How many different teams are possible with no restrictions?
- **b** How many different teams contain at least one person of each sex?

12 Find the coefficient of x^{-6} in the expansion of $\left(2x-\frac{3}{x^2}\right)^{12}$.

13 Find the coefficient of x^5 in the expansion of $(2x+3)(x-2)^6$.

14 Find the possible values of a if the coefficient of x^3 in $\left(2x+\frac{1}{ax^2}\right)^9$ is 288.

15 In the expansion of $(kx-1)^6$, the coefficient of x^4 is equal to four times the coefficient of x^2. Find the possible values of k.

Review set 10B

1 Alpha-numeric number plates have two letters followed by four digits. How many plates are possible if:

a there are no restrictions

b the first letter must be a vowel

c no letter or digit may be repeated?

2 **a** How many committees of five can be selected from eight men and seven women?

b How many of the committees contain two men and three women?

c How many of the committees contain at least one man?

3 Use the binomial expansion to find:

a $(x-2y)^3$

b $(3x+2)^4$

4 Find the coefficient of x^3 in the expansion of $(2x+5)^6$.

5 Find the term independent of x in the expansion of $\left(2x^2-\frac{1}{x}\right)^6$.

6 Find $\frac{(3-\sqrt{2})^3}{\sqrt{2}+1}$, giving your answer in the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Z}$.

7 Kristen's school offers 6 Group A subjects, 8 Group B subjects, and 5 Group C subjects. Kristen must select 2 Group A, 3 Group B, and 1 Group C subject to study. In how many ways can she make her selection?

8 Find the coefficient of x^4 in the expansion of $(x-3)(2x+1)^5$.

9 How many arrangements containing 4 different letters from the word DRAGONFLY are possible if:

a there are no restrictions

b the letters G and Y must not be included

c the arrangement must start with R and end with N?

10 Find the possible values of q if the constant terms in the expansions of $\left(x^3+\frac{q}{x^3}\right)^8$ and $\left(x^3+\frac{q}{x^3}\right)^4$ are equal.

11 Eight people enter a room and sit in a row of eight chairs. In how many ways can the sisters Cathy, Robyn, and Jane sit together in the row?

12 Find k in the expansion $(m-2n)^{10} = m^{10} - 20m^9n + km^8n^2 - .... + 1024n^{10}$.

13 A team of eight is chosen from 11 men and 7 women. How many different teams are possible if there:

a are no restrictions

b must be four of each sex on the team

c must be at least two women on the team

d must be more women than men?

14 The coefficient of x^2 in the expansion of $\left(1+\frac{x}{2}\right)^n$ is $\frac{21}{4}$. Find n.

15 The first three terms in the expansion of $(1+kx)^n$ are $1-4x+\frac{15}{2}x^2$. Find k and n.

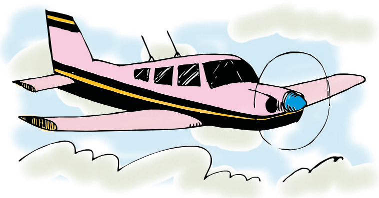
11

Vectors

Contents:

Opening problem

An aeroplane in calm conditions is flying at 800 km h^{-1} due east. A cold wind suddenly blows from the south-west at 35 km h^{-1}, pushing the aeroplane slightly off course.



Things to think about:

- **a** How can we use an array of numbers to represent the speed *and* direction of the plane?
- **b** What operation do we need to perform to find the effect of the wind on the aeroplane?
- **c** Can you determine the resulting speed and direction of the aeroplane?

A VECTORS AND SCALARS

In the **Opening Problem**, the effect of the wind on the aeroplane is determined by both its speed *and* its direction. The effect would be different if the wind was blowing against the aeroplane rather than from behind it.

Quantities which have only magnitude are called **scalars**.

Quantities which have both magnitude and direction are called **vectors**.

The *speed* of the plane is a scalar. It describes its size or strength.

The *velocity* of the plane is a vector. It includes both its speed and also its direction.

Other examples of vector quantities are:

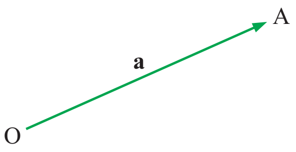
- acceleration
- force
- displacement
- momentum

For example, farmer Giles needs to remove a fence post. He starts by pushing on the post sideways to loosen the ground. Giles has a choice of how hard to push the post, and in which direction. The force he applies is therefore a vector.

From previous courses, you should have seen how we can represent a vector quantity using a **directed line segment** or **arrow**. The **length of the arrow** represents the size or magnitude of the quantity, and the **arrowhead** shows its direction.

POSITION VECTORS

Consider the vector from the origin O to the point A. We call this the **position vector** of point A.



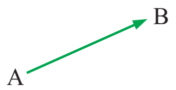
- This **position vector** could be represented by $\overrightarrow{OA}$ or $\mathbf{a}$ or $\vec{a}$.

  $\mathbf{a}$: bold used in textbooks

  $\vec{a}$: used by students

- The **magnitude** or **length** could be represented by $|\overrightarrow{OA}|$ or OA or $|\mathbf{a}|$ or $|\vec{a}|$.

Now consider the vector from point A to point B. We say that:

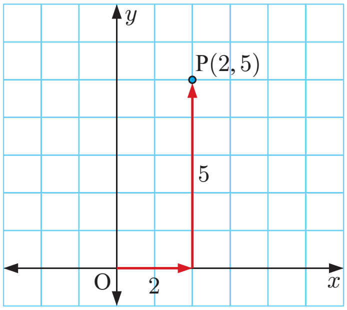


- $\overrightarrow{\mathrm{AB}}$ is the vector which **originates** at A and **terminates** at B
- $\overrightarrow{\mathrm{AB}}$ is the **position vector** of B relative to A.

When we plot points in the Cartesian plane, we move first in the x-direction and then in the y-direction.

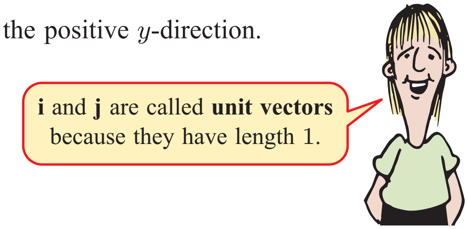
For example, to plot the point P(2, 5), we start at the origin, move 2 units in the x-direction, and then 5 units in the y-direction.

We therefore say that the vector from O to P is $\overrightarrow{\mathrm{OP}} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.



Suppose that $\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is a vector of length 1 unit in the positive x-direction

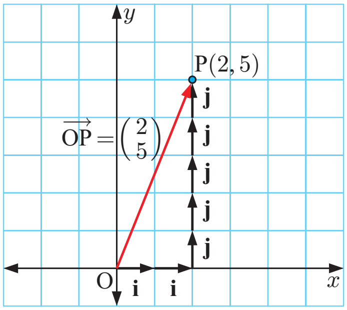
and that $\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is a vector of length 1 unit in the positive y-direction.



We can see that moving from O to P is equivalent to 2 lots of **i** plus 5 lots of **j**.

$$\overrightarrow{\mathrm{OP}} = 2\mathbf{i} + 5\mathbf{j}$$

$$\therefore \begin{pmatrix} 2 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 5\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



The point P(x, y) has **position vector** $\overrightarrow{\mathrm{OP}} = \begin{pmatrix} x \\ y \end{pmatrix} = x\mathbf{i} + y\mathbf{j}$.

component form unit vector form

$\mathbf{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the **base unit vector** in the x-direction.

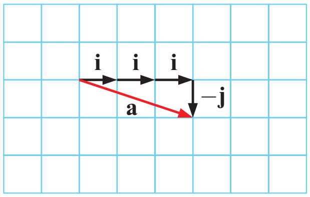
$\mathbf{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the **base unit vector** in the y-direction.

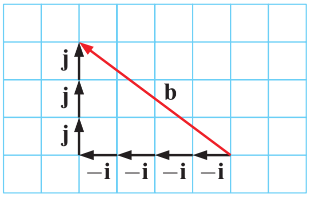
The set of vectors $\{\mathbf{i}, \mathbf{j}\}$ is the **standard basis** for the 2-dimensional (x, y) coordinate system.

All vectors in the plane can be described in terms of the base unit vectors **i** and **j**.

For example: $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$

$\mathbf{b} = -4\mathbf{i} + 3\mathbf{j}$



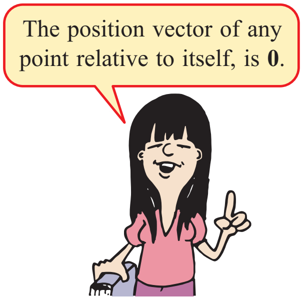


THE ZERO VECTOR

The **zero vector**, **0**, is a vector of length 0.

It is the only vector with no direction.

In component form, $\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.



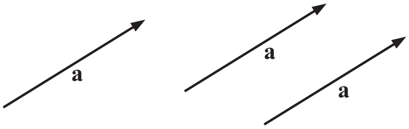
When we write the zero vector by hand, we usually write $\overrightarrow{0}$.

VECTOR EQUALITY

Two vectors are **equal** if they have the same magnitude and direction.

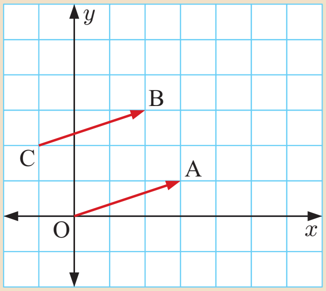
In component form, their x-components are equal *and* their y-components are equal.

Equal vectors are **parallel** and in the same direction, and are **equal in length**. The arrows that represent them are translations of one another.



Example 1

Self Tutor



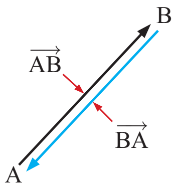
a Write $\overrightarrow{\text{OA}}$ and $\overrightarrow{\text{CB}}$ in component form and in unit vector form.

b Comment on your answers in **a**.

a $\overrightarrow{\text{OA}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3\mathbf{i} + \mathbf{j}$ $\overrightarrow{\text{CB}} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3\mathbf{i} + \mathbf{j}$

b The vectors $\overrightarrow{\text{OA}}$ and $\overrightarrow{\text{CB}}$ are equal.

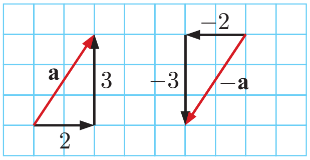
NEGATIVE VECTORS



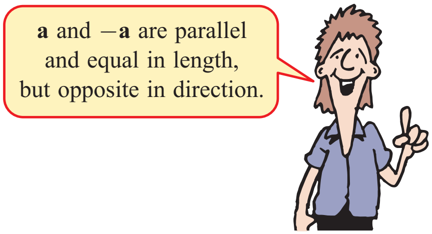
$\overrightarrow{AB}$ and $\overrightarrow{BA}$ have the same length, but they have opposite directions.

We say that $\overrightarrow{BA}$ is the **negative** of $\overrightarrow{AB}$, and write $\overrightarrow{BA} = -\overrightarrow{AB}$.

In the diagram we see the vector $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and its negative $-\mathbf{a} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$.



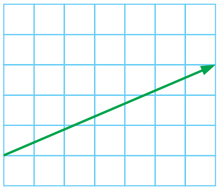
If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ then $-\mathbf{a} = \begin{pmatrix} -a_1 \\ -a_2 \end{pmatrix}$.



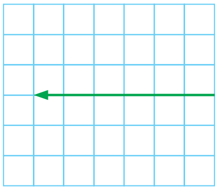
EXERCISE 11A

1 Write the illustrated vectors in component form and in unit vector form:

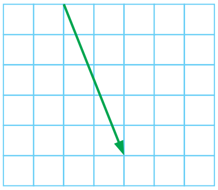
a



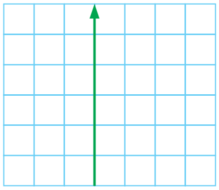
b



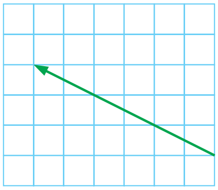
c



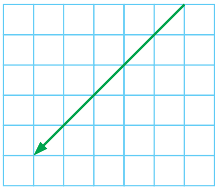
d



e



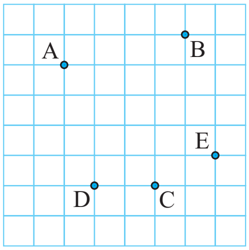
f



2 Write each vector in unit vector form, and illustrate it using an arrow diagram:

a $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$ **d** $\begin{pmatrix} -1 \\ -3 \end{pmatrix}$

3



a Find in component form and in unit vector form:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BA}$ **iii** $\overrightarrow{BC}$
iv $\overrightarrow{DC}$ **v** $\overrightarrow{AC}$ **vi** $\overrightarrow{DE}$

b Which two vectors in **a** are equal? Explain your answer.

c Which two vectors in **a** are negatives? Explain your answer.

4 Write in component form and illustrate using a directed line segment:

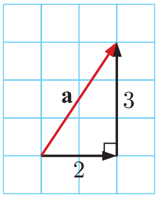
a $\mathbf{i}+2\mathbf{j}$ **b** $-\mathbf{i}+3\mathbf{j}$ **c** $-5\mathbf{j}$ **d** $4\mathbf{i}-2\mathbf{j}$

5 Write down the negative of:

a $\begin{pmatrix}1\\4\end{pmatrix}$ **b** $\begin{pmatrix}-2\\3\end{pmatrix}$ **c** $\begin{pmatrix}5\\-2\end{pmatrix}$ **d** $\begin{pmatrix}0\\-6\end{pmatrix}$

B THE MAGNITUDE OF A VECTOR

Consider vector $\mathbf{a}=\begin{pmatrix}2\\3\end{pmatrix}=2\mathbf{i}+3\mathbf{j}$.



The **magnitude** or **length** of **a** is represented by $|\,\mathbf{a}\,|$.

By Pythagoras, $|\,\mathbf{a}\,|^2=2^2+3^2=4+9=13$

$\therefore\ |\,\mathbf{a}\,|=\sqrt{13}$ units {since $|\,\mathbf{a}\,|>0$}

If $\mathbf{a}=\begin{pmatrix}a_1\\a_2\end{pmatrix}=a_1\mathbf{i}+a_2\mathbf{j}$, the **magnitude** or **length** of **a** is $|\,\mathbf{a}\,|=\sqrt{a_1^2+a_2^2}$.

Example 2 Self Tutor

If $\mathbf{p}=\begin{pmatrix}3\\-5\end{pmatrix}$ and $\mathbf{q}=2\mathbf{i}-5\mathbf{j}$, find:

a $|\,\mathbf{p}\,|$ **b** $|\,\mathbf{q}\,|$

a $\mathbf{p}=\begin{pmatrix}3\\-5\end{pmatrix}$

$\therefore\ |\,\mathbf{p}\,|=\sqrt{3^2+(-5)^2}$

$=\sqrt{34}$ units

b $\mathbf{q}=2\mathbf{i}-5\mathbf{j}=\begin{pmatrix}2\\-5\end{pmatrix}$

$\therefore\ |\,\mathbf{q}\,|=\sqrt{2^2+(-5)^2}$

$=\sqrt{29}$ units

UNIT VECTORS

A **unit vector** is any vector which has a length of one unit.

$\mathbf{i}=\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{j}=\begin{pmatrix}0\\1\end{pmatrix}$ are the base unit vectors in the positive x and y-directions respectively.

Example 3

Self Tutor

Find k given that $\begin{pmatrix} -\frac{1}{3} \\ k \end{pmatrix}$ is a unit vector.

Since $\begin{pmatrix} -\frac{1}{3} \\ k \end{pmatrix}$ is a unit vector, $\sqrt{(-\frac{1}{3})^2 + k^2} = 1$

$$\therefore \quad \sqrt{\tfrac{1}{9} + k^2} = 1$$

$$\therefore \quad \tfrac{1}{9} + k^2 = 1 \quad \{\text{squaring both sides}\}$$

$$\therefore \quad k^2 = \tfrac{8}{9}$$

$$\therefore \quad k = \pm\tfrac{\sqrt{8}}{3}$$

EXERCISE 11B

1 Find the magnitude of:

a $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **b** $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$ **c** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ **d** $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$ **e** $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$

2 Find the length of:

a $\mathbf{i} + \mathbf{j}$ **b** $5\mathbf{i} - 12\mathbf{j}$ **c** $-\mathbf{i} + 4\mathbf{j}$ **d** $3\mathbf{i}$ **e** $k\mathbf{j}$

3 Which of the following are unit vectors?

a $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ **b** $\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ **c** $\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$ **d** $\begin{pmatrix} -\frac{3}{5} \\ -\frac{4}{5} \end{pmatrix}$ **e** $\begin{pmatrix} \frac{2}{7} \\ -\frac{5}{7} \end{pmatrix}$

4 Find k for the unit vectors:

a $\begin{pmatrix} 0 \\ k \end{pmatrix}$ **b** $\begin{pmatrix} k \\ 0 \end{pmatrix}$ **c** $\begin{pmatrix} k \\ 1 \end{pmatrix}$ **d** $\begin{pmatrix} k \\ k \end{pmatrix}$ **e** $\begin{pmatrix} \frac{1}{2} \\ k \end{pmatrix}$

5 Given $\mathbf{v} = \begin{pmatrix} 8 \\ p \end{pmatrix}$ and $|\mathbf{v}| = \sqrt{73}$ units, find the possible values of p.

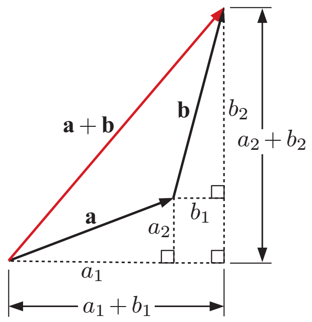
C OPERATIONS WITH PLANE VECTORS

VECTOR ADDITION

Consider adding vectors $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Notice that:

- the horizontal step for $\mathbf{a} + \mathbf{b}$ is $a_1 + b_1$
- the vertical step for $\mathbf{a} + \mathbf{b}$ is $a_2 + b_2$.



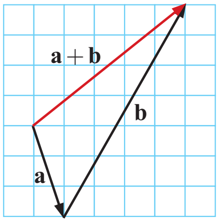
If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ then $\mathbf{a} + \mathbf{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \end{pmatrix}$.

Example 4 Self Tutor

If $\mathbf{a} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$, find $\mathbf{a} + \mathbf{b}$. Check your answer graphically.

$$\begin{aligned}\mathbf{a} + \mathbf{b} &= \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ 7 \end{pmatrix} \\ &= \begin{pmatrix} 1+4 \\ -3+7 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 4 \end{pmatrix}\end{aligned}$$

Graphical check:



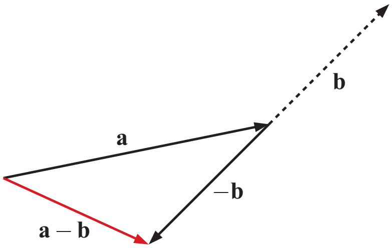
VECTOR SUBTRACTION

To subtract one vector from another, we simply **add its negative**.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

then

$$\begin{aligned}\mathbf{a} - \mathbf{b} &= \mathbf{a} + (-\mathbf{b}) \\ &= \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} -b_1 \\ -b_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}\end{aligned}$$



If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, then $\mathbf{a} - \mathbf{b} = \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix}$.

Example 5 Self Tutor

Given $\mathbf{p} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix} -2 \\ -5 \end{pmatrix}$, find:

a $\mathbf{q} - \mathbf{p}$

b $\mathbf{p} - \mathbf{q} - \mathbf{r}$

a

$$\begin{aligned}&\mathbf{q} - \mathbf{p} \\ &= \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 1-3 \\ 4+2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 6 \end{pmatrix}\end{aligned}$$

b

$$\begin{aligned}&\mathbf{p} - \mathbf{q} - \mathbf{r} \\ &= \begin{pmatrix} 3 \\ -2 \end{pmatrix} - \begin{pmatrix} 1 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} 3-1+2 \\ -2-4+5 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -1 \end{pmatrix}\end{aligned}$$

SCALAR MULTIPLICATION

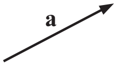
A **scalar** is a non-vector quantity. It has a size but no direction.

We can multiply vectors by scalars such as 2 and -3, or in fact any $k \in \mathbb{R}$.

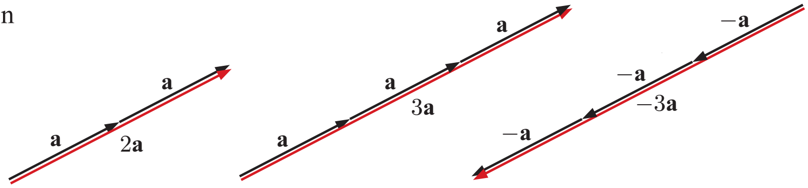
If $\mathbf{a}$ is a vector, we define $2\mathbf{a} = \mathbf{a} + \mathbf{a}$ and $3\mathbf{a} = \mathbf{a} + \mathbf{a} + \mathbf{a}$

so $-3\mathbf{a} = 3(-\mathbf{a}) = (-\mathbf{a}) + (-\mathbf{a}) + (-\mathbf{a})$.

If $\mathbf{a}$ is



then



So, $2\mathbf{a}$ is in the same direction as $\mathbf{a}$ but is twice as long as $\mathbf{a}$

$3\mathbf{a}$ is in the same direction as $\mathbf{a}$ but is three times longer than $\mathbf{a}$

$-3\mathbf{a}$ has the opposite direction to $\mathbf{a}$ and is three times longer than $\mathbf{a}$.

If $\mathbf{a}$ is a vector and k is a scalar, then $k\mathbf{a}$ is also a vector and we are performing **scalar multiplication**.

If $k > 0$, $k\mathbf{a}$ and $\mathbf{a}$ have the same direction.
If $k < 0$, $k\mathbf{a}$ and $\mathbf{a}$ have opposite directions.
If $k = 0$, $k\mathbf{a} = \mathbf{0}$, the zero vector.

If k is any scalar and $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, then $k\mathbf{v} = \begin{pmatrix} kv_1 \\ kv_2 \end{pmatrix}$.

VECTOR SCALAR MULTIPLICATION

Notice that:

- $(-1)\mathbf{v} = \begin{pmatrix} (-1)\,v_1 \\ (-1)\,v_2 \end{pmatrix} = \begin{pmatrix} -v_1 \\ -v_2 \end{pmatrix} = -\mathbf{v}$
- $(0)\mathbf{v} = \begin{pmatrix} (0)\,v_1 \\ (0)\,v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \mathbf{0}$

Example 6

Self Tutor

If $\mathbf{p} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, find: **a** $3\mathbf{q}$ **b** $\mathbf{p} + 2\mathbf{q}$ **c** $\frac{1}{2}\mathbf{p} - 3\mathbf{q}$

a $3\mathbf{q}$

$$= 3\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 6 \\ -9 \end{pmatrix}$$

b $\mathbf{p} + 2\mathbf{q}$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 4 + 2(2) \\ 1 + 2(-3) \end{pmatrix}$$

$$= \begin{pmatrix} 8 \\ -5 \end{pmatrix}$$

c $\frac{1}{2}\mathbf{p} - 3\mathbf{q}$

$$= \tfrac{1}{2}\begin{pmatrix} 4 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(4) - 3(2) \\ \frac{1}{2}(1) - 3(-3) \end{pmatrix}$$

$$= \begin{pmatrix} -4 \\ 9\frac{1}{2} \end{pmatrix}$$

Example 7

Self Tutor

If $\mathbf{p} = 3\mathbf{i} - 5\mathbf{j}$ and $\mathbf{q} = -\mathbf{i} - 2\mathbf{j}$, find $|\mathbf{p} - 2\mathbf{q}|$.

$$\begin{aligned}\mathbf{p} - 2\mathbf{q} &= 3\mathbf{i} - 5\mathbf{j} - 2(-\mathbf{i} - 2\mathbf{j})\\ &= 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{i} + 4\mathbf{j}\\ &= 5\mathbf{i} - \mathbf{j}\end{aligned}$$

$$\therefore\ |\mathbf{p} - 2\mathbf{q}| = \sqrt{5^2 + (-1)^2} = \sqrt{26}\text{ units}$$

EXERCISE 11C

VECTOR RACE GAME

1 If $\mathbf{a} = \begin{pmatrix}-3\\2\end{pmatrix}$, $\mathbf{b} = \begin{pmatrix}1\\4\end{pmatrix}$, and $\mathbf{c} = \begin{pmatrix}-2\\-5\end{pmatrix}$ find:

a $\mathbf{a} + \mathbf{b}$ **b** $\mathbf{b} + \mathbf{a}$ **c** $\mathbf{b} + \mathbf{c}$ **d** $\mathbf{c} + \mathbf{b}$
e $\mathbf{a} + \mathbf{c}$ **f** $\mathbf{c} + \mathbf{a}$ **g** $\mathbf{a} + \mathbf{a}$ **h** $\mathbf{b} + \mathbf{a} + \mathbf{c}$

2 Given $\mathbf{p} = \begin{pmatrix}-4\\2\end{pmatrix}$, $\mathbf{q} = \begin{pmatrix}-1\\-5\end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix}3\\-2\end{pmatrix}$ find:

a $\mathbf{p} - \mathbf{q}$ **b** $\mathbf{q} - \mathbf{r}$ **c** $\mathbf{p} + \mathbf{q} - \mathbf{r}$
d $\mathbf{p} - \mathbf{q} - \mathbf{r}$ **e** $\mathbf{q} - \mathbf{r} - \mathbf{p}$ **f** $\mathbf{r} + \mathbf{q} - \mathbf{p}$

3 Consider $\mathbf{a} = \begin{pmatrix}a_1\\a_2\end{pmatrix}$.

a Use vector addition to show that $\mathbf{a} + \mathbf{0} = \mathbf{a}$.
b Use vector subtraction to show that $\mathbf{a} - \mathbf{a} = \mathbf{0}$.

4 For $\mathbf{p} = \begin{pmatrix}1\\5\end{pmatrix}$, $\mathbf{q} = \begin{pmatrix}-2\\4\end{pmatrix}$, and $\mathbf{r} = \begin{pmatrix}-3\\-1\end{pmatrix}$ find:

a $-3\mathbf{p}$ **b** $\frac{1}{2}\mathbf{q}$ **c** $2\mathbf{p} + \mathbf{q}$ **d** $\mathbf{p} - 2\mathbf{q}$
e $\mathbf{p} - \frac{1}{2}\mathbf{r}$ **f** $2\mathbf{p} + 3\mathbf{r}$ **g** $2\mathbf{q} - 3\mathbf{r}$ **h** $2\mathbf{p} - \mathbf{q} + \frac{1}{3}\mathbf{r}$

5 Consider $\mathbf{p} = \begin{pmatrix}1\\1\end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix}2\\-1\end{pmatrix}$. Find geometrically and then comment on the results:

a $\mathbf{p} + \mathbf{p} + \mathbf{q} + \mathbf{q} + \mathbf{q}$ **b** $\mathbf{p} + \mathbf{q} + \mathbf{p} + \mathbf{q} + \mathbf{q}$ **c** $\mathbf{q} + \mathbf{p} + \mathbf{q} + \mathbf{p} + \mathbf{q}$

6 For $\mathbf{r} = \begin{pmatrix}2\\3\end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix}-1\\4\end{pmatrix}$ find:

a $|\mathbf{r}|$ **b** $|\mathbf{s}|$ **c** $|\mathbf{r} + \mathbf{s}|$ **d** $|\mathbf{r} - \mathbf{s}|$ **e** $|\mathbf{s} - 2\mathbf{r}|$

7 If $\mathbf{p} = \begin{pmatrix}1\\3\end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix}-2\\4\end{pmatrix}$ find:

a $|\mathbf{p}|$ **b** $|2\mathbf{p}|$ **c** $|-2\mathbf{p}|$ **d** $|3\mathbf{p}|$ **e** $|-3\mathbf{p}|$
f $|\mathbf{q}|$ **g** $|4\mathbf{q}|$ **h** $|-4\mathbf{q}|$ **i** $\left|\frac{1}{2}\mathbf{q}\right|$ **j** $\left|-\frac{1}{2}\mathbf{q}\right|$

8 Suppose $\mathbf{a} = 2\mathbf{i} - \mathbf{j}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j}$, and $\mathbf{c} = -4\mathbf{i}$. Find:

a $\mathbf{a} + \mathbf{b}$ **b** $3\mathbf{b} + \mathbf{c}$ **c** $\mathbf{a} - \mathbf{c}$

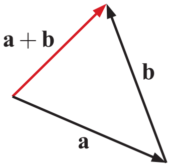
d $2\mathbf{b} - \mathbf{a}$ **e** $|\mathbf{c} + 2\mathbf{a}|$ **f** $|-2\mathbf{b}|$

9 Suppose $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Prove that:

a if $k\mathbf{a} = \mathbf{b}$, $k \neq 0$, then $\mathbf{a} = \frac{1}{k}\mathbf{b}$ **b** $|k\mathbf{a}| = |k|\,|\mathbf{a}|$

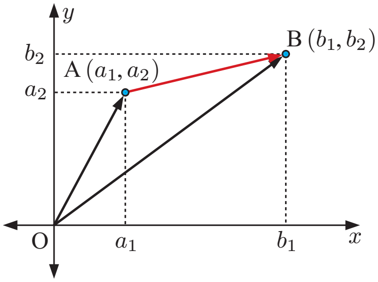
10 Prove that $|\mathbf{a} + \mathbf{b}| \leqslant |\mathbf{a}| + |\mathbf{b}|$:

a using a geometric argument and the diagram



b by letting $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

D THE VECTOR BETWEEN TWO POINTS



In the diagram, point A has position vector $\overrightarrow{OA} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$,

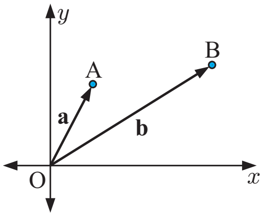
and point B has position vector $\overrightarrow{OB} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

$$\begin{aligned} \therefore\ \overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} - \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ &= \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} \end{aligned}$$

The **position vector of B relative to A** is $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix}$.

In general, for two points A and B with position vectors **a** and **b** respectively, we observe

$$\begin{aligned} \overrightarrow{AB} &= -\mathbf{a} + \mathbf{b} \\ &= \mathbf{b} - \mathbf{a} \\ &= \begin{pmatrix} b_1 - a_1 \\ b_2 - a_2 \end{pmatrix} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \overrightarrow{BA} &= -\mathbf{b} + \mathbf{a} \\ &= \mathbf{a} - \mathbf{b} \\ &= \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix} \end{aligned}$$



Example 8

Self Tutor

Given points A$(-1, 2)$, B$(3, 4)$, and C$(4, -5)$, find the position vector of:

a B from O **b** B from A **c** A from C

a The position vector of B relative to O is $\overrightarrow{OB} = \begin{pmatrix} 3-0 \\ 4-0 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

b The position vector of B relative to A is $\overrightarrow{AB} = \begin{pmatrix} 3--1 \\ 4-2 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

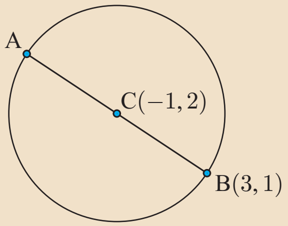
c The position vector of A relative to C is $\overrightarrow{CA} = \begin{pmatrix} -1-4 \\ 2--5 \end{pmatrix} = \begin{pmatrix} -5 \\ 7 \end{pmatrix}$.

Example 9

Self Tutor

[AB] is the diameter of a circle with centre C$(-1, 2)$. If B is $(3, 1)$, find:

a $\overrightarrow{BC}$ **b** the coordinates of A.



a $\overrightarrow{BC} = \begin{pmatrix} -1-3 \\ 2-1 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

b If A has coordinates (a, b), then $\overrightarrow{CA} = \begin{pmatrix} a-(-1) \\ b-2 \end{pmatrix} = \begin{pmatrix} a+1 \\ b-2 \end{pmatrix}$

But $\overrightarrow{CA} = \overrightarrow{BC}$, so $\begin{pmatrix} a+1 \\ b-2 \end{pmatrix} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

$$\therefore \quad a+1=-4 \quad \text{and} \quad b-2=1$$

$$\therefore \quad a=-5 \quad \text{and} \quad b=3$$

$\therefore$ A is $(-5, 3)$.

EXERCISE 11D

1 Find $\overrightarrow{AB}$ given:

a A$(2, 3)$ and B$(4, 7)$
b A$(3, -1)$ and B$(1, 4)$
c A$(-2, 7)$ and B$(1, 4)$
d B$(3, 0)$ and A$(2, 5)$
e B$(6, -1)$ and A$(0, 4)$
f B$(0, 0)$ and A$(-1, -3)$

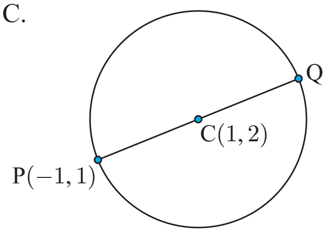
2 Consider the point A$(1, 4)$. Find the coordinates of:

a B given $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$
b C given $\overrightarrow{CA} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.

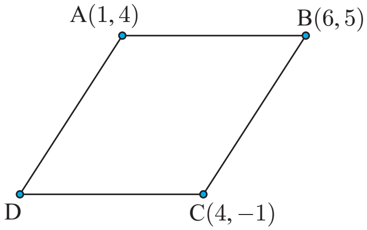
3 [PQ] is the diameter of a circle with centre C.

a Find $\overrightarrow{PC}$.

b Hence find the coordinates of Q.



4



ABCD is a parallelogram.

a Find $\overrightarrow{AB}$.

b Find $\overrightarrow{CD}$.

c Hence find the coordinates of D.

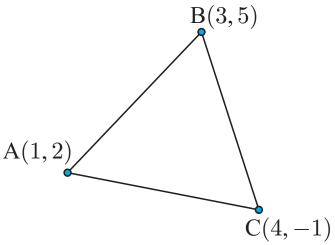
5 $A(-1,\ 3)$ and $B(3,\ k)$ are two points which are 5 units apart.

a Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

b Hence, find the two possible values of k.

c Show, by illustration, why k should have two possible values.

6



a Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

b Explain why $\overrightarrow{BC} = -\overrightarrow{AB} + \overrightarrow{AC}$.

c Hence find $\overrightarrow{BC}$.

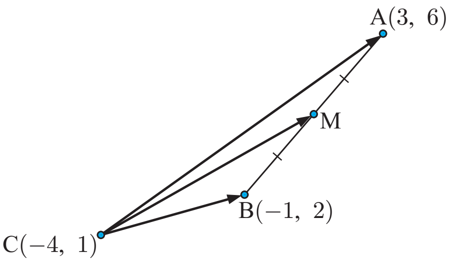
d Check your answer to **c** by direct evaluation.

7 **a** Given $\overrightarrow{BA} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$, find $\overrightarrow{AC}$.

b Given $\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\overrightarrow{CA} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, find $\overrightarrow{CB}$.

c Given $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, and $\overrightarrow{RS} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$, find $\overrightarrow{SP}$.

8

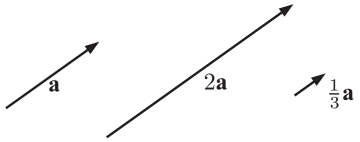


a Find the coordinates of M.

b Find vectors $\overrightarrow{CA}$, $\overrightarrow{CM}$, and $\overrightarrow{CB}$.

c Verify that $\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB}$.

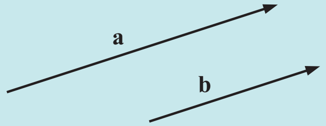
E PARALLELISM



are parallel vectors of different length.

Two non-zero vectors are **parallel** if and only if one is a scalar multiple of the other.

Given any non-zero vector $\mathbf{a}$ and non-zero scalar k, the vector $k\mathbf{a}$ is parallel to $\mathbf{a}$.



- If $\mathbf{a}$ is parallel to $\mathbf{b}$, then there exists a scalar k such that $\mathbf{a} = k\mathbf{b}$.
- If $\mathbf{a} = k\mathbf{b}$ for some scalar k, then
  - $\mathbf{a}$ is parallel to $\mathbf{b}$, and
  - $|\mathbf{a}| = |k|\,|\mathbf{b}|$.

$|k|$ is the modulus of k, whereas $|\mathbf{a}|$ is the length of vector $\mathbf{a}$.

Example 10

Self Tutor

Find r given that $\mathbf{a} = \begin{pmatrix} -1 \\ r \end{pmatrix}$ is parallel to $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

Since $\mathbf{a}$ and $\mathbf{b}$ are parallel, $\mathbf{a} = k\mathbf{b}$ for some scalar k.

$$\therefore \begin{pmatrix} -1 \\ r \end{pmatrix} = k\begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$\therefore -1 = 2k \text{ and } r = -3k$$

$$\therefore k = -\tfrac{1}{2} \text{ and hence } r = -3(-\tfrac{1}{2}) = \tfrac{3}{2}$$

UNIT VECTORS

Given a non-zero vector $\mathbf{a}$, its magnitude $|\mathbf{a}|$ is a scalar quantity.

If we multiply $\mathbf{a}$ by the scalar $\dfrac{1}{|\mathbf{a}|}$, we obtain the parallel vector $\dfrac{1}{|\mathbf{a}|}\mathbf{a}$ with length 1.

- A unit vector in the direction of $\mathbf{a}$ is $\dfrac{1}{|\mathbf{a}|}\mathbf{a}$.
- A vector of length k in the same direction as $\mathbf{a}$ is $\dfrac{k}{|\mathbf{a}|}\mathbf{a}$.
- A vector of length k which is *parallel to* $\mathbf{a}$ could be $\pm\dfrac{k}{|\mathbf{a}|}\mathbf{a}$.

Example 11

Self Tutor

If $\mathbf{a} = 3\mathbf{i} - \mathbf{j}$, find:

a a unit vector in the direction of $\mathbf{a}$

b a vector of length 4 units in the direction of $\mathbf{a}$

c vectors of length 4 units which are parallel to $\mathbf{a}$.

a $$\begin{aligned} |\,\mathbf{a}\,| &= \sqrt{3^2+(-1)^2} \\ &= \sqrt{9+1} \\ &= \sqrt{10} \text{ units} \end{aligned}$$

$\therefore$ the unit vector is $$\begin{aligned} &\tfrac{1}{\sqrt{10}}(3\mathbf{i}-\mathbf{j}) \\ =\; &\tfrac{3}{\sqrt{10}}\,\mathbf{i} - \tfrac{1}{\sqrt{10}}\,\mathbf{j} \end{aligned}$$

b This vector is $$\begin{aligned} &\tfrac{4}{\sqrt{10}}(3\mathbf{i}-\mathbf{j}) \\ =\; &\tfrac{12}{\sqrt{10}}\,\mathbf{i} - \tfrac{4}{\sqrt{10}}\,\mathbf{j} \end{aligned}$$

c The vectors are $\tfrac{12}{\sqrt{10}}\,\mathbf{i} - \tfrac{4}{\sqrt{10}}\,\mathbf{j}$ and $-\tfrac{12}{\sqrt{10}}\,\mathbf{i} + \tfrac{4}{\sqrt{10}}\,\mathbf{j}$.

EXERCISE 11E

1 Find r given that $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -6 \\ r \end{pmatrix}$ are parallel.

2 Find a given that $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} a \\ 2 \end{pmatrix}$ are parallel.

3 What can be deduced from the following?

a $\overrightarrow{AB} = 3\overrightarrow{CD}$ **b** $\overrightarrow{RS} = -\frac{1}{2}\overrightarrow{KL}$ **c** $\overrightarrow{AB} = 2\overrightarrow{BC}$

4 If $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, write down the vector:

a in the same direction as $\mathbf{a}$ and twice its length

b in the opposite direction to $\mathbf{a}$ and half its length.

5 Find the unit vector in the direction of:

a $\mathbf{i} + 2\mathbf{j}$ **b** $\mathbf{i} - 3\mathbf{j}$ **c** $2\mathbf{i} - \mathbf{j}$

6 Find a vector $\mathbf{v}$ which has:

a the same direction as $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and length 3 units

b the opposite direction to $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$ and length 2 units.

7 A is $(3, 2)$ and point B is 4 units from A in the direction $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

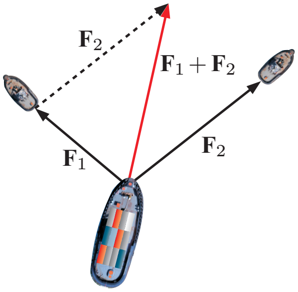
a Find $\overrightarrow{AB}$. **b** Find $\overrightarrow{OB}$ using $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$.

c Hence deduce the coordinates of B.

F PROBLEMS INVOLVING VECTOR OPERATIONS

When we apply vectors to problems in the real world, we often consider the combined effect when vectors are added together. This sum is called the **resultant vector**.

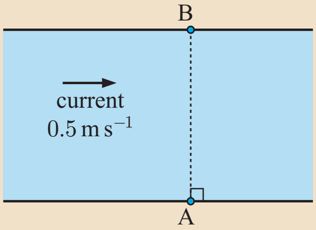
The diagram shows an example of vector addition. Two tug boats are being used to pull a ship into port. If the tugs tow with forces $\mathbf{F}_1$ and $\mathbf{F}_2$ then the resultant force is $\mathbf{F}_1 + \mathbf{F}_2$.



Example 12

Self Tutor

In still water, Jacques can swim at $1.5\ \text{m}\,\text{s}^{-1}$. Jacques is at point A on the edge of a canal, and considers point B directly opposite. A current is flowing from the left at a constant speed of $0.5\ \text{m}\,\text{s}^{-1}$.



a If Jacques dives in straight towards B, and swims without allowing for the current, what will his actual speed and direction be?

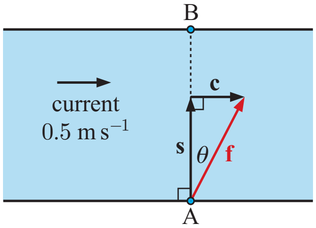
b Jacques wants to swim directly across the canal to point B.

  i At what angle should Jacques *aim* to swim in order that the current will correct his direction?

  ii What will Jacques' actual speed be?

Suppose $\mathbf{c}$ is the current's velocity vector,
$\mathbf{s}$ is the velocity vector Jacques would have if the water was still, and
$\mathbf{f} = \mathbf{c} + \mathbf{s}$ is Jacques' resultant velocity vector.

a Jacques aims directly across the river, but the current takes him downstream to the right.



$$|\mathbf{f}|^2 = |\mathbf{c}|^2 + |\mathbf{s}|^2 = 0.5^2 + 1.5^2 = 2.5$$

$$\therefore\ |\mathbf{f}| \approx 1.58$$

$$\tan\theta = \frac{0.5}{1.5}$$

$$\therefore\ \theta \approx 18.4°$$

Jacques has an actual speed of approximately $1.58\ \text{m}\,\text{s}^{-1}$ and his direction of motion is approximately 18.4° to the right of his intended line.

b Jacques needs to aim to the left of B so the current will correct his direction.

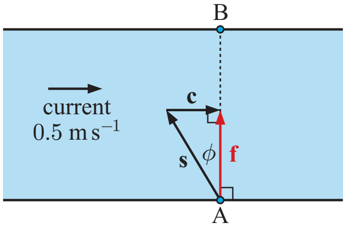
i $\sin\phi = \dfrac{0.5}{1.5}$

$\therefore\ \phi \approx 19.5°$

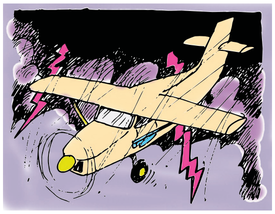
Jacques needs to aim approximately 19.5° to the left of B.

ii $$|\mathbf{f}|^2+|\mathbf{c}|^2=|\mathbf{s}|^2$$
$$\therefore\ |\mathbf{f}|^2+0.5^2=1.5^2$$
$$\therefore\ |\mathbf{f}|^2=2$$
$$\therefore\ |\mathbf{f}|\approx 1.41$$

In these conditions, Jacques' actual speed towards B is approximately 1.41 m s^{-1}.



Another example of vector addition is when an aircraft is affected by wind. A pilot needs to know how to compensate for the wind, especially during take-off and landing.



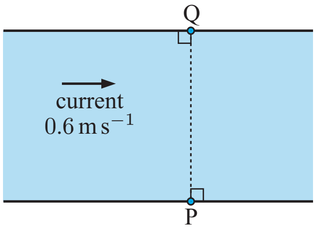
SIMULATION

EXERCISE 11F

1 A bird can normally fly with constant speed 6 m s^{-1}. Using a vector diagram to illustrate each situation, find the bird's speed if:

- **a** it is assisted by a wind of 1 m s^{-1} from directly behind it
- **b** it flies into a head wind of 1 m s^{-1}.

2 In still water, Mary can swim at 1.2 m s^{-1}. She is standing at point P on the edge of a canal, directly opposite point Q. The water is flowing to the right at a constant speed of 0.6 m s^{-1}.



- **a** If Mary tries to swim directly from P to Q without allowing for the current, what will her actual velocity be?
- **b** Mary wants to swim directly across the canal to point Q.
  - **i** At what angle should she *aim* to swim in order that the current corrects her direction?
  - **ii** What will Mary's actual speed be?

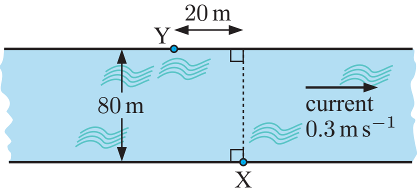
3 A boat needs to travel south at a speed of 20 km h^{-1}. However, a constant current of 6 km h^{-1} is flowing from the south-east. Use vectors to find:

- **a** the equivalent speed in still water for the boat to achieve the actual speed of 20 km h^{-1}
- **b** the direction in which the boat must head to compensate for the current.

4 As part of an endurance race, Stephanie needs to swim from X to Y across a wide river.

Stephanie swims at 1.8 m s^{-1} in still water.

The river flows with a consistent current of 0.3 m s^{-1} as shown.



- **a** Find the distance from X to Y.
- **b** In which direction should Stephanie *aim* so that the current will push her onto a path directly towards Y?
- **c** Find the time Stephanie will take to cross the river.

5 An aeroplane needs to fly due east from one city to another at a speed of 400 km h^{-1}. However, a 50 km h^{-1} wind blows constantly from the north-east.

a How does the wind affect the speed of the aeroplane?

b In what direction must the aeroplane head to compensate for the wind?

G LINES

We have seen in Cartesian geometry that we can determine the **equation of a line** using its **direction** and any **fixed point** on the line. We can do the same using vectors.

Suppose a line passes through a fixed point A with position vector **a**, and that the line is parallel to the vector **b**.

DEMO

Consider a point R on the line so that $\overrightarrow{OR} = \mathbf{r}$.

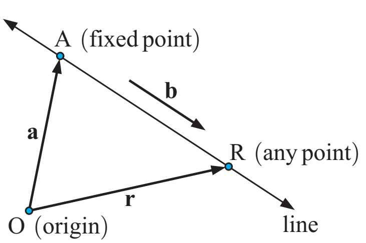
By vector addition, $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$

$\therefore \quad \mathbf{r} = \mathbf{a} + \overrightarrow{AR}$.

Since $\overrightarrow{AR}$ is parallel to **b**,

$\overrightarrow{AR} = t\mathbf{b}$ for some scalar $t \in \mathbb{R}$

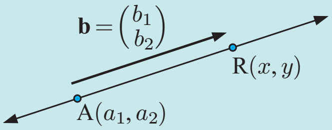
$\therefore \quad \mathbf{r} = \mathbf{a} + t\mathbf{b}$



Suppose a line passes through a fixed point $A(a_1, a_2)$ with position vector **a**, and that the line is parallel to the vector $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. If $R(x, y)$ with position vector **r** is any point on the line, then:

- $\mathbf{r} = \mathbf{a} + t\mathbf{b}, \; t \in \mathbb{R}$ or $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + t \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

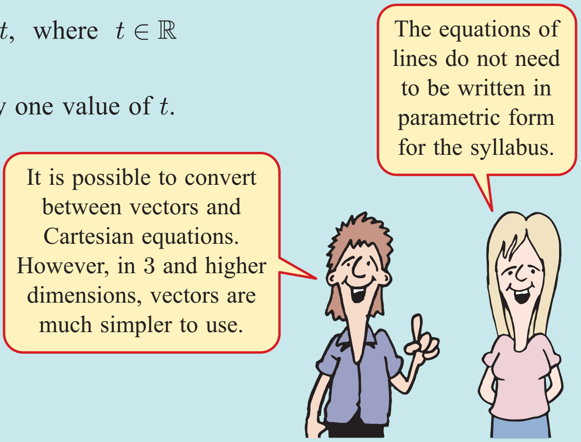
  is the **vector equation** of the line.



- The gradient of the line is $m = \dfrac{b_2}{b_1}$.
- Since $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_1 + b_1 t \\ a_2 + b_2 t \end{pmatrix}$, the **parametric equations** of the line are $x = a_1 + b_1 t$ and $y = a_2 + b_2 t$, where $t \in \mathbb{R}$ is the **parameter**.

  Each point on the line corresponds to exactly one value of t.
- We can convert these equations into Cartesian form by equating t values.

  Using $t = \dfrac{x - a_1}{b_1} = \dfrac{y - a_2}{b_2}$ we obtain $b_2 x - b_1 y = b_2 a_1 - b_1 a_2$ which is the **Cartesian equation** of the line.



Example 13

Self Tutor

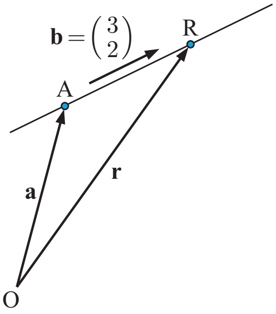
A line passes through the point A(1, 5) and has direction vector $\begin{pmatrix}3\\2\end{pmatrix}$. Describe the line using:

a a vector equation **b** parametric equations **c** a Cartesian equation.

a The vector equation is $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ where

$\mathbf{a} = \overrightarrow{OA} = \begin{pmatrix}1\\5\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}3\\2\end{pmatrix}$

$\therefore \begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\5\end{pmatrix} + t\begin{pmatrix}3\\2\end{pmatrix}, \quad t \in \mathbb{R}$



b $x = 1 + 3t$ and $y = 5 + 2t, \quad t \in \mathbb{R}$

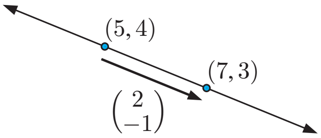
c Now $t = \dfrac{x-1}{3} = \dfrac{y-5}{2}$

$\therefore \; 2x - 2 = 3y - 15$

$\therefore \; 2x - 3y = -13$

NON-UNIQUENESS OF THE VECTOR EQUATION OF A LINE

Consider the line passing through (5, 4) and (7, 3). When writing the equation of the line, we could use either point to give the position vector **a**.



Similarly, we could use the direction vector $\begin{pmatrix}2\\-1\end{pmatrix}$, but we could also use $\begin{pmatrix}-2\\1\end{pmatrix}$ or indeed any non-zero scalar multiple of these vectors.

We could thus write the equation of the line as

$$\mathbf{x} = \begin{pmatrix}5\\4\end{pmatrix} + t\begin{pmatrix}2\\-1\end{pmatrix}, \quad t \in \mathbb{R} \quad \text{or} \quad \mathbf{x} = \begin{pmatrix}7\\3\end{pmatrix} + s\begin{pmatrix}-2\\1\end{pmatrix}, \quad s \in \mathbb{R} \quad \text{and so on.}$$

Notice how we use different parameters t and s when we write these equations. This is because the parameters are clearly not the same: when $t = 0$, we have the point (5, 4)
when $s = 0$, we have the point (7, 3).

In fact, the parameters are related by $s = 1 - t$.

EXERCISE 11G

1 Describe each of the following lines using:

i a vector equation **ii** parametric equations **iii** a Cartesian equation

a a line with direction $\begin{pmatrix}1\\4\end{pmatrix}$ which passes through (3, −4)

b a line parallel to $3\mathbf{i} + 7\mathbf{j}$ which cuts the x-axis at −6

c a line passing through (−1, 11) and (−3, 12).

2 A line passes through $(-1, 4)$ with direction vector $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

- **a** Write parametric equations for the line using the parameter t.
- **b** Find the points on the line for which $t = 0, 1, 3, -1$, and -4.

3 **a** Does $(3, -2)$ lie on the line with vector equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \end{pmatrix}$?

- **b** $(k, 4)$ lies on the line with parametric equations $x = 1 - 2t$, $y = 1 + t$. Find k.

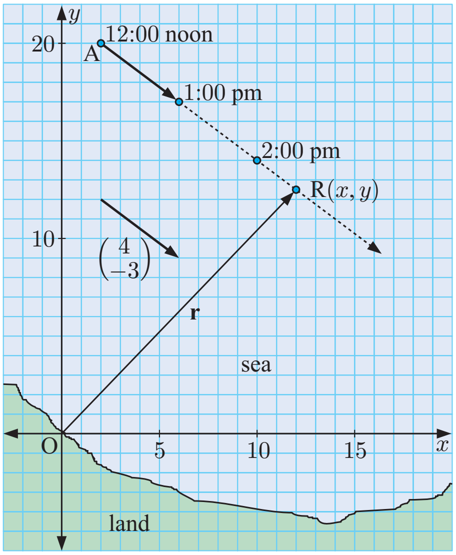
4 Line L has vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 5 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

- **a** Locate the point on the line corresponding to $t = 1$.
- **b** Explain why the direction of the line could also be described by $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$.
- **c** Use your answers to **a** and **b** to write an alternative vector equation for line L.

H CONSTANT VELOCITY PROBLEMS

A yacht club is situated at $(0, 0)$. At 12:00 noon a yacht is at point A$(2, 20)$. The yacht is moving with constant speed in the straight path shown in the diagram. The grid intervals are kilometres.

At 1:00 pm the yacht is at $(6, 17)$.
At 2:00 pm it is at $(10, 14)$.



In this case:

- the **initial position** of the yacht is given by the position vector $\mathbf{a} = \begin{pmatrix} 2 \\ 20 \end{pmatrix}$
- the direction of the yacht is given by the vector $\mathbf{b} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

  Suppose that t hours after leaving A, the yacht is at R(x, y).

$$\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$$

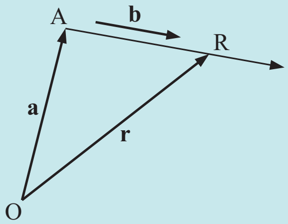
$$\therefore\ \mathbf{r} = \begin{pmatrix} 2 \\ 20 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix} \quad \text{for } t \geqslant 0$$

$$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 20 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix}$$ is the **vector equation** of the yacht's path.

If an object has initial position vector **a** and moves with constant velocity **b**, its position at time t is given by

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \quad \text{for} \quad t \geqslant 0.$$

The **speed** of the object is $|\,\mathbf{b}\,|$.



Example 14

Self Tutor

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix} + t\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ is the vector equation of the path of an object.

The time t is in seconds, $t \geqslant 0$. The distance units are metres.

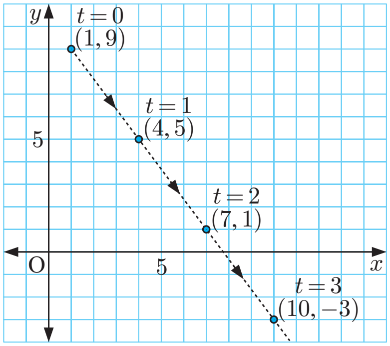
- **a** Find the object's initial position.
- **b** Plot the path of the object for $t = 0, 1, 2, 3$.
- **c** Find the velocity vector of the object.
- **d** Find the object's speed.
- **e** If the object continues in the same direction but increases its speed to $30\ \text{m}\,\text{s}^{-1}$, state its new velocity vector.

a At $t = 0$, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \end{pmatrix}$

$\therefore$ the object is at $(1, 9)$.

DEMO

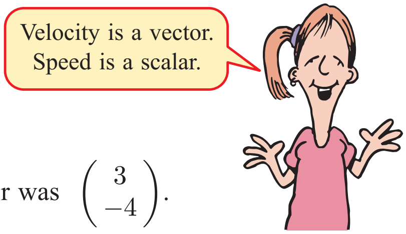
b



c The velocity vector is $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

d The speed is $\left|\begin{pmatrix} 3 \\ -4 \end{pmatrix}\right| = \sqrt{3^2 + (-4)^2}$

$= 5\ \text{m}\,\text{s}^{-1}$.



e Previously, the speed was $5\ \text{m}\,\text{s}^{-1}$ and the velocity vector was $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$.

$\therefore$ the new velocity vector is $6\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 18 \\ -24 \end{pmatrix}$.

Example 15

Self Tutor

An object is initially at $(5,\ 10)$ and moves with velocity vector $3\mathbf{i} - \mathbf{j}$ metres per minute. Find:

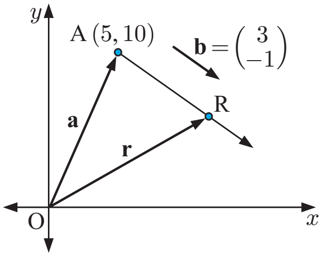
a the position of the object at time t minutes

b the speed of the object

c the position of the object at $t = 3$ minutes

d the time when the object is due east of $(0,\ 0)$.

a



$$\mathbf{r} = \mathbf{a} + t\mathbf{b}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad t \in \mathbb{R}$$

$$\therefore \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5+3t \\ 10-t \end{pmatrix}$$

After t minutes, the object is at $(5 + 3t,\ 10 - t)$.

b The speed of the object is $|\,\mathbf{b}\,| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$ metres per minute.

c At $t = 3$ minutes, $5 + 3t = 14$ and $10 - t = 7$. The object is at $(14,\ 7)$.

d When the object is due east of $(0,\ 0)$, y must be zero.

$\therefore\ 10 - t = 0$

$\therefore\ t = 10$

The object is due east of $(0,\ 0)$ after 10 minutes.

EXERCISE 11H

1 A particle at $P(x(t),\ y(t))$ moves such that $x(t) = 1 + 2t$ and $y(t) = 2 - 5t$, $t \geqslant 0$. The distances are in centimetres and t is in seconds.

- **a** Find the initial position of P.
- **b** Illustrate the initial part of the motion of P where $t = 0,\ 1,\ 2,\ 3$.
- **c** Find the velocity vector of P.
- **d** Find the speed of P.

2 **a** Find the vector equation of a boat initially at $(2,\ 3)$, which travels with velocity vector $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$. The grid units are kilometres and the time is in hours.

- **b** Locate the boat's position after 90 minutes.
- **c** How long will it take for the boat to reach the point $(5,\ -0.75)$?

3 A remote controlled toy car is initially at $(-3,\ -2)$. It moves with constant velocity $2\mathbf{i} + 4\mathbf{j}$. The distance units are centimetres, and the time is in seconds.

- **a** Write an expression for the position vector of the car at any time $t \geqslant 0$.
- **b** Find the position vector of the car at time $t = 2.5$.

c Find when the car is **i** due north **ii** due west of the observation point $(0, 0)$.

d Plot the car's positions at times $t = 0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, 2\frac{1}{2}, ...$

4 Each of the following vector equations represents the path of a moving object. t is measured in seconds, and $t \geqslant 0$. Distances are measured in metres. In each case, find:

i the initial position **ii** the velocity vector **iii** the speed of the object.

a $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 12 \\ 5 \end{pmatrix}$ **b** $x = 3 + 2t,\ \ y = -t$

5 Find the velocity vector of a speed boat moving parallel to:

a $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ with a speed of $150\ \text{km h}^{-1}$ **b** $2\mathbf{i} + \mathbf{j}$ with a speed of $50\ \text{km h}^{-1}$.

6 Find the velocity vector of a swooping eagle moving in the direction $5\mathbf{i} - 12\mathbf{j}$ with a speed of $91\ \text{km h}^{-1}$.

7 Yacht A moves according to $x(t) = 4 + t,\ \ y(t) = 5 - 2t$ where the distance units are kilometres and the time units are hours. Yacht B moves according to $x(t) = 1 + 2t,\ \ y(t) = -8 + t,\ \ t \geqslant 0$.

a Find the initial position of each yacht.

b Find the velocity vector of each yacht.

c Show that the speed of each yacht is constant, and state these speeds.

d Find the Cartesian equation of the path of each yacht.

e *Hence* show that the paths of the yachts intersect at right angles.

f Will the yachts collide?

8 Submarine P is at $(-5, 4)$. It fires a torpedo with velocity vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ at 1:34 pm.

Submarine Q is at $(15, 7)$. a minutes after 1:34 pm, it fires a torpedo with velocity vector $\begin{pmatrix} -4 \\ -3 \end{pmatrix}$.

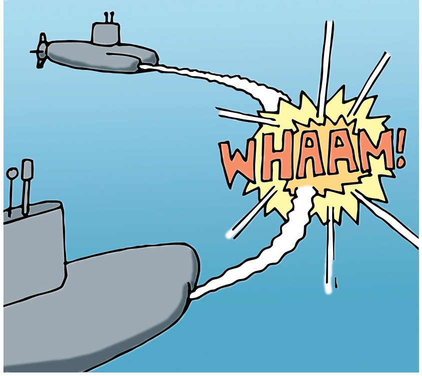
Distances are measured in kilometres, and time is in minutes.

a Show that the position of P's torpedo can be written as $P(x_1(t),\ y_1(t))$ where $x_1(t) = -5 + 3t$ and $y_1(t) = 4 - t$.

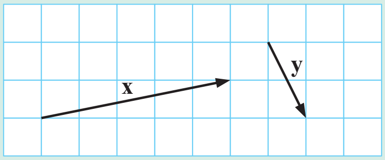
b What is the speed of P's torpedo?

c Show that the position of Q's torpedo can be written as $Q(x_2(t),\ y_2(t))$ where $x_2(t) = 15 - 4(t - a)$ and $y_2(t) = 7 - 3(t - a)$.

d Q's torpedo is successful in knocking out P's torpedo. At what time did Q fire its torpedo, and at what time did the explosion occur?



Review set 11A



1 **a** Write the given vectors in component form and in unit vector form.

b Find, in unit vector form:

i $\mathbf{x}+\mathbf{y}$ **ii** $\mathbf{y}-2\mathbf{x}$

2 Consider the vector $3\mathbf{i}-\mathbf{j}$.

a Write the vector in component form.

b Illustrate the vector using a directed line segment.

c Write the negative of the vector.

d Find the length of the vector.

3 **a** Find k given that $\begin{pmatrix} k \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ is a unit vector.

b Find the vector which is 5 units long and has the opposite direction to $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$.

4 For $\mathbf{m}=\begin{pmatrix} 6 \\ -3 \end{pmatrix}$, $\mathbf{n}=\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, and $\mathbf{p}=\begin{pmatrix} -1 \\ 3 \end{pmatrix}$, find:

a $\mathbf{m}-\mathbf{n}+\mathbf{p}$ **b** $2\mathbf{n}-3\mathbf{p}$ **c** $|\mathbf{m}+\mathbf{p}|$

5 Given points A(3, 1), B(5, −2), and C(8, 4), find:

a $\overrightarrow{AB}$ **b** $\overrightarrow{CB}$ **c** $|\overrightarrow{AC}|$

6 B(−3, −1) and C(k, 2) are 5 units apart.

a Find $\overrightarrow{BC}$ and $|\overrightarrow{BC}|$.

b Hence, find the two possible values of k.

c Show, by illustration, why k should have two possible values.

7 A small plane can fly at 350 km h^{-1} in still conditions. Its pilot needs to fly due north, but needs to deal with a 70 km h^{-1} wind from the east.

a In what direction should the pilot face the plane in order that its resultant velocity is due north?

b What will the speed of the plane be?

8 For the line that passes through (−6, 3) with direction $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, write down the corresponding:

a vector equation **b** parametric equations **c** Cartesian equation.

9 $(-3, m)$ lies on the line with vector equation $\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 18 \\ -2 \end{pmatrix}+t\begin{pmatrix} -7 \\ 4 \end{pmatrix}$. Find m.

10 Find the velocity vector of an object moving in the direction $3\mathbf{i}-\mathbf{j}$ with speed 20 km h^{-1}.

11 Line L has equation $\mathbf{r} = \begin{pmatrix} 3 \\ -3 \end{pmatrix} + t \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

a Locate the point on the line corresponding to $t = 1$.

b Explain why the direction of the line could also be described by $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$.

c Use your answers to **a** and **b** to write an alternative vector equation for line L.

12 A moving particle has coordinates $P(x(t), y(t))$ where $x(t) = -4 + 8t$ and $y(t) = 3 + 6t$. The distance units are metres, and $t \geqslant 0$ is the time in seconds. Find the:

a initial position of the particle

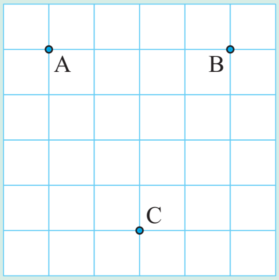
b position of the particle after 4 seconds

c particle's velocity vector

d speed of the particle.

Review set 11B

1



a Find in component form and in unit vector form:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CA}$

b Which two vectors in **a** have the same length? Explain your answer.

c Write the negative vector of $\overrightarrow{CA}$ in *three* different ways.

2 If $\mathbf{r} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and $\mathbf{s} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ find:

a $|\mathbf{s}|$ **b** $|\mathbf{r} + \mathbf{s}|$ **c** $|2\mathbf{s} - \mathbf{r}|$

3 Find k if the following are unit vectors:

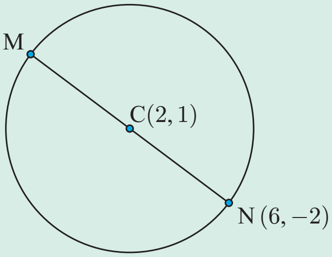
a $\begin{pmatrix} \frac{5}{13} \\ k \end{pmatrix}$ **b** $\begin{pmatrix} k \\ -k \end{pmatrix}$

4 If $\overrightarrow{PQ} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, and $\overrightarrow{RS} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, find $\overrightarrow{SP}$.

5 [MN] is the diameter of a circle with centre C.

a Find the coordinates of M.

b Find the radius of the circle.



6 Find m if $\begin{pmatrix} 3 \\ m \end{pmatrix}$ and $\begin{pmatrix} -12 \\ -20 \end{pmatrix}$ are parallel vectors.

7 When an archer fires an arrow, he is suddenly aware of a breeze which pushes his shot off-target. The speed of the shot $|\mathbf{v}|$ is *not* affected by the wind, but the arrow's flight is 2° off-line.

- **a** Draw a vector diagram to represent the situation.
- **b** Hence explain why:
  - **i** the breeze must be 91° to the intended direction of the arrow
  - **ii** the speed of the breeze must be $2\,|\mathbf{v}|\sin 1^\circ$.

8 Find the vector equation of the line which cuts the y-axis at $(0,\,8)$ and has direction $5\mathbf{i}+4\mathbf{j}$.

9 A yacht is sailing with constant speed $5\sqrt{10}$ km h^{-1} in the direction $-\mathbf{i}-3\mathbf{j}$. Initially it is at point $(-6,\,10)$. A beacon is at $(0,\,0)$ at the centre of a tiny atoll. Distances are in kilometres.

- **a** Find, in terms of **i** and **j** :
  - **i** the initial position vector of the yacht
  - **ii** the velocity vector of the yacht
  - **iii** the position vector of the yacht at any time t hours, $t \geqslant 0$.
- **b** Find the time when the yacht is due west of the beacon. How far away from the beacon is the yacht at this time?

10 Write down **i** a vector equation **ii** parametric equations for the line passing through:

- **a** $(2,\,-3)$ with direction $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$
- **b** $(-1,\,6)$ and $(5,\,-2)$.

11 Submarine X23 is at $(2,\,4)$. It fires a torpedo with velocity vector $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ at exactly 2:17 pm. Submarine Y18 is at $(11,\,3)$. It fires a torpedo with velocity vector $\begin{pmatrix} -1 \\ a \end{pmatrix}$ at 2:19 pm to intercept the torpedo from X23. Distance units are kilometres. t is in minutes.

- **a** Find $x_1(t)$ and $y_1(t)$ for the torpedo fired from submarine X23.
- **b** Find $x_2(t)$ and $y_2(t)$ for the torpedo fired from submarine Y18.
- **c** At what time does the interception occur?
- **d** What was the direction and speed of the interception torpedo?

12

Matrices

Contents:

Opening problem

Aakriti owns a stationery shop. She sells two brands of pen, in three colours. Her sales for one week are shown in the table below.

	Brand	
Colour	*Pentex*	*Rollerball*
Blue	32	24
Black	25	16
Red	13	9

Pentex pens sell for \$1.19 each, and Rollerball pens sell for \$1.55 each.

Things to think about:

a How can we convert the table into a 3×2 quantities matrix $\mathbf{Q}$?

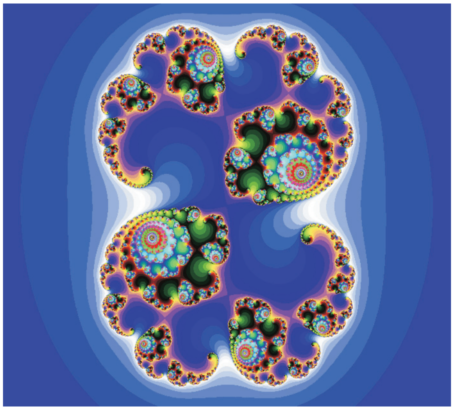
b How can we display the prices in a price matrix $\mathbf{P}$?

c How can we multiply the matrices $\mathbf{Q}$ and $\mathbf{P}$? What does the matrix $\mathbf{QP}$ represent?

d Can you find the total revenue for Aakriti in pen sales for the week?

Matrices are rectangular arrays of numbers which are used to organise numerical information. They are used in a wide range of fields, including:

- solving systems of equations in business, physics, and engineering
- linear programming where we may wish to optimise a linear expression subject to linear constraints
- business inventories involving stock control, cost, revenue, and profit calculations
- Markov chains for predicting long term probabilities such as in weather
- strategies in games where we wish to maximise our chance of winning
- economic modelling where the input from suppliers is needed to help a business be successful
- graph (network) theory used to determine routes for trucks and airlines to minimise distance travelled and therefore costs
- assignment problems to direct resources in the most cost-effective way
- forestry and fisheries management where we need to select an appropriate sustainable harvesting policy
- cubic spline interpolation used to construct curves and fonts
- computer graphics
- flight simulation
- Computer Aided Tomography (CAT scanning) and Magnetic Resonance Imaging (MRI)
- fractals and chaos
- genetics
- cryptography including coding, code breaking, and computer confidentiality.



The Julia set

A MATRIX STRUCTURE

A **matrix** is a rectangular array of numbers arranged in **rows** and **columns**.

Each number within a matrix has a particular meaning.

You have been using matrices for many years without realising it. For example, a football premiership table and a recipe can each be written as matrices.

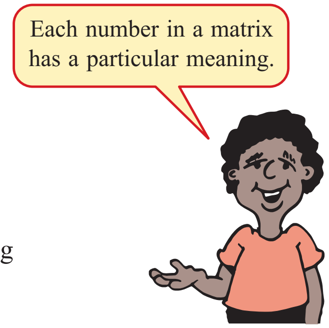
	Won	*Lost*	*Drew*	*Points*
Manchester United	28	5	5	89
Manchester City	23	9	6	78
Chelsea	22	9	7	75
Arsenal	21	10	7	73
⋮				

Ingredients	*Amount*
sugar	1 tspn
flour	1 cup
milk	200 mL
salt	1 pinch

Consider these two items of information:

Shopping list	
Bread	2 loaves
Juice	1 carton
Eggs	6
Cheese	1

Furniture inventory			
	chairs	tables	beds
Flat	6	1	2
Unit	9	2	3
House	10	3	4



We can write these tables as matrices by extracting the numbers and placing them in round brackets:

$$\begin{array}{cc} & \text{number} \\ \begin{array}{c} B \\ J \\ E \\ C \end{array} & \begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{cc} & \begin{array}{ccc} C & T & B \end{array} \\ \begin{array}{c} F \\ U \\ H \end{array} & \begin{pmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & 3 & 4 \end{pmatrix} \end{array} \quad \text{or simply} \quad \begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & 3 & 4 \end{pmatrix}$$

Notice how the organisation of the data is maintained in matrix form.

$$\begin{pmatrix} 2 \\ 1 \\ 6 \\ 1 \end{pmatrix}$$

has 4 rows and 1 column, and we say that this is a 4×1 **column matrix** or **column vector**.

column 2

$$\text{row 3} \longrightarrow \begin{pmatrix} 6 & 1 & 2 \\ 9 & 2 & 3 \\ 10 & \textcircled{3} & 4 \end{pmatrix}$$

this element is in row 3, column 2

has 3 rows and 3 columns, and is called a 3×3 **square matrix**.

$$\begin{pmatrix} 3 & 0 & -1 & 2 \end{pmatrix}$$

has 1 row and 4 columns, and is called a 1×4 **row matrix** or **row vector**.

An $m \times n$ matrix has m rows and n columns.

$m \times n$ specifies the **order** of a matrix.

Example 1

Self Tutor

Lisa goes shopping at store A to buy 2 loaves of bread at \$2.65 each, 3 litres of milk at \$1.55 per litre, and one 500 g tub of butter at \$2.35.

a Represent the quantities purchased in a row matrix $\mathbf{Q}$, and the costs in a column matrix $\mathbf{A}$.

b When Lisa goes to a different supermarket (store B), she finds that the prices for the same items are \$2.25 for bread, \$1.50 for milk, and \$2.20 for butter.
Write the costs for both stores in a single costs matrix $\mathbf{C}$.

a The quantities matrix is $\mathbf{Q} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$

$$\begin{array}{ccc} 2 & 3 & 1 \\ \uparrow & \uparrow & \uparrow \\ \text{bread} & \text{milk} & \text{butter} \end{array}$$

The costs matrix is
$$\mathbf{A} = \begin{array}{cl} \begin{pmatrix} 2.65 \\ 1.55 \\ 2.35 \end{pmatrix} & \begin{array}{l} \leftarrow \text{bread} \\ \leftarrow \text{milk} \\ \leftarrow \text{butter} \end{array} \end{array}$$

b We write the costs for each store in separate columns.

The new costs matrix is
$$\mathbf{C} = \begin{array}{cl} \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix} & \begin{array}{l} \leftarrow \text{bread} \\ \leftarrow \text{milk} \\ \leftarrow \text{butter} \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \text{store A} & \text{store B} \end{array} & \end{array}$$

EXERCISE 12A

1 Write down the order of:

a $\begin{pmatrix} 5 & 1 & 0 & 2 \end{pmatrix}$ **b** $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 5 & 1 & 0 \end{pmatrix}$

2 A grocery list consists of 2 loaves of bread, 1 kg of butter, 6 eggs, and 1 carton of cream. Each loaf of bread costs \$1.95, each kilogram of butter costs \$2.35, each egg costs \$0.45, and each carton of cream costs \$2.95.

a Construct a row matrix showing quantities.

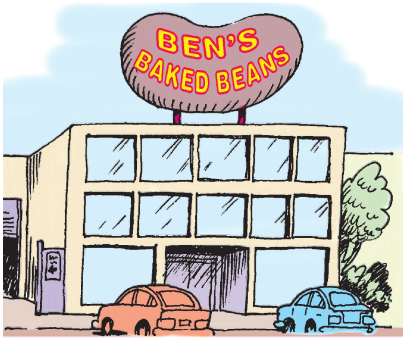
b Construct a column matrix showing prices.

c What is the significance of $(2 \times 1.95) + (1 \times 2.35) + (6 \times 0.45) + (1 \times 2.95)$?

3 A food processing factory produces cans of beans in three sizes: 200 g, 300 g, and 500 g. In February they produced respectively:

- 1000, 1500, and 1250 cans of each in week 1
- 1500, 1000, and 1000 cans of each in week 2
- 800, 2300, and 1300 cans of each in week 3
- 1200 cans of each in week 4.

Construct a matrix to show February's production levels.



4 Over a long weekend holiday, a baker produced the following food items: On Friday he baked 40 dozen pies, 50 dozen pasties, 55 dozen rolls, and 40 dozen buns. On Saturday he baked 25 dozen pies, 65 dozen pasties, 30 dozen buns, and 44 dozen rolls. On Sunday he baked 40 dozen pasties, 40 dozen rolls, and 35 dozen of each of pies and buns. On Monday he baked 40 dozen pasties, 50 dozen buns, and 35 dozen of each of pies and rolls. Represent this information as a matrix.

B MATRIX OPERATIONS AND DEFINITIONS

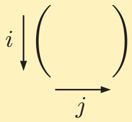
MATRIX NOTATION

Consider a matrix **A** which has order $m \times n$.

We can write

$$\mathbf{A} = (a_{ij}) \quad \text{where} \quad i = 1, 2, 3, \ldots., m$$
$$j = 1, 2, 3, \ldots., n$$
and a_{ij} is the element in the ith row, jth column.

By convention, the a_{ij} are labelled down then across.



For example, a_{23} is the number in row 2 and column 3 of matrix **A**.

EQUALITY

Two matrices are **equal** if they have the **same order** *and* the elements in corresponding positions are equal.

$$\mathbf{A} = \mathbf{B} \quad \Leftrightarrow \quad a_{ij} = b_{ij} \quad \text{for all } i, j.$$

For example, if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ then $a = w,\ b = x,\ c = y,$ and $d = z$.

MATRIX ADDITION

Thao has three stores: A, B, and C. Her stock levels for dresses, skirts, and blouses are given by the matrix:

$$\begin{array}{cc} \begin{array}{c}\text{Store}\\ \begin{array}{ccc}\text{A} & \text{B} & \text{C}\end{array}\end{array} & \\ \begin{pmatrix} 23 & 41 & 68 \\ 28 & 39 & 79 \\ 46 & 17 & 62 \end{pmatrix} & \begin{array}{l}\text{dresses}\\ \text{skirts}\\ \text{blouses}\end{array} \end{array}$$

Some newly ordered stock has just arrived. 20 dresses, 30 skirts, and 50 blouses must be added to the stock levels of each store. Her stock order is given by the matrix:

$$\begin{pmatrix} 20 & 20 & 20 \\ 30 & 30 & 30 \\ 50 & 50 & 50 \end{pmatrix}$$

Clearly the new levels are: $\begin{pmatrix} 23 & 41 & 68 \\ 28 & 39 & 79 \\ 46 & 17 & 62 \end{pmatrix} + \begin{pmatrix} 20 & 20 & 20 \\ 30 & 30 & 30 \\ 50 & 50 & 50 \end{pmatrix} = \begin{pmatrix} 43 & 61 & 88 \\ 58 & 69 & 109 \\ 96 & 67 & 112 \end{pmatrix}$

To **add** two matrices, they must be of the **same order**, and we **add corresponding elements**.

MATRIX SUBTRACTION

Suppose Thao's stock levels were $\begin{pmatrix} 29 & 51 & 19 \\ 31 & 28 & 32 \\ 40 & 17 & 29 \end{pmatrix}$ and her sales matrix for the week was $\begin{pmatrix} 15 & 12 & 6 \\ 20 & 16 & 19 \\ 19 & 8 & 14 \end{pmatrix}$.

Thao will be left with her original stock levels less what she has sold. Clearly, we need to subtract corresponding elements:

$$\begin{pmatrix} 29 & 51 & 19 \\ 31 & 28 & 32 \\ 40 & 17 & 29 \end{pmatrix} - \begin{pmatrix} 15 & 12 & 6 \\ 20 & 16 & 19 \\ 19 & 8 & 14 \end{pmatrix} = \begin{pmatrix} 14 & 39 & 13 \\ 11 & 12 & 13 \\ 21 & 9 & 15 \end{pmatrix}$$

To **subtract** matrices, they must be of the **same order**, and we **subtract corresponding elements**.

Summary:

- $\mathbf{A} \pm \mathbf{B} = (a_{ij}) \pm (b_{ij}) = (a_{ij} \pm b_{ij})$
- We can only add or subtract matrices of the same order.
- We add or subtract corresponding elements.
- The result of addition or subtraction is another matrix of the same order.

Example 2

Self Tutor

If $\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$

b $\mathbf{A} + \mathbf{C}$

a $$\begin{aligned} \mathbf{A} + \mathbf{B} &= \begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 4 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 6 \\ 0 & 3 & 5 \end{pmatrix} \\ &= \begin{pmatrix} 1+2 & 2+1 & 3+6 \\ 6+0 & 5+3 & 4+5 \end{pmatrix} \\ &= \begin{pmatrix} 3 & 3 & 9 \\ 6 & 8 & 9 \end{pmatrix} \end{aligned}$$

b $\mathbf{A} + \mathbf{C}$ cannot be found as the matrices do not have the same order.

Example 3

Self Tutor

If $\mathbf{A} = \begin{pmatrix} 3 & 4 & 8 \\ 2 & 1 & 0 \\ 1 & 4 & 7 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 0 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 3 \end{pmatrix}$,

find $\mathbf{A} - \mathbf{B}$.

$$\begin{aligned} \mathbf{A} - \mathbf{B} &= \begin{pmatrix} 3 & 4 & 8 \\ 2 & 1 & 0 \\ 1 & 4 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 6 \\ 3 & 0 & 4 \\ 5 & 2 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 3-2 & 4-0 & 8-6 \\ 2-3 & 1-0 & 0-4 \\ 1-5 & 4-2 & 7-3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 4 & 2 \\ -1 & 1 & -4 \\ -4 & 2 & 4 \end{pmatrix} \end{aligned}$$

EXERCISE 12B.1

1 If $\mathbf{A}=\begin{pmatrix}3&4\\5&2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}6&-3\\-2&1\end{pmatrix}$, and $\mathbf{C}=\begin{pmatrix}-3&7\\-4&-2\end{pmatrix}$, find:

a $\mathbf{A}+\mathbf{B}$ **b** $\mathbf{A}+\mathbf{B}+\mathbf{C}$ **c** $\mathbf{B}+\mathbf{C}$ **d** $\mathbf{C}+\mathbf{B}-\mathbf{A}$

2 If $\mathbf{P}=\begin{pmatrix}3&5&-11\\10&2&6\\-2&-1&7\end{pmatrix}$ and $\mathbf{Q}=\begin{pmatrix}17&-4&3\\-2&8&-8\\3&-4&11\end{pmatrix}$, find:

a $\mathbf{P}+\mathbf{Q}$ **b** $\mathbf{P}-\mathbf{Q}$ **c** $\mathbf{Q}-\mathbf{P}$

3 A restaurant served 85 men, 92 women, and 52 children on Friday night. On Saturday night they served 102 men, 137 women, and 49 children.

a Express this information in *two* column matrices.

b Use the matrices to find the totals of men, women, and children served over the two nights.

4 David bought shares in five companies on Monday, and he sold them on Friday. The details are shown in the table alongside.

a Write down David's column matrix for:

i cost price **ii** selling price.

b What matrix operation is needed to find David's profit or loss on each type of share?

c Find David's profit or loss matrix.

	Cost price per share	*Selling price per share*
A	\$1.72	\$1.79
B	\$27.85	\$28.75
C	\$0.92	\$1.33
D	\$2.53	\$2.25
E	\$3.56	\$3.51

5 In November, Lou E Gee sold 23 fridges, 17 stoves, and 31 microwave ovens. His partner Rose A Lee sold 19 fridges, 29 stoves, and 24 microwave ovens.

In December, Lou sold 18 fridges, 7 stoves, and 36 microwaves, and Rose sold 25 fridges, 13 stoves, and 19 microwaves.

a Write their sales for November as a 3×2 matrix.

b Write their sales for December as a 3×2 matrix.

c Write their total sales for November and December as a 3×2 matrix.

6 Find x and y if:

a $\begin{pmatrix}x&x^2\\3&-1\end{pmatrix}=\begin{pmatrix}y&4\\3&y+1\end{pmatrix}$ **b** $\begin{pmatrix}x&y\\y&x\end{pmatrix}=\begin{pmatrix}-y&x\\x&-y\end{pmatrix}$

7 **a** If $\mathbf{A}=\begin{pmatrix}2&1\\3&-1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}-1&2\\2&3\end{pmatrix}$, find $\mathbf{A}+\mathbf{B}$ and $\mathbf{B}+\mathbf{A}$.

b Explain why $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$.

8 **a** For $\mathbf{A} = \begin{pmatrix} -1 & 0 \\ 1 & 5 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & 4 \\ -1 & -2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 4 & -1 \\ -1 & 3 \end{pmatrix}$, find $(\mathbf{A}+\mathbf{B})+\mathbf{C}$ and $\mathbf{A}+(\mathbf{B}+\mathbf{C})$.

b Prove that, if **A**, **B**, and **C** are any 2×2 matrices, then $(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$.

Hint: Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

MULTIPLES OF MATRICES

In the pantry there are 6 cans of peaches, 4 cans of apricots, and 8 cans of pears. We represent this by the column vector $\mathbf{C} = \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix}$.

If we doubled the cans in the pantry, we would have $\begin{pmatrix} 12 \\ 8 \\ 16 \end{pmatrix}$ which is $\mathbf{C}+\mathbf{C}$ or $2\mathbf{C}$.

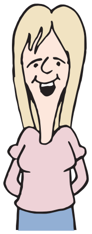
Notice that to get $2\mathbf{C}$ from **C** we simply multiply all the matrix elements by 2.

Likewise, trebling the fruit cans in the pantry gives $3\mathbf{C} = \begin{pmatrix} 3 \times 6 \\ 3 \times 4 \\ 3 \times 8 \end{pmatrix} = \begin{pmatrix} 18 \\ 12 \\ 24 \end{pmatrix}$

and halving them gives $\frac{1}{2}\mathbf{C} = \begin{pmatrix} \frac{1}{2} \times 6 \\ \frac{1}{2} \times 4 \\ \frac{1}{2} \times 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$.

If $\mathbf{A} = (a_{ij})$ has order $m \times n$, and k is a scalar, then $k\mathbf{A} = (ka_{ij})$.
So, to find $k\mathbf{A}$, we multiply each element in **A** by k.
The result is another matrix of order $m \times n$.

We use capital letters for matrices and lower-case letters for scalars.



Example 4

Self Tutor

If **A** is $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix}$, find:

a $3\mathbf{A}$ **b** $\frac{1}{2}\mathbf{A}$

a $$3\mathbf{A} = 3\begin{pmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 \times 1 & 3 \times 2 & 3 \times 5 \\ 3 \times 2 & 3 \times 0 & 3 \times 1 \end{pmatrix} = \begin{pmatrix} 3 & 6 & 15 \\ 6 & 0 & 3 \end{pmatrix}$$

b $$\tfrac{1}{2}\mathbf{A} = \tfrac{1}{2}\begin{pmatrix} 1 & 2 & 5 \\ 2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \times 1 & \frac{1}{2} \times 2 & \frac{1}{2} \times 5 \\ \frac{1}{2} \times 2 & \frac{1}{2} \times 0 & \frac{1}{2} \times 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 1 & 2\frac{1}{2} \\ 1 & 0 & \frac{1}{2} \end{pmatrix}$$

EXERCISE 12B.2

1 If $\mathbf{B} = \begin{pmatrix} 6 & 12 \\ 24 & 6 \end{pmatrix}$, find:

a $2\mathbf{B}$ **b** $\frac{1}{3}\mathbf{B}$ **c** $\frac{1}{12}\mathbf{B}$ **d** $-\frac{1}{2}\mathbf{B}$

2 If $\mathbf{A} = \begin{pmatrix} 2 & 3 & 5 \\ 1 & 6 & 4 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 3 \end{pmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} - \mathbf{B}$ **c** $2\mathbf{A} + \mathbf{B}$ **d** $3\mathbf{A} - \mathbf{B}$

3 A builder builds a block of 12 identical flats. Each flat is to contain 1 table, 4 chairs, 2 beds, and 1 wardrobe.

Let $\mathbf{F} = \begin{pmatrix} 1 \\ 4 \\ 2 \\ 1 \end{pmatrix}$ be the matrix representing the furniture in one flat.

In terms of **F**, what is the matrix representing the furniture in **all** flats? Evaluate this matrix.

4 On weekdays, a video store finds that its average daily hirings are 75 DVD movies, 27 Blu-ray movies, and 102 games. On weekends, the average daily hirings are 43 Blu-ray movies, 136 DVD movies, and 129 games.

a Represent the data using *two* column matrices **A** and **B**.

b Find $5\mathbf{A} + 2\mathbf{B}$.

c What does the matrix in **b** represent?

$$\begin{pmatrix} \\ \\ \end{pmatrix} \begin{array}{l} \leftarrow \text{DVD movies} \\ \leftarrow \text{Blu-ray movies} \\ \leftarrow \text{games} \end{array}$$

5 Isabelle sells clothing made by four different companies which we will call A, B, C, and D.
Her usual monthly order is:

$$\begin{array}{r c} & \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{array}{r} \text{skirt} \\ \text{dress} \\ \text{evening} \\ \text{suit} \end{array} & \begin{pmatrix} 30 & 40 & 40 & 60 \\ 50 & 40 & 30 & 75 \\ 40 & 40 & 50 & 50 \\ 10 & 20 & 20 & 15 \end{pmatrix} \end{array}$$

Find her order, to the nearest whole number, if:

a she increases her total order by 15%

b she decreases her total order by 15%.

ZERO OR NULL MATRIX

A **zero matrix** is a matrix in which all the elements are zero.

For example, the 2×2 zero matrix is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and the 2×3 zero matrix is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

If **A** is a matrix of any order and **O** is the corresponding **zero matrix**, then $\mathbf{A} + \mathbf{O} = \mathbf{O} + \mathbf{A} = \mathbf{A}$.

For example: $\begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 4 & -1 \end{pmatrix}$.

NEGATIVE MATRICES

The **negative** matrix $\mathbf{A}$, denoted $-\mathbf{A}$, is actually $-1\mathbf{A}$.

$-\mathbf{A}$ is obtained from $\mathbf{A}$ by reversing the sign of each element of $\mathbf{A}$.

For example, if $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$, then $-\mathbf{A} = \begin{pmatrix} -1\times 3 & -1\times -1 \\ -1\times 2 & -1\times 4 \end{pmatrix} = \begin{pmatrix} -3 & 1 \\ -2 & -4 \end{pmatrix}$

The addition of a matrix and its negative always produces a zero matrix.

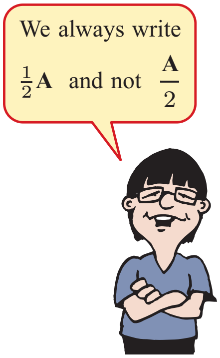
$$\mathbf{A} + (-\mathbf{A}) = (-\mathbf{A}) + \mathbf{A} = \mathbf{O}$$

For example: $\begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 1 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

MATRIX ALGEBRA

We now compare our discoveries about matrices so far with ordinary algebra. We assume that $\mathbf{A}$ and $\mathbf{B}$ are matrices of the same order.

Ordinary algebra	Matrix algebra
• If a and b are real numbers then $a+b$ is also a real number.	• If $\mathbf{A}$ and $\mathbf{B}$ are matrices then $\mathbf{A}+\mathbf{B}$ is a matrix of the same order.
• $a+b=b+a$	• $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$
• $(a+b)+c=a+(b+c)$	• $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$
• $a+0=0+a=a$	• $\mathbf{A}+\mathbf{O}=\mathbf{O}+\mathbf{A}=\mathbf{A}$
• $a+(-a)=(-a)+a=0$	• $\mathbf{A}+(-\mathbf{A})=(-\mathbf{A})+\mathbf{A}=\mathbf{O}$
• a half of a is $\frac{a}{2}$	• a half of $\mathbf{A}$ is $\frac{1}{2}\mathbf{A}$


We always write $\frac{1}{2}\mathbf{A}$ and not $\frac{\mathbf{A}}{2}$

Example 5

Self Tutor

Show that:

a if $\mathbf{X}+\mathbf{A}=\mathbf{B}$ then $\mathbf{X}=\mathbf{B}-\mathbf{A}$ **b** if $3\mathbf{X}=\mathbf{A}$ then $\mathbf{X}=\frac{1}{3}\mathbf{A}$

a
$$\begin{aligned} \mathbf{X}+\mathbf{A} &= \mathbf{B} \\ \therefore\ \mathbf{X}+\mathbf{A}+(-\mathbf{A}) &= \mathbf{B}+(-\mathbf{A}) \\ \therefore\ \mathbf{X}+\mathbf{O} &= \mathbf{B}-\mathbf{A} \\ \therefore\ \mathbf{X} &= \mathbf{B}-\mathbf{A} \end{aligned}$$

b
$$\begin{aligned} 3\mathbf{X} &= \mathbf{A} \\ \therefore\ \tfrac{1}{3}(3\mathbf{X}) &= \tfrac{1}{3}\mathbf{A} \\ \therefore\ 1\mathbf{X} &= \tfrac{1}{3}\mathbf{A} \\ \therefore\ \mathbf{X} &= \tfrac{1}{3}\mathbf{A} \end{aligned}$$

EXERCISE 12B.3

1 Simplify:

a $\mathbf{A}+2\mathbf{A}$
b $3\mathbf{B}-3\mathbf{B}$
c $\mathbf{C}-2\mathbf{C}$
d $-\mathbf{B}+\mathbf{B}$
e $2(\mathbf{A}+\mathbf{B})$
f $-(\mathbf{A}+\mathbf{B})$
g $-(2\mathbf{A}-\mathbf{C})$
h $3\mathbf{A}-(\mathbf{B}-\mathbf{A})$
i $\mathbf{A}+2\mathbf{B}-(\mathbf{A}-\mathbf{B})$

2 Find $\mathbf{X}$ in terms of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ if:

a $\mathbf{X}+\mathbf{B}=\mathbf{A}$
b $\mathbf{B}+\mathbf{X}=\mathbf{C}$
c $4\mathbf{B}+\mathbf{X}=2\mathbf{C}$
d $2\mathbf{X}=\mathbf{A}$
e $3\mathbf{X}=\mathbf{B}$
f $\mathbf{A}-\mathbf{X}=\mathbf{B}$
g $\frac{1}{2}\mathbf{X}=\mathbf{C}$
h $2(\mathbf{X}+\mathbf{A})=\mathbf{B}$
i $\mathbf{A}-4\mathbf{X}=\mathbf{C}$

3 **a** Suppose $\mathbf{M}=\begin{pmatrix}1 & 2\\ 3 & 6\end{pmatrix}$ and $\frac{1}{3}\mathbf{X}=\mathbf{M}$. Find $\mathbf{X}$.

b Suppose $\mathbf{N}=\begin{pmatrix}2 & -1\\ 3 & 5\end{pmatrix}$ and $4\mathbf{X}=\mathbf{N}$. Find $\mathbf{X}$.

c Suppose $\mathbf{A}=\begin{pmatrix}1 & 0\\ -1 & 2\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & 4\\ -1 & 1\end{pmatrix}$, and $\mathbf{A}-2\mathbf{X}=3\mathbf{B}$. Find $\mathbf{X}$.

C MATRIX MULTIPLICATION

Suppose you go to a shop and purchase 3 cans of soft drink, 4 chocolate bars, and 2 ice creams.

The prices are:

soft drink cans	chocolate bars	ice creams
\$1.30	\$0.90	\$1.20

We can represent this by the quantities matrix $\mathbf{A}=\begin{pmatrix}3 & 4 & 2\end{pmatrix}$ and the costs matrix $\mathbf{B}=\begin{pmatrix}1.30\\ 0.90\\ 1.20\end{pmatrix}$.

We can find the total cost of the items by multiplying the number of each item by its respective cost, and then adding the results:

$$3\times\$1.30+4\times\$0.90+2\times\$1.20=\$9.90$$

We can also determine the total cost by the **matrix multiplication**:

$$\begin{aligned}\mathbf{AB}&=\begin{pmatrix}3 & 4 & 2\end{pmatrix}\begin{pmatrix}1.30\\ 0.90\\ 1.20\end{pmatrix}\\ &=(3\times 1.30)+(4\times 0.90)+(2\times 1.20)\\ &=9.90\end{aligned}$$

Notice that we write the **row matrix** first and the **column matrix** second.

In general,

$$\begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}p\\ q\\ r\end{pmatrix}=ap+bq+cr.$$

EXERCISE 12C.1

1 Determine:

a $\begin{pmatrix}3 & -1\end{pmatrix}\begin{pmatrix}5\\ 4\end{pmatrix}$
b $\begin{pmatrix}1 & 3 & 2\end{pmatrix}\begin{pmatrix}5\\ 1\\ 7\end{pmatrix}$
c $\begin{pmatrix}6 & -1 & 2 & 3\end{pmatrix}\begin{pmatrix}1\\ 0\\ -1\\ 4\end{pmatrix}$

2 **a** Show that the sum of w, x, y, and z is given by $\begin{pmatrix} w & x & y & z \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

b Represent the *average* of w, x, y, and z in a similar way.

3 Lucy buys 4 shirts, 3 skirts, and 2 blouses costing \$27, \$35, and \$39 each respectively.

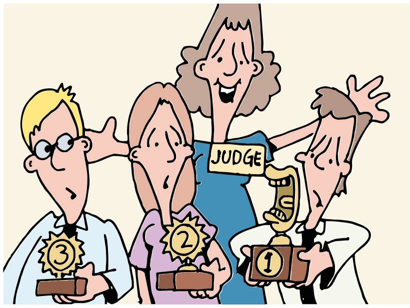
a Write down a quantities matrix **Q** and a price matrix **P**.

b Show how to use **P** and **Q** to determine the total cost of Lucy's clothes.

4 In the interschool public speaking competition, a first place is awarded 10 points, second place 6 points, third place 3 points, and fourth place 1 point. One school won 3 first places, 2 seconds, 4 thirds, and 2 fourths.

a Write down this information in terms of a points matrix **P** and a numbers matrix **N**.

b Show how to use **P** and **N** to find the total number of points awarded to the school.



MORE COMPLICATED MULTIPLICATIONS

Consider again **Example 1** on page **308** where Lisa needed 2 loaves of bread, 3 litres of milk, and 1 tub of butter.

We represented this by the quantities matrix $\mathbf{Q} = \begin{pmatrix} 2 & 3 & 1 \end{pmatrix}$.

The prices for each store were summarised in the costs matrix $\mathbf{C} = \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix}$.

To find the *total cost* of the items in each store, Lisa needs to multiply the number of items by their respective cost.

In Store A, a loaf of bread is \$2.65, a litre of milk is \$1.55, and a tub of butter is \$2.35, so the total cost is $2 \times \$2.65 + 3 \times \$1.55 + 1 \times \$2.35 = \$12.30$.

In Store B, a loaf of bread is \$2.25, a litre of milk is \$1.50, and a tub of butter is \$2.20, so the total cost is $2 \times \$2.25 + 3 \times \$1.50 + 1 \times \$2.20 = \$11.20$.

To do this using matrices notice that:

$$\mathbf{QC} = \underset{1 \times 3}{\begin{pmatrix} 2 & 3 & 1 \end{pmatrix}} \times \underset{3 \times 2}{\begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix}} = \underset{1 \times 2}{\begin{pmatrix} \overset{\text{row } \mathbf{Q} \times \text{column } 1}{\downarrow \; 12.30} & \overset{\text{row } \mathbf{Q} \times \text{column } 2}{\downarrow \; 11.20} \end{pmatrix}}$$

$1 \times 3 \longleftarrow$ the same $\longrightarrow 3 \times 2$

resultant matrix

Now suppose Lisa's friend Olu needs 1 loaf of bread, 2 litres of milk, and 2 tubs of butter.

The quantities matrix for both Lisa and Olu would be

$$\begin{array}{c} \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} \begin{array}{l} \longleftarrow \text{Lisa} \\ \longleftarrow \text{Olu} \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{bread} & \text{milk} & \text{butter} \end{array} \end{array}$$

Lisa's *total cost* at Store A is \$12.30, and at store B is \$11.20 .

Olu's *total cost* at Store A is $1 \times \$2.65 + 2 \times \$1.55 + 2 \times \$2.35 = \$10.45$,
and at Store B is $1 \times \$2.25 + 2 \times \$1.50 + 2 \times \$2.20 = \$9.65$.

So, using matrices we require that

row 1 × column 1

row 1 × column 2

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} \times \begin{pmatrix} 2.65 & 2.25 \\ 1.55 & 1.50 \\ 2.35 & 2.20 \end{pmatrix} = \begin{pmatrix} 12.30 & 11.20 \\ 10.45 & 9.65 \end{pmatrix}$$

row 2 × column 2

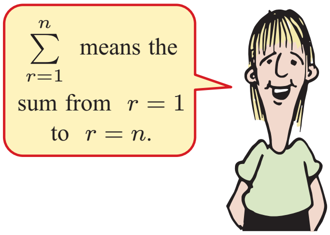
row 2 × column 1

$2 \times 3 \leftarrow$ the same $\rightarrow 3 \times 2$ $\qquad 2 \times 2$

resultant matrix

Having observed the usefulness of multiplying matrices in the contextual examples above, we now define matrix multiplication more formally.

> The **product** of an $m \times n$ matrix **A** with an $n \times p$ matrix **B**, is the $m \times p$ matrix **AB** in which the element in the rth row and cth column is the sum of the products of the elements in the rth row of **A** with the corresponding elements in the cth column of **B**.
>
> If $\mathbf{C} = \mathbf{AB}$ then $c_{ij} = \sum_{r=1}^{n} a_{ir} b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}$
>
> for each pair i and j with $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant p$.
>
> Note that the product **AB** exists *only* if the number of columns of **A** equals the number of rows of **B**.



For example:

$$\text{If } \mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \text{ then } \mathbf{AB} = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}.$$

$$\text{If } \mathbf{C} = \underset{2 \times 3}{\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}} \text{ and } \mathbf{D} = \underset{3 \times 1}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}, \text{ then } \mathbf{CD} = \underset{2 \times 1}{\begin{pmatrix} ax + by + cz \\ dx + ey + fz \end{pmatrix}}.$$

To get the matrix **AB** you multiply **rows by columns**. To get the element in the 5th row and 3rd column of **AB** (if it exists), multiply the 5th row of **A** by the 3rd column of **B**.

Example 6

Self Tutor

For $\mathbf{A}=\begin{pmatrix}1 & 3 & 5\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1 & 3 & 5\\ 2 & 1 & 3\end{pmatrix}$, and $\mathbf{C}=\begin{pmatrix}1 & 0\\ 2 & 3\\ 1 & 4\end{pmatrix}$, find: **a** **AC** **b** **BC**

a **A** is 1×3 and **C** is 3×2 $\therefore$ **AC** is 1×2

✓

$$\mathbf{AC}=\begin{pmatrix}1 & 3 & 5\end{pmatrix}\begin{pmatrix}1 & 0\\ 2 & 3\\ 1 & 4\end{pmatrix}$$

$$=\begin{pmatrix}1\times1+3\times2+5\times1 & 1\times0+3\times3+5\times4\end{pmatrix}$$

$$=\begin{pmatrix}12 & 29\end{pmatrix}$$

b **B** is 2×3 and **C** is 3×2 $\therefore$ **BC** is 2×2

✓

$$\mathbf{BC}=\begin{pmatrix}1 & 3 & 5\\ 2 & 1 & 3\end{pmatrix}\begin{pmatrix}1 & 0\\ 2 & 3\\ 1 & 4\end{pmatrix}$$

$$=\begin{pmatrix}1\times1+3\times2+5\times1 & 1\times0+3\times3+5\times4\\ 2\times1+1\times2+3\times1 & 2\times0+1\times3+3\times4\end{pmatrix}$$

$$=\begin{pmatrix}12 & 29\\ 7 & 15\end{pmatrix}$$

To get the element in the 2nd row and 1st column of **BC**, multiply the 2nd row of **B** by the 1st column of **C**.

EXERCISE 12C.2

1 Explain why **AB** cannot be found for $\mathbf{A}=\begin{pmatrix}4 & 2 & 1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1 & 2 & 1\\ 0 & 1 & 0\end{pmatrix}$.

2 Suppose **A** is $2\times n$ and **B** is $m\times 3$.

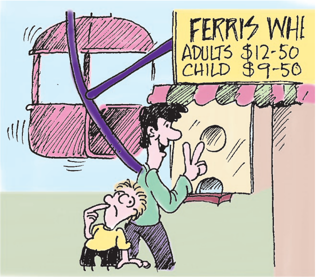
- **a** When can we find **AB**?
- **b** If **AB** can be found, what is its order?
- **c** Explain why **BA** cannot be found.

3 For $\mathbf{A}=\begin{pmatrix}2 & 1\\ 3 & 4\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}5 & 6\end{pmatrix}$, find: **a** **AB** **b** **BA**

4 For $\mathbf{A}=\begin{pmatrix}2 & 0 & 3\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1\\ 4\\ 2\end{pmatrix}$, find: **a** **AB** **b** **BA**

5 Find: **a** $\begin{pmatrix}1 & 2 & 1\end{pmatrix}\begin{pmatrix}2 & 3 & 1\\ 0 & 1 & 0\\ 1 & 0 & 2\end{pmatrix}$ **b** $\begin{pmatrix}1 & 0 & -1\\ -1 & 1 & 0\\ 0 & -1 & 1\end{pmatrix}\begin{pmatrix}2\\ 3\\ 4\end{pmatrix}$

6 Answer the **Opening Problem** on page **306**.

7 

At a fair, tickets for the Ferris wheel are \$12.50 per adult and \$9.50 per child. On the first day of the fair, 2375 adults and 5156 children ride this wheel. On the second day, 2502 adults and 3612 children ride the wheel.

- **a** Write the costs as a 2×1 matrix **C**, and the numbers as a 2×2 matrix **N**.
- **b** Find **NC** and interpret the resulting matrix.
- **c** Find the total income for the two days.

8 You and your friend each go to your local hardware stores A and B to price items you wish to purchase. You want to buy 1 hammer, 1 screwdriver, and 2 cans of white paint. Your friend wants 1 hammer, 2 screwdrivers, and 3 cans of white paint. The prices of these goods are:

	Hammer	*Screwdriver*	*Can of paint*
Store A	\$7	\$3	\$19
Store B	\$6	\$2	\$22

- **a** Write the requirements matrix **R** as a 3×2 matrix.
- **b** Write the prices matrix **P** as a 2×3 matrix.
- **c** Find **PR**.
- **d** Find:
  - **i** your costs at store A
  - **ii** your friend's costs at store B.
- **e** Do any of the elements of **PR** tell you and your friend the cheapest way to buy all your items? Explain your answer.

PROPERTIES OF MATRIX MULTIPLICATION

Discovery 1 — Matrix multiplication

In this Discovery we find the properties of 2×2 matrix multiplication which are like those of ordinary number multiplication, and those which are not.

What to do:

1 For ordinary arithmetic $2 \times 3 = 3 \times 2$, and in algebra $ab = ba$.
For matrices, does **AB** always equal **BA**?

Hint: Try $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 1 \\ 0 & 3 \end{pmatrix}$.

2 If $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, find **AO** and **OA**.

3 Find **AB** for:

- **a** $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$
- **b** $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -3 \\ 2 & -6 \end{pmatrix}$

4 For all real numbers a, b, and c, we have the **distributive law** $a(b+c) = ab + ac$.

- **a** Use any three 2×2 matrices **A** **B** and **C** to verify that $\mathbf{A(B+C)} = \mathbf{AB} + \mathbf{AC}$

b Now let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

Prove that in general, $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$.

c Use the matrices you 'made up' in **a** to verify that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

d Prove that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

5 **a** If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, show that $w = z = 1$ and $x = y = 0$ is a solution for any values of a, b, c, and d.

b For any real number a, we know that $a \times 1 = 1 \times a = a$.
Is there a matrix $\mathbf{I}$ such that $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$?

6 Suppose $\mathbf{A}^2 = \mathbf{AA} = \mathbf{A} \times \mathbf{A}$ and that $\mathbf{A}^3 = \mathbf{AAA}$.

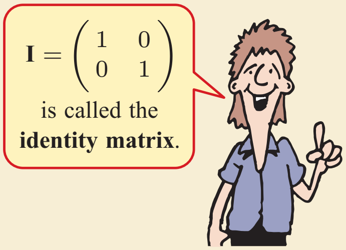
a Find $\mathbf{A}^2$ if $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}$.

b Find $\mathbf{A}^3$ if $\mathbf{A} = \begin{pmatrix} 5 & -1 \\ 2 & 4 \end{pmatrix}$.

c If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ try to find $\mathbf{A}^2$.

d Under what conditions can we square a matrix?

7 Show that if $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $\mathbf{I}^2 = \mathbf{I}$ and $\mathbf{I}^3 = \mathbf{I}$.



In the **Discovery** you should have found that:

Ordinary algebra	**Matrix algebra**
• If a and b are real numbers then so is ab. {closure}	• If $\mathbf{A}$ and $\mathbf{B}$ are matrices that can be multiplied then $\mathbf{AB}$ is also a matrix. {closure}
• $ab = ba$ for all a, b {commutative}	• In general $\mathbf{AB} \neq \mathbf{BA}$. {non-commutative}
• $a0 = 0a = 0$ for all a	• If $\mathbf{O}$ is a zero matrix then $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$ for all $\mathbf{A}$.
• $ab = 0 \Leftrightarrow a = 0$ or $b = 0$ {Null Factor law}	• $\mathbf{AB}$ may be $\mathbf{O}$ without requiring $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$.
• $a(b+c) = ab + ac$ {distributive law}	• $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$ {distributive law}
• $a \times 1 = 1 \times a = a$ {identity law}	• If $\mathbf{I}$ is the **identity matrix** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ then $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$ for all 2×2 matrices $\mathbf{A}$. {identity law}
• a^n exists for all $a \geqslant 0$ and $n \in \mathbb{R}$.	• $\mathbf{A}^n$ exists provided $\mathbf{A}$ is square and $n \in \mathbb{Z}^+$.

Note that in general, $\mathbf{A}(k\mathbf{B}) = k(\mathbf{AB}) \neq k\mathbf{BA}$. We can change the order in which we multiply by a scalar, but we cannot reverse the order in which we multiply matrices.

Example 7

Self Tutor

Expand and simplify where possible:

a $(\mathbf{A}+2\mathbf{I})^2$ **b** $(\mathbf{A}-\mathbf{B})^2$

a $(\mathbf{A}+2\mathbf{I})^2$

$= (\mathbf{A}+2\mathbf{I})(\mathbf{A}+2\mathbf{I})$ $\{\mathbf{X}^2=\mathbf{XX}$ by definition$\}$

$= (\mathbf{A}+2\mathbf{I})\mathbf{A}+(\mathbf{A}+2\mathbf{I})2\mathbf{I}$ $\{\mathbf{B}(\mathbf{C}+\mathbf{D})=\mathbf{BC}+\mathbf{BD}\}$

$= \mathbf{A}^2+2\mathbf{IA}+2\mathbf{AI}+4\mathbf{I}^2$ $\{(\mathbf{C}+\mathbf{D})\mathbf{B}=\mathbf{CB}+\mathbf{DB}\}$

$= \mathbf{A}^2+2\mathbf{A}+2\mathbf{A}+4\mathbf{I}$ $\{\mathbf{AI}=\mathbf{IA}=\mathbf{A}$ and $\mathbf{I}^2=\mathbf{I}\}$

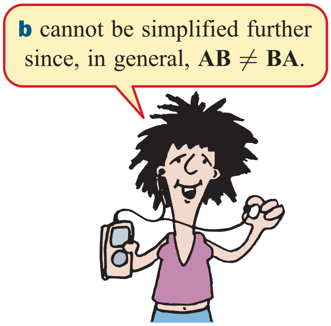
$= \mathbf{A}^2+4\mathbf{A}+4\mathbf{I}$

b $(\mathbf{A}-\mathbf{B})^2$

$= (\mathbf{A}-\mathbf{B})(\mathbf{A}-\mathbf{B})$ $\{\mathbf{X}^2=\mathbf{XX}$ by definition$\}$

$= (\mathbf{A}-\mathbf{B})\mathbf{A}-(\mathbf{A}-\mathbf{B})\mathbf{B}$ $\{\mathbf{C}(\mathbf{D}-\mathbf{E})=\mathbf{CD}-\mathbf{CE}\}$

$= \mathbf{A}^2-\mathbf{BA}-\mathbf{AB}+\mathbf{B}^2$ $\{(\mathbf{D}-\mathbf{E})\mathbf{C}=\mathbf{DC}-\mathbf{EC}\}$



Example 8

Self Tutor

If $\mathbf{A}^2 = 2\mathbf{A}+3\mathbf{I}$, find $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $k\mathbf{A}+l\mathbf{I}$ where k and l are scalars.

$\mathbf{A}^3 = \mathbf{A}\times\mathbf{A}^2$

$= \mathbf{A}(2\mathbf{A}+3\mathbf{I})$

$= 2\mathbf{A}^2+3\mathbf{AI}$

$= 2(2\mathbf{A}+3\mathbf{I})+3\mathbf{AI}$

$= 7\mathbf{A}+6\mathbf{I}$

$\mathbf{A}^4 = \mathbf{A}\times\mathbf{A}^3$

$= \mathbf{A}(7\mathbf{A}+6\mathbf{I})$

$= 7\mathbf{A}^2+6\mathbf{AI}$

$= 7(2\mathbf{A}+3\mathbf{I})+6\mathbf{A}$

$= 20\mathbf{A}+21\mathbf{I}$

EXERCISE 12C.3

1 Given that all matrices are 2×2 and $\mathbf{I}$ is the identity matrix, expand and simplify:

a $\mathbf{A}(\mathbf{A}+\mathbf{I})$ **b** $(\mathbf{B}+2\mathbf{I})\mathbf{B}$ **c** $\mathbf{A}(\mathbf{A}^2-2\mathbf{A}+\mathbf{I})$

d $\mathbf{A}(\mathbf{A}^2+\mathbf{A}-2\mathbf{I})$ **e** $(\mathbf{A}+\mathbf{B})(\mathbf{C}+\mathbf{D})$ **f** $(\mathbf{A}+\mathbf{B})^2$

g $(\mathbf{A}+\mathbf{B})(\mathbf{A}-\mathbf{B})$ **h** $(\mathbf{A}+\mathbf{I})^2$ **i** $(3\mathbf{I}-\mathbf{B})^2$

2 **a** If $\mathbf{A}^2=2\mathbf{A}-\mathbf{I}$, find $\mathbf{A}^3$ and $\mathbf{A}^4$ in the linear form $k\mathbf{A}+l\mathbf{I}$ where k and l are scalars.

b If $\mathbf{B}^2=2\mathbf{I}-\mathbf{B}$, find $\mathbf{B}^3$, $\mathbf{B}^4$, and $\mathbf{B}^5$ in linear form.

c If $\mathbf{C}^2=4\mathbf{C}-3\mathbf{I}$, find $\mathbf{C}^3$ and $\mathbf{C}^5$ in linear form.

3 **a** If $\mathbf{A}^2=\mathbf{I}$, simplify:

i $\mathbf{A}(\mathbf{A}+2\mathbf{I})$ **ii** $(\mathbf{A}-\mathbf{I})^2$ **iii** $\mathbf{A}(\mathbf{A}+3\mathbf{I})^2$

b If $\mathbf{A}^3=\mathbf{I}$, simplify $\mathbf{A}^2(\mathbf{A}+\mathbf{I})^2$.

c If $\mathbf{A}^2=\mathbf{O}$, simplify:

i $\mathbf{A}(2\mathbf{A}-3\mathbf{I})$ **ii** $\mathbf{A}(\mathbf{A}+2\mathbf{I})(\mathbf{A}-\mathbf{I})$ **iii** $\mathbf{A}(\mathbf{A}+\mathbf{I})^3$

4 **a** If $\mathbf{A} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, determine $\mathbf{A}^2$.

b Comment on the following argument for a 2×2 matrix $\mathbf{A}$ such that $\mathbf{A}^2 = \mathbf{A}$:

$$\begin{aligned} \mathbf{A}^2 &= \mathbf{A} \\ \therefore\ \mathbf{A}^2 - \mathbf{A} &= \mathbf{O} \\ \therefore\ \mathbf{A}(\mathbf{A} - \mathbf{I}) &= \mathbf{O} \\ \therefore\ \mathbf{A} = \mathbf{O} &\text{ or } \mathbf{A} - \mathbf{I} = \mathbf{O} \\ \therefore\ \mathbf{A} = \mathbf{O} &\text{ or } \mathbf{I} \end{aligned}$$

c Find *all* 2×2 matrices $\mathbf{A}$ for which $\mathbf{A}^2 = \mathbf{A}$. **Hint:** Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

5 Give *one* example which shows that "if $\mathbf{A}^2 = \mathbf{O}$ then $\mathbf{A} = \mathbf{O}$" is a *false* statement.

Example 9

Self Tutor

For $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$.

Since $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = a\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1+6 & 2+8 \\ 3+12 & 6+16 \end{pmatrix} = \begin{pmatrix} a & 2a \\ 3a & 4a \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$$

$$\therefore \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} = \begin{pmatrix} a+b & 2a \\ 3a & 4a+b \end{pmatrix}$$

Thus $a + b = 7$ and $2a = 10$

$\therefore$ $a = 5$ and $b = 2$

Checking for consistency:

$3a = 3(5) = 15$ ✓ $4a + b = 4(5) + (2) = 22$ ✓

6 Find constants a and b such that $\mathbf{A}^2 = a\mathbf{A} + b\mathbf{I}$, given:

a $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$

b $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 2 & -2 \end{pmatrix}$

7 **a** For $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & -3 \end{pmatrix}$, find constants p and q such that $\mathbf{A}^2 = p\mathbf{A} + q\mathbf{I}$.

b Hence, write $\mathbf{A}^3$ in the linear form $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.

c Write $\mathbf{A}^4$ in linear form.

D THE INVERSE OF A 2×2 MATRIX

The real numbers 5 and $\frac{1}{5}$ are called **multiplicative inverses** because when they are multiplied together, the result is the multiplicative identity 1: $5 \times \frac{1}{5} = \frac{1}{5} \times 5 = 1$

For the matrices $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$, we notice that $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$ and $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$.

We say that $\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$ are *multiplicative inverses* of each other.

The **multiplicative inverse** of $\mathbf{A}$, denoted $\mathbf{A}^{-1}$, satisfies $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

To find the multiplicative inverse of a matrix $\mathbf{A}$, we need a matrix which, when multiplied by $\mathbf{A}$, gives the identity matrix $\mathbf{I}$.

We will now determine how to find the inverse of a matrix $\mathbf{A}$.

Suppose $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{A}^{-1} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$

$$\therefore \quad \mathbf{A}\mathbf{A}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \mathbf{I}$$

$$\therefore \quad \begin{pmatrix} aw+by & ax+bz \\ cw+dy & cx+dz \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \quad \begin{cases} aw+by=1 & \text{.... (1)} \\ cw+dy=0 & \text{.... (2)} \end{cases} \quad \text{and} \quad \begin{cases} ax+bz=0 & \text{.... (3)} \\ cx+dz=1 & \text{.... (4)} \end{cases}$$

Solving (1) and (2) simultaneously for w and y gives: $w = \frac{d}{ad-bc}$ and $y = \frac{-c}{ad-bc}$.

Solving (3) and (4) simultaneously for x and z gives: $x = \frac{-b}{ad-bc}$ and $z = \frac{a}{ad-bc}$.

So, if $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $ad - bc \neq 0$, then $\mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

In this case
$$\begin{aligned}\mathbf{A}^{-1}\mathbf{A} &= \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \frac{1}{ad-bc}\begin{pmatrix} ad-bc & bd-bd \\ ac-ac & -bc+ad \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \mathbf{I} \quad \text{also,}\end{aligned}$$

so $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$

Just as the real number 0 does not have a multiplicative inverse, some matrices do not have a multiplicative inverse. This occurs when $\det \mathbf{A} = ad - bc = 0$.

For the matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

- the value $ad - bc$ is called the **determinant** of matrix $\mathbf{A}$, denoted $\det \mathbf{A}$
- if $\det \mathbf{A} \neq 0$, then $\mathbf{A}$ is **invertible** or **non-singular**, and $\mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- if $\det \mathbf{A} = 0$, then $\mathbf{A}$ is **singular**, and $\mathbf{A}^{-1}$ does not exist.

Example 10

Self Tutor

Find, if it exists, the inverse matrix of:

a $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$ **b** $\mathbf{B} = \begin{pmatrix} 6 & 3 \\ -4 & -2 \end{pmatrix}$

a $\mathbf{A} = \begin{pmatrix} 5 & 6 \\ 3 & 4 \end{pmatrix}$

$\therefore \det \mathbf{A} = 5(4) - 6(3) = 2$

$\therefore \mathbf{A}^{-1} = \frac{1}{2} \begin{pmatrix} 4 & -6 \\ -3 & 5 \end{pmatrix}$

$= \begin{pmatrix} 2 & -3 \\ -\frac{3}{2} & \frac{5}{2} \end{pmatrix}$

b $\mathbf{B} = \begin{pmatrix} 6 & 3 \\ -4 & -2 \end{pmatrix}$

$\therefore \det \mathbf{B} = 6(-2) - 3(-4)$

$= -12 + 12$

$= 0$

$\therefore \mathbf{B}^{-1}$ does not exist.

EXERCISE 12D.1

1 **a** Find $\begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 3 & -6 \\ -2 & 5 \end{pmatrix}$, and hence find the inverse of $\begin{pmatrix} 5 & 6 \\ 2 & 3 \end{pmatrix}$.

b Find $\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ -1 & 3 \end{pmatrix}$, and hence find the inverse of $\begin{pmatrix} 3 & -4 \\ 1 & 2 \end{pmatrix}$.

2 Find $\det \mathbf{A}$ for $\mathbf{A}$ equal to:

a $\begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$ **c** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

3 Find $\det \mathbf{B}$ for $\mathbf{B}$ equal to:

a $\begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ **c** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **d** $\begin{pmatrix} a & -a \\ 1 & a \end{pmatrix}$

4 For $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & -1 \end{pmatrix}$, find:

a $\det \mathbf{A}$ **b** $\det(-\mathbf{A})$ **c** $\det(2\mathbf{A})$

5 Prove that if $\mathbf{A}$ is any 2×2 matrix and k is a constant, then $\det(k\mathbf{A}) = k^2 \times \det \mathbf{A}$.

6 Suppose $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} w & x \\ y & z \end{pmatrix}$.

a Find:

i $\det \mathbf{A}$ **ii** $\det \mathbf{B}$ **iii** $\mathbf{AB}$ **iv** $\det(\mathbf{AB})$

b Hence show that $\det(\mathbf{AB})=\det \mathbf{A}\times\det \mathbf{B}$ for all 2×2 matrices $\mathbf{A}$ and $\mathbf{B}$.

7 Suppose $\mathbf{A}=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$.

a Find $\det \mathbf{A}$ and $\det \mathbf{B}$.

b Find:

i $\det(2\mathbf{A})$ **ii** $\det(-\mathbf{A})$ **iii** $\det(-3\mathbf{B})$ **iv** $\det(\mathbf{AB})$

8 Find, if it exists, the inverse matrix of:

a $\begin{pmatrix} 2 & 4 \\ -1 & 5 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

d $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **e** $\begin{pmatrix} 5 & 0 \\ -1 & 2 \end{pmatrix}$ **f** $\begin{pmatrix} 3 & 5 \\ -6 & -10 \end{pmatrix}$

g $\begin{pmatrix} -1 & 2 \\ 4 & 7 \end{pmatrix}$ **h** $\begin{pmatrix} 3 & 4 \\ -1 & 2 \end{pmatrix}$ **i** $\begin{pmatrix} -1 & -1 \\ 2 & 3 \end{pmatrix}$

Example 11

Self Tutor

Suppose $\mathbf{A}=\begin{pmatrix} 4 & k \\ 2 & -1 \end{pmatrix}$.

Find $\mathbf{A}^{-1}$ and state the values of k for which $\mathbf{A}^{-1}$ exists.

$$\mathbf{A}^{-1}=\frac{1}{-4-2k}\begin{pmatrix} -1 & -k \\ -2 & 4 \end{pmatrix}=\begin{pmatrix} \frac{1}{2k+4} & \frac{k}{2k+4} \\ \frac{2}{2k+4} & \frac{-4}{2k+4} \end{pmatrix}$$

$\mathbf{A}^{-1}$ exists provided that $2k+4\neq 0$

$\therefore\ k\neq -2$

If $\det \mathbf{A}=0$, the matrix $\mathbf{A}$ is singular.

9 For each of the following matrices $\mathbf{A}$, find $\mathbf{A}^{-1}$ and state the values of k for which $\mathbf{A}^{-1}$ exists.

a $\mathbf{A}=\begin{pmatrix} k & 1 \\ -6 & 2 \end{pmatrix}$ **b** $\mathbf{A}=\begin{pmatrix} 3 & -1 \\ 0 & k \end{pmatrix}$ **c** $\mathbf{A}=\begin{pmatrix} k+1 & 2 \\ 1 & k \end{pmatrix}$

d $\mathbf{A}=\begin{pmatrix} k-2 & k \\ -3 & k \end{pmatrix}$ **e** $\mathbf{A}=\begin{pmatrix} k^2 & k-1 \\ 2k & 1 \end{pmatrix}$ **f** $\mathbf{A}=\begin{pmatrix} k+1 & 2 \\ k^2+2 & 3k \end{pmatrix}$

FURTHER MATRIX ALGEBRA

In this section we consider matrix algebra with inverse matrices. Be careful that you use multiplication correctly. In particular, remember that:

- We can only perform matrix multiplication if the orders of the matrices allow it.
- If we *pre*multiply on one side then we must *pre*multiply on the other. This is important because, in general, $\mathbf{AB} \neq \mathbf{BA}$. The same applies if we *post*multiply.

Premultiply means multiply on the left of each side.
Postmultiply means multiply on the right of each side.

Discovery 2 — Properties of inverse matrices

In this Discovery, we consider some properties of invertible 2×2 matrices.

What to do:

1 A matrix $\mathbf{A}$ is **self-inverse** when $\mathbf{A} = \mathbf{A}^{-1}$.

For example, if $\mathbf{A} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ then $\mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \mathbf{A}$.

- **a** Show that if $\mathbf{A} = \mathbf{A}^{-1}$, then $\mathbf{A}^2 = \mathbf{I}$.
- **b** Show that there are exactly 4 self-inverse matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$.

2
- **a** Given $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, find $\mathbf{A}^{-1}$ and $(\mathbf{A}^{-1})^{-1}$.
- **b** If $\mathbf{A}$ is any invertible matrix, simplify $(\mathbf{A}^{-1})^{-1}(\mathbf{A}^{-1})$ and $(\mathbf{A}^{-1})(\mathbf{A}^{-1})^{-1}$ by replacing $\mathbf{A}^{-1}$ by $\mathbf{B}$.
- **c** What can be deduced from **b**?

3 Suppose k is a non-zero number and $\mathbf{A}$ is an invertible matrix.

- **a** Simplify $(k\mathbf{A})(\frac{1}{k}\mathbf{A}^{-1})$ and $(\frac{1}{k}\mathbf{A}^{-1})(k\mathbf{A})$.
- **b** What can you conclude from your results?

4
- **a** If $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}$, find in simplest form:
  - **i** $\mathbf{A}^{-1}$
  - **ii** $\mathbf{B}^{-1}$
  - **iii** $(\mathbf{AB})^{-1}$
  - **iv** $(\mathbf{BA})^{-1}$
  - **v** $\mathbf{A}^{-1}\mathbf{B}^{-1}$
  - **vi** $\mathbf{B}^{-1}\mathbf{A}^{-1}$
- **b** Choose any two invertible matrices and repeat **a**.
- **c** What do the results of **a** and **b** suggest?
- **d** Simplify $(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1})$ and $(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB})$ given that $\mathbf{A}^{-1}$ and $\mathbf{B}^{-1}$ exist. What can you conclude from your results?

From the **Discovery** you should have found that if $\mathbf{A}$ and $\mathbf{B}$ are invertible, then:

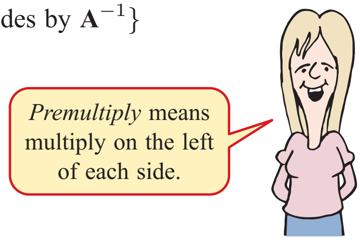
- $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
- $(k\mathbf{A})^{-1} = \frac{1}{k}\mathbf{A}^{-1}$
- $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Example 12

Self Tutor

If $\mathbf{A}^2 = 2\mathbf{A} + 3\mathbf{I}$, find $\mathbf{A}^{-1}$ in the linear form $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.

$$\begin{aligned}
\mathbf{A}^2 &= 2\mathbf{A} + 3\mathbf{I} \\
\therefore\ \mathbf{A}^{-1}\mathbf{A}^2 &= \mathbf{A}^{-1}(2\mathbf{A} + 3\mathbf{I}) \qquad \{\text{premultiplying both sides by } \mathbf{A}^{-1}\} \\
\therefore\ \mathbf{A}^{-1}\mathbf{A}\mathbf{A} &= 2\mathbf{A}^{-1}\mathbf{A} + 3\mathbf{A}^{-1}\mathbf{I} \\
\therefore\ \mathbf{I}\mathbf{A} &= 2\mathbf{I} + 3\mathbf{A}^{-1} \\
\therefore\ \mathbf{A} - 2\mathbf{I} &= 3\mathbf{A}^{-1} \\
\therefore\ \mathbf{A}^{-1} &= \tfrac{1}{3}(\mathbf{A} - 2\mathbf{I}) \\
\therefore\ \mathbf{A}^{-1} &= \tfrac{1}{3}\mathbf{A} - \tfrac{2}{3}\mathbf{I}
\end{aligned}$$



EXERCISE 12D.2

1 Suppose $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$, $\mathbf{C} = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$, and $\mathbf{AXB} = \mathbf{C}$. Find $\mathbf{X}$.

2 Suppose $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are 2×1 matrices, and $\mathbf{A}$ and $\mathbf{B}$ are invertible 2×2 matrices.
If $\mathbf{X} = \mathbf{AY}$ and $\mathbf{Y} = \mathbf{BZ}$, write:

a $\mathbf{X}$ in terms of $\mathbf{Z}$ **b** $\mathbf{Z}$ in terms of $\mathbf{X}$.

3 If $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix}$, write $\mathbf{A}^2$ in the linear form $p\mathbf{A} + q\mathbf{I}$ where p and q are scalars.

Hence write $\mathbf{A}^{-1}$ in the form $r\mathbf{A} + s\mathbf{I}$ where r and s are scalars.

4 Write $\mathbf{A}^{-1}$ in linear form given that:

a $\mathbf{A}^2 = 4\mathbf{A} - \mathbf{I}$ **b** $5\mathbf{A} = \mathbf{I} - \mathbf{A}^2$ **c** $2\mathbf{I} = 3\mathbf{A}^2 - 4\mathbf{A}$

5 It is known that $\mathbf{AB} = \mathbf{A}$ and $\mathbf{BA} = \mathbf{B}$ where the matrices $\mathbf{A}$ and $\mathbf{B}$ are not necessarily invertible.
Prove that $\mathbf{A}^2 = \mathbf{A}$.
Hint: From $\mathbf{AB} = \mathbf{A}$, you cannot deduce that $\mathbf{B} = \mathbf{I}$.

6 Under what condition is it true that "if $\mathbf{AB} = \mathbf{AC}$ then $\mathbf{B} = \mathbf{C}$"?

7 If $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$ and $\mathbf{A}^3 = \mathbf{I}$, prove that $\mathbf{X}^3 = \mathbf{I}$.

8 If $a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{O}$ and $\mathbf{X} = \mathbf{P}^{-1}\mathbf{AP}$, prove that $a\mathbf{X}^2 + b\mathbf{X} + c\mathbf{I} = \mathbf{O}$.

E SIMULTANEOUS LINEAR EQUATIONS

We can solve $\begin{cases} 2x+3y=4 \\ 5x+4y=17 \end{cases}$ algebraically to get $x=5,\quad y=-2$.

Notice that this system can be written as a matrix equation $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 4 \\ 17 \end{pmatrix}$.

The solution $x=5,\quad y=-2$ is easily checked as

$$\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ -2 \end{pmatrix}=\begin{pmatrix} 2(5)+3(-2) \\ 5(5)+4(-2) \end{pmatrix}=\begin{pmatrix} 4 \\ 17 \end{pmatrix}\quad \checkmark$$

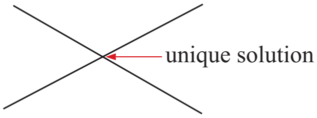
In general, a system of linear equations can be written in the form $\mathbf{AX}=\mathbf{B}$ where $\mathbf{A}$ is the matrix of coefficients, $\mathbf{X}$ is the unknown column matrix, and $\mathbf{B}$ is a column matrix of constants.

We can use inverses to solve the matrix equation $\mathbf{AX}=\mathbf{B}$ for $\mathbf{X}$.

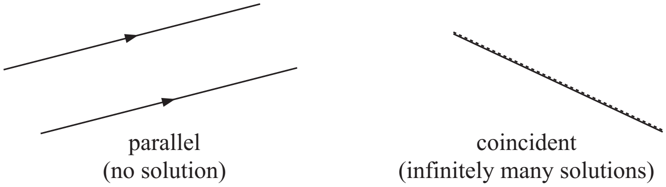
If we premultiply each side of $\mathbf{AX}=\mathbf{B}$ by $\mathbf{A}^{-1}$, we get

$$\begin{aligned} \mathbf{A}^{-1}(\mathbf{AX}) &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore\quad (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \\ \therefore\quad \mathbf{IX} &= \mathbf{A}^{-1}\mathbf{B} \\ \text{and so}\quad \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \end{aligned}$$

If the matrix of coefficients $\mathbf{A}$ is invertible, then calculating $\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$ will give a unique solution to the pair of linear equations. This indicates that the lines intersect at a single point.



If the matrix of coefficients $\mathbf{A}$ is singular, then we cannot calculate $\mathbf{X}=\mathbf{A}^{-1}\mathbf{B}$. This indicates that either the lines are parallel and there are no solutions, or that the lines are coincident and there are infinitely many solutions.



Example 13

Self Tutor

a If $\mathbf{A}=\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$, find $\mathbf{A}^{-1}$.

b Write the system $\begin{cases} 2x+3y=4 \\ 5x+4y=17 \end{cases}$ in matrix form.

c Hence, solve the simultaneous linear equations.

a $\det \mathbf{A}=2(4)-3(5)$
$=-7$

$\therefore\ \mathbf{A}^{-1}=\frac{1}{-7}\begin{pmatrix} 4 & -3 \\ -5 & 2 \end{pmatrix}$

b In matrix form, the system is $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 17 \end{pmatrix}$ which has the form $\mathbf{AX} = \mathbf{B}$.

c Premultiplying by $\mathbf{A}^{-1}$, $\mathbf{A}^{-1}\mathbf{AX} = \mathbf{A}^{-1}\mathbf{B}$

$$\therefore\ \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$$

$$\therefore\ \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{-7}\begin{pmatrix} 4 & -3 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 17 \end{pmatrix}$$

$$= \tfrac{1}{-7}\begin{pmatrix} -35 \\ 14 \end{pmatrix}$$

$$= \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

$\therefore$ $x = 5$ and $y = -2$.

EXERCISE 12E

1 Convert into matrix equations:

a $\begin{cases} 3x - y = 8 \\ 2x + 3y = 6 \end{cases}$ **b** $\begin{cases} 4x - 3y = 11 \\ 3x + 2y = -5 \end{cases}$ **c** $\begin{cases} 3a - b = 6 \\ 2a + 7b = -4 \end{cases}$

2 Use matrix algebra to solve the system:

a $\begin{cases} 2x - y = 6 \\ x + 3y = 14 \end{cases}$ **b** $\begin{cases} 5x - 4y = 5 \\ 2x + 3y = -13 \end{cases}$ **c** $\begin{cases} x - 2y = 7 \\ 5x + 3y = -2 \end{cases}$

d $\begin{cases} 3x + 5y = 4 \\ 2x - y = 11 \end{cases}$ **e** $\begin{cases} 4x - 7y = 8 \\ 3x - 5y = 0 \end{cases}$ **f** $\begin{cases} 7x + 11y = 18 \\ 11x - 7y = -11 \end{cases}$

3 **a** Show that if $\mathbf{AX} = \mathbf{B}$ then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$, whereas if $\mathbf{XA} = \mathbf{B}$ then $\mathbf{X} = \mathbf{BA}^{-1}$.

b Find $\mathbf{X}$ if:

i $\begin{pmatrix} -6 & 5 \\ -3 & 4 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$ **ii** $\mathbf{X}\begin{pmatrix} 1 & 2 \\ 5 & -1 \end{pmatrix} = \begin{pmatrix} 14 & -5 \\ 22 & 0 \end{pmatrix}$

iii $\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 1 & -3 \\ 4 & 2 \end{pmatrix}$ **iv** $\mathbf{X}\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 8 & 10 \\ -5 & 15 \end{pmatrix}$

4 **a** Consider the system $\begin{cases} 2x - 3y = 8 \\ 4x - y = 11 \end{cases}$.

i Write the equations in the form $\mathbf{AX} = \mathbf{B}$, and find $\det \mathbf{A}$.
ii Does the system have a unique solution? If so, find it.

b Consider the system $\begin{cases} 2x + ky = 8 \\ 4x - y = 11 \end{cases}$.

i Write the system in the form $\mathbf{AX} = \mathbf{B}$, and find $\det \mathbf{A}$.
ii For what value(s) of k does the system have a unique solution? Find the unique solution.
iii Find k when the system does not have a unique solution. How many solutions does the system have in this case?

Review set 12A

1 If $\mathbf{A}=\begin{pmatrix}3 & 2\\ 0 & -1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1 & 0\\ -2 & 4\end{pmatrix}$, find:

a $\mathbf{A}+\mathbf{B}$ **b** $3\mathbf{A}$ **c** $-2\mathbf{B}$ **d** $\mathbf{A}-\mathbf{B}$

e $\mathbf{B}-2\mathbf{A}$ **f** $3\mathbf{A}-2\mathbf{B}$ **g** $\mathbf{AB}$ **h** $\mathbf{BA}$

i $\mathbf{A}^{-1}$ **j** $\mathbf{A}^2$ **k** $\mathbf{ABA}$ **l** $(\mathbf{AB})^{-1}$

2 Find a, b, c, and d if:

a $\begin{pmatrix}a & b-2\\ c & d\end{pmatrix}=\begin{pmatrix}-a & 3\\ 2-c & -4\end{pmatrix}$ **b** $\begin{pmatrix}3 & 2a\\ b & -2\end{pmatrix}+\begin{pmatrix}b & -a\\ c & d\end{pmatrix}=\begin{pmatrix}a & 2\\ 2 & 6\end{pmatrix}$

3 Write $\mathbf{Y}$ in terms of $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$:

a $\mathbf{B}-\mathbf{Y}=\mathbf{A}$ **b** $2\mathbf{Y}+\mathbf{C}=\mathbf{A}$ **c** $\mathbf{AY}=\mathbf{B}$

d $\mathbf{YB}=\mathbf{C}$ **e** $\mathbf{C}-\mathbf{AY}=\mathbf{B}$ **f** $\mathbf{AY}^{-1}=\mathbf{B}$

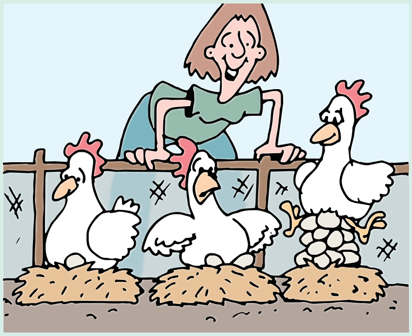
4 Susan keeps 3 hens in a pen. She calls them Anya, Betsy, and Charise. Each week the hens lay eggs according to the matrix

$$\mathbf{L}=\begin{pmatrix}a\\ b\\ c\end{pmatrix}.$$

Write, in terms of $\mathbf{L}$, a matrix to describe:

a the eggs laid by the hens over a 4 week period

b the eggs each hen loses each fortnight when Susan collects the eggs.



5 Suppose $\mathbf{A}=\begin{pmatrix}-2 & 3\\ 4 & -1\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-7 & 9\\ 9 & -3\end{pmatrix}$, and $\mathbf{C}=\begin{pmatrix}-1 & 0 & 3\\ 0 & 2 & 1\end{pmatrix}$.

Evaluate, if possible:

a $2\mathbf{A}-2\mathbf{B}$ **b** $\mathbf{AC}$ **c** $\mathbf{CB}$

6 Given that all matrices are 2×2 and $\mathbf{I}$ is the identity matrix, expand and simplify:

a $\mathbf{A}(\mathbf{I}-\mathbf{A})$ **b** $(\mathbf{A}-\mathbf{B})(\mathbf{B}+\mathbf{A})$ **c** $(2\mathbf{A}-\mathbf{I})^2$

7 If $\mathbf{A}^2=5\mathbf{A}+2\mathbf{I}$, write $\mathbf{A}^3$ and $\mathbf{A}^4$ in the form $r\mathbf{A}+s\mathbf{I}$.

8 If $\mathbf{A}=\begin{pmatrix}2 & -1\\ 3 & 2\end{pmatrix}$, find constants a and b such that $\mathbf{A}^2=a\mathbf{A}+b\mathbf{I}$.

9 Find, if possible, the inverse matrix of:

a $\begin{pmatrix}6 & 8\\ 5 & 7\end{pmatrix}$ **b** $\begin{pmatrix}4 & -3\\ 8 & -6\end{pmatrix}$ **c** $\begin{pmatrix}11 & 5\\ -6 & -3\end{pmatrix}$

10 For what values of k does $\begin{cases}x+4y=2\\ kx+3y=-6\end{cases}$ have a unique solution?

11 Solve using an inverse matrix:

a $\begin{cases} 3x - 4y = 2 \\ 5x + 2y = -1 \end{cases}$ **b** $\begin{cases} 4x - y = 5 \\ 2x + 3y = 9 \end{cases}$

12 Suppose $\mathbf{A} = 2\mathbf{A}^{-1}$.

a Show that $\mathbf{A}^2 = 2\mathbf{I}$.

b Simplify $(\mathbf{A} - \mathbf{I})(\mathbf{A} + 3\mathbf{I})$, giving your answer in the form $r\mathbf{A} + s\mathbf{I}$ where r and s are real numbers.

Review set 12B

1 For $\mathbf{P} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 2 & 3 \end{pmatrix}$ and $\mathbf{Q} = \begin{pmatrix} 3 & 0 \\ 1 & 4 \\ 1 & 1 \end{pmatrix}$, find:

a $\mathbf{P} + \mathbf{Q}$ **b** $\mathbf{Q} - \mathbf{P}$ **c** $\frac{3}{2}\mathbf{P} - \mathbf{Q}$

2 A library owns several copies of a popular trilogy of novels, according to the matrix:

$$\mathbf{A} = \begin{array}{cc} \begin{array}{cc} \text{paperback} & \text{hard cover} \\ \downarrow & \downarrow \end{array} & \\ \begin{pmatrix} 4 & 2 \\ 5 & 2 \\ 6 & 3 \end{pmatrix} & \begin{array}{l} \leftarrow \text{book 1} \\ \leftarrow \text{book 2} \\ \leftarrow \text{book 3} \end{array} \end{array}$$

a At present, the books on loan are described by the matrix $\mathbf{B} = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 3 & 2 \end{pmatrix}$. Write a matrix to describe the books currently on the shelves.

b The values of the books (in dollars) are described by the matrix $\mathbf{C} = \begin{pmatrix} 7 & 7 & 8 \\ 15 & 16 & 20 \end{pmatrix}$.

i Which book has value \$16?

ii Find the total value of the books currently on loan.

3 Prove that for any square matrix $\mathbf{A}$, $\mathbf{AO} = \mathbf{OA} = \mathbf{O}$.

4 Write $\mathbf{X}$ in terms of $\mathbf{A}$ and $\mathbf{B}$ if:

a $2\mathbf{X} = \mathbf{B} - \mathbf{A}$ **b** $3(\mathbf{A} + \mathbf{X}) = 2\mathbf{B}$ **c** $\mathbf{B} - 4\mathbf{X} = \mathbf{A}$

5 Suppose $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -2 & 2 \\ 1 & -3 \end{pmatrix}$, and $\mathbf{A} + 2\mathbf{X} = -\mathbf{B}$. Find $\mathbf{X}$.

6 If $\mathbf{A}$ is $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and $\mathbf{B}$ is $\begin{pmatrix} 2 & 4 \\ 0 & 1 \\ 3 & 2 \end{pmatrix}$, find, if possible:

a $2\mathbf{B}$ **b** $\frac{1}{2}\mathbf{B}$ **c** $\mathbf{AB}$ **d** $\mathbf{BA}$

7 If $\mathbf{A}$ and $\mathbf{B}$ are square matrices, under what conditions are the following true?

a If $\mathbf{AB} = \mathbf{B}$ then $\mathbf{A} = \mathbf{I}$. **b** $(\mathbf{A} + \mathbf{B})^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$

8 For $\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$, find:

a $\det \mathbf{A}$ **b** $\det(-2\mathbf{A})$ **c** $\det(\mathbf{A}^2)$

9 Solve using an inverse matrix:

a $\begin{cases} x + y = 5 \\ x - 2y = 4 \end{cases}$ **b** $\begin{cases} 3x + 2y = 3 \\ 5x + 3y = 4 \end{cases}$

10 If $\mathbf{M} = \begin{pmatrix} k & 2 \\ 2 & k \end{pmatrix} \begin{pmatrix} k-1 & -2 \\ -3 & k \end{pmatrix}$ has an inverse $\mathbf{M}^{-1}$, what values can k have?

11 For what values of k does the system $\begin{cases} kx + 3y = -6 \\ x + (k+2)y = 2 \end{cases}$ have a unique solution?

State the solution in this case.

12 Write $5\mathbf{A}^2 - 6\mathbf{A} = 3\mathbf{I}$ in the form $\mathbf{AB} = \mathbf{I}$. Hence write $\mathbf{A}^{-1}$ in terms of $\mathbf{A}$ and $\mathbf{I}$.

13 Prove that for any 2×2 matrix $\mathbf{A}$, $\mathbf{A}^2$ can be written in the linear form $a\mathbf{A} + b\mathbf{I}$.

13

Introduction to differential calculus

Contents:

Opening problem

In a BASE jumping competition from the Petronas Towers in Kuala Lumpur, the altitude of a professional jumper in the first 3 seconds is given by $f(t) = 452 - 4.8t^2$ metres, where $0 \leqslant t \leqslant 3$ seconds.

Things to think about:

a What will a graph of the altitude of the jumper in the first 3 seconds look like?

b Does the jumper travel with constant speed?

c Can you find the speed of the jumper when:

- **i** $t = 0$ seconds
- **ii** $t = 1$ second
- **iii** $t = 2$ seconds
- **iv** $t = 3$ seconds?

Calculus is a major branch of mathematics which builds on algebra, trigonometry, and analytic geometry. It has widespread applications in science, engineering, and financial mathematics.

The study of calculus is divided into two fields, **differential calculus** and **integral calculus**. These fields are linked by the **Fundamental Theorem of Calculus** which we will study later in the course.

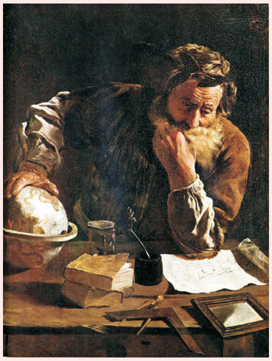
Historical note

Calculus is a Latin word meaning 'pebble'. Ancient Romans used stones for counting.

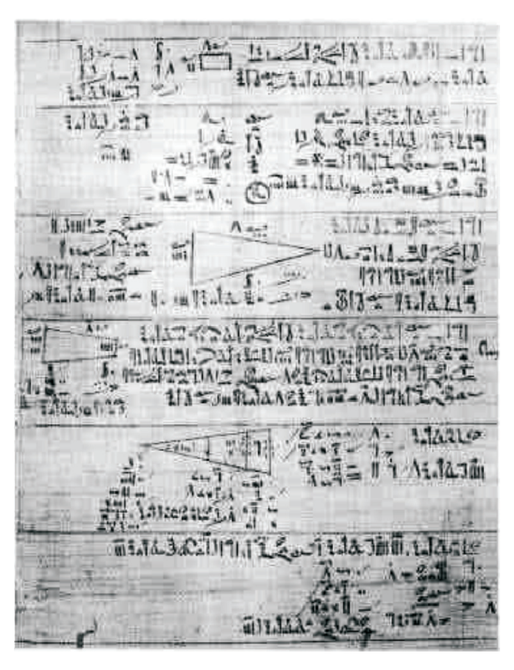
The history of calculus begins with the **Egyptian Moscow papyrus** from about 1850 BC.

The Greek mathematicians **Democritus**, **Zeno of Elea**, **Antiphon**, and **Eudoxes** studied **infinitesimals**, dividing objects into an infinite number of pieces in order to calculate the area of regions, and volume of solids.

Archimedes of Syracuse was the first to find the tangent to a curve other than a circle. His methods were the foundation of modern calculus developed almost 2000 years later.



Archimedes



Egyptian Moscow papyrus

A LIMITS

The concept of a **limit** is essential to differential calculus. We will see that calculating limits is necessary for finding the gradient of a tangent to a curve at any point on the curve.

The table alongside shows values for $f(x) = x^2$ where x is less than 2, but increasing and getting closer and closer to 2.

x	1	1.9	1.99	1.999	1.9999
$f(x)$	1	3.61	3.9601	3.996 00	3.999 60

We say that as x approaches 2 from the left, $f(x)$ approaches 4 from below.

We can construct a similar table of values where x is greater than 2, but decreasing and getting closer and closer to 2:

x	3	2.1	2.01	2.001	2.0001
$f(x)$	9	4.41	4.0401	4.004 00	4.000 40

We say that as x approaches 2 from the right, $f(x)$ approaches 4 from above.

So, as x approaches 2 from either direction, $f(x)$ approaches a limit of 4. We write this as $\lim\limits_{x\to 2} x^2 = 4$.

INFORMAL DEFINITION OF A LIMIT

The following definition of a limit is informal but adequate for the purposes of this course:

> If $f(x)$ can be made as close as we like to some real number A by making x sufficiently close to (but not equal to) a, then we say that $f(x)$ has a **limit** of A as x approaches a, and we write
>
> $$\lim_{x\to a} f(x) = A.$$
>
> In this case, $f(x)$ is said to **converge** to A as x approaches a.

Notice that the limit is defined for x close to but *not equal to* a. Whether the function f is defined or not at $x = a$ is not important to the definition of the limit of f as x approaches a. What *is* important is the behaviour of the function as x gets *very close to* a.

For example, if $f(x) = \dfrac{5x + x^2}{x}$ and we wish to find the limit as $x \to 0$, it is tempting for us to simply substitute $x = 0$ into $f(x)$. However, in doing this, not only do we get the meaningless value of $\frac{0}{0}$, but also we destroy the basic limit method.

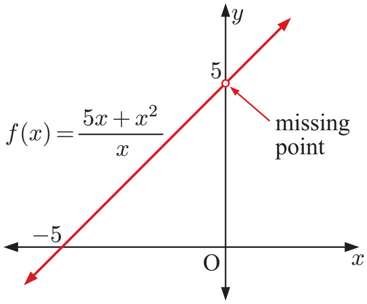
Observe that if $f(x) = \dfrac{5x + x^2}{x} = \dfrac{x(5+x)}{x}$ then $f(x) = \begin{cases} 5 + x & \text{if } x \neq 0 \\ \text{is undefined if } x = 0. \end{cases}$

The graph of $y = f(x)$ is shown alongside. It is the straight line $y = x + 5$ with the point $(0, 5)$ missing, called a **point of discontinuity** of the function.

However, even though this point is missing, the *limit* of $f(x)$ as x approaches 0 does exist. In particular, as $x \to 0$ from either direction, $f(x) \to 5$.

We write $\lim\limits_{x\to 0} \dfrac{5x + x^2}{x} = 5$ which reads:

"the limit as x approaches 0, of $f(x) = \dfrac{5x+x^2}{x}$, is 5".



In practice we do not need to graph functions each time to determine limits, and most can be found algebraically.

Example 1

Self Tutor

Evaluate: **a** $\lim\limits_{x\to 2} x^2$ **b** $\lim\limits_{x\to 0} \dfrac{x^2+3x}{x}$ **c** $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3}$

a x^2 can be made as close as we like to 4 by making x sufficiently close to 2.

$\therefore \lim\limits_{x\to 2} x^2 = 4.$

b
$$\begin{aligned} &\lim_{x\to 0} \frac{x^2+3x}{x} \\ &= \lim_{x\to 0} \frac{{}^1\cancel{x}(x+3)}{\cancel{x}_1} \\ &= \lim_{x\to 0} (x+3) \quad \text{since } x \neq 0 \\ &= 3 \end{aligned}$$

c
$$\begin{aligned} &\lim_{x\to 3} \frac{x^2-9}{x-3} \\ &= \lim_{x\to 3} \frac{(x+3)\cancel{(x-3)}^1}{\cancel{x-3}_1} \\ &= \lim_{x\to 3} (x+3) \quad \text{since } x \neq 3 \\ &= 6 \end{aligned}$$

EXERCISE 13A

1 Evaluate:

a $\lim\limits_{x\to 3} (x+4)$ **b** $\lim\limits_{x\to -1} (5-2x)$ **c** $\lim\limits_{x\to 4} (3x-1)$

d $\lim\limits_{x\to 2} (5x^2-3x+2)$ **e** $\lim\limits_{h\to 0} h^2(1-h)$ **f** $\lim\limits_{x\to 0} (x^2+5)$

2 Evaluate:

a $\lim\limits_{x\to 0} 5$ **b** $\lim\limits_{h\to 2} 7$ **c** $\lim\limits_{x\to 0} c$, c a constant

3 Evaluate:

a $\lim\limits_{x\to 1} \dfrac{x^2-3x}{x}$ **b** $\lim\limits_{h\to 2} \dfrac{h^2+5h}{h}$ **c** $\lim\limits_{x\to 0} \dfrac{x-1}{x+1}$ **d** $\lim\limits_{x\to 0} \dfrac{x}{x}$

4 Evaluate the following limits:

a $\lim\limits_{x\to 0} \dfrac{x^2-3x}{x}$ **b** $\lim\limits_{x\to 0} \dfrac{x^2+5x}{x}$ **c** $\lim\limits_{x\to 0} \dfrac{2x^2-x}{x}$

d $\lim\limits_{h\to 0} \dfrac{2h^2+6h}{h}$ **e** $\lim\limits_{h\to 0} \dfrac{3h^2-4h}{h}$ **f** $\lim\limits_{h\to 0} \dfrac{h^3-8h}{h}$

g $\lim\limits_{x\to 1} \dfrac{x^2-x}{x-1}$ **h** $\lim\limits_{x\to 2} \dfrac{x^2-2x}{x-2}$ **i** $\lim\limits_{x\to 3} \dfrac{x^2-x-6}{x-3}$

B RATES OF CHANGE

A **rate** is a comparison between two quantities with different units.

We often judge performances by rates. For example:

- Sir Donald Bradman's average batting rate at Test cricket level was 99.94 *runs per innings*.
- Michael Jordan's average basketball scoring rate was 20.0 *points per game*.
- Rangi's average typing rate is 63 *words per minute* with an error rate of 2.3 *errors per page*.

Speed is a commonly used rate. It is the rate of change in distance per unit of time.

We are familiar with the formula:

$$\textbf{average speed} = \frac{\textbf{distance travelled}}{\textbf{time taken}}$$

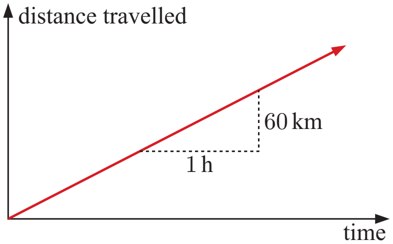
However, if a car has an average speed of 60 km h^{-1} for a journey, it does not mean that the car travels at exactly 60 km h^{-1} for the whole time.

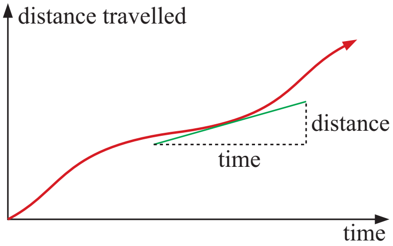
In fact, the speed will probably vary continuously throughout the journey.

So, how can we calculate the car's speed at any particular time?

Suppose we are given a graph of the car's distance travelled against time taken. If this graph is a straight line, then we know the speed is constant and is given by the *gradient* of the line.

If the graph is a curve, then the car's instantaneous speed is given by the *gradient of the tangent* to the curve at that time.





Historical note

The modern study of **differential calculus** originated in the 17th century with the work of **Sir Isaac Newton** and **Gottfried Wilhelm Leibniz**. They developed the necessary theory while attempting to find algebraic methods for solving problems dealing with the **gradients of tangents** to curves, and finding the **rate of change** in one variable with respect to another.

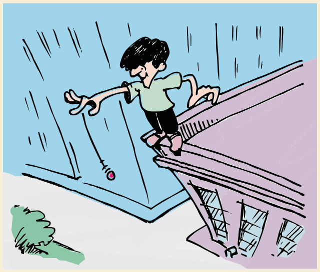
Isaac Newton 1642 – 1727

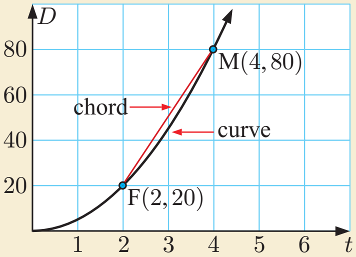
Gottfried Leibniz 1646 – 1716

Discovery 1 — Instantaneous speed

A ball bearing is dropped from the top of a tall building. The distance D it has fallen after t seconds is recorded, and the following graph of distance against time obtained.

We choose a fixed point F on the curve when $t = 2$ seconds. We then choose another point M on the curve, and draw in the line segment or **chord** FM between the two points. To start with, we let M be the point when $t = 4$ seconds.





The *average* speed in the time interval $2 \leqslant t \leqslant 4$

$$= \frac{\text{distance travelled}}{\text{time taken}}$$

$$= \frac{(80 - 20)\text{ m}}{(4 - 2)\text{ s}}$$

$$= \frac{60}{2}\text{ m s}^{-1}$$

$$= 30\text{ m s}^{-1}$$

In this Discovery we will try to measure the *instantaneous* speed of the ball when $t = 2$ seconds.

What to do:

DEMO

1. Click on the icon to start the demonstration.
   F is the point where $t = 2$ seconds, and M is another point on the curve.
   To start with, M is at $t = 4$ seconds.
   The number in the box marked *gradient* is the gradient of the chord FM. This is the *average speed* of the ball bearing in the interval from F to M. For M at $t = 4$ seconds, you should see the average speed is 30 m s^{-1}.
2. Click on M and drag it slowly towards F. Copy and complete the table alongside with the gradient of the chord FM for M being the points on the curve at the given varying times t.

t	gradient of FM
3	
2.5	
2.1	
2.01	

3. Observe what happens as M reaches F. Explain why this is so.
4. Now move M to the origin, and then slide it towards F from the left. Copy and complete the table with the gradient of the chord FM for various times t.

t	gradient of FM
0	
1.5	
1.9	
1.99	

5. **a** What can you say about the gradient of FM in the limit as $t \to 2$?
   b What is the instantaneous speed of the ball bearing when $t = 2$ seconds? Explain your answer.

THE TANGENT TO A CURVE

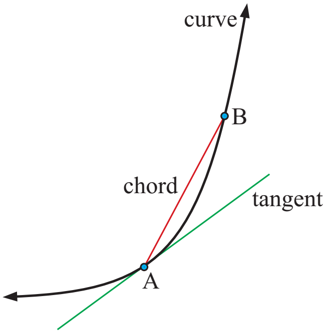
A **chord** of a curve is a straight line segment which joins any two points on the curve.

The gradient of the chord AB measures the average rate of change of the function values for the given change in x-values.

A **tangent** is a straight line which *touches* a curve at a single point. The tangent is the best approximating straight line to the curve through A.

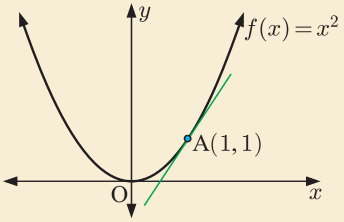
The gradient of the tangent at point A measures the instantaneous rate of change of the function at point A.

As B approaches A, the limit of the gradient of the chord AB will be the gradient of the tangent at A.



The **gradient of the tangent** to $y = f(x)$ at $x = a$ is the **instantaneous rate of change** in $f(x)$ with respect to x at that point.

Discovery 2 — The gradient of a tangent



Given a curve $f(x)$, we wish to find the gradient of the tangent at the point $(a, f(a))$.

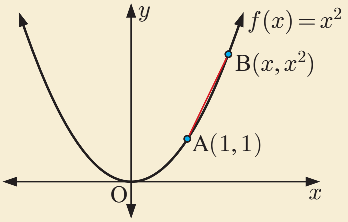
In this Discovery we find the gradient of the tangent to $f(x) = x^2$ at the point A(1, 1).

DEMO

What to do:

1 Suppose B lies on $f(x) = x^2$, and B has coordinates (x, x^2).

a Show that the chord AB has gradient $\dfrac{x^2 - 1}{x - 1}$.



x	Point B	gradient of AB
5	(5, 25)	6
3		
2		
1.5		
1.1		
1.01		
1.001		

b Copy and complete the table shown.

c Comment on the gradient of AB as x gets closer to 1.

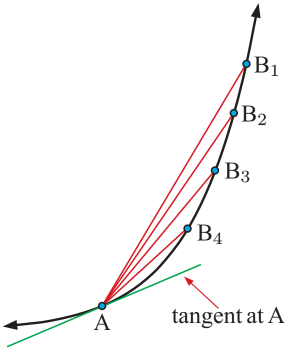
2 Repeat the process letting x get closer to 1, but from the left of A. Use the points where $x = 0$, 0.8, 0.9, 0.99, and 0.999.

3 Click on the icon to view a demonstration of the process.

4 What do you suspect is the gradient of the tangent at A?

Fortunately we do not have to use a graph and table of values each time we wish to find the gradient of a tangent. Instead we can use an algebraic and geometric approach which involves **limits**.

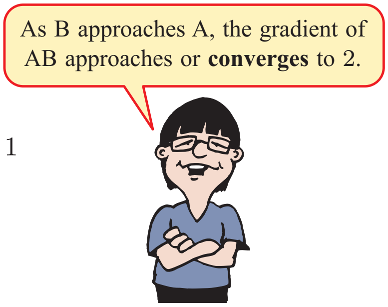
From **Discovery 2**, the gradient of AB $= \dfrac{x^2 - 1}{x - 1}$.



As B approaches A, $x \to 1$ and the gradient of AB → the gradient of the tangent at A.

So, the gradient of the tangent at the point A is

$$\begin{aligned} m_T &= \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \to 1} \frac{(x+1)(x-1)}{x - 1} \\ &= \lim_{x \to 1} (x + 1) \quad \text{since } x \neq 1 \\ &= 2 \end{aligned}$$



EXERCISE 13B

1 Use the method in **Discovery 1** to answer the **Opening Problem** on page **334**.

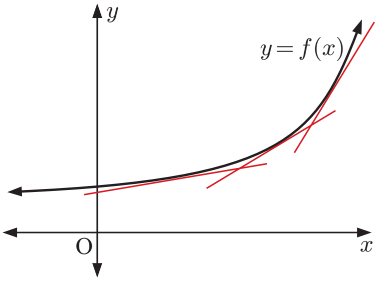
2 **a** Use the method in **Discovery 2** to find the gradient of the tangent to $y = x^2$ at the point $(2, 4)$.

b Evaluate $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$, and provide a geometric interpretation of this result.

C THE DERIVATIVE FUNCTION

For a non-linear function with equation $y = f(x)$, the gradients of the tangents at various points are different.

Our task is to determine a **gradient function** which gives the gradient of the tangent to $y = f(x)$ at $x = a$, for any point a in the domain of f.



The gradient function of $y = f(x)$ is called its **derivative function** and is labelled $f'(x)$.

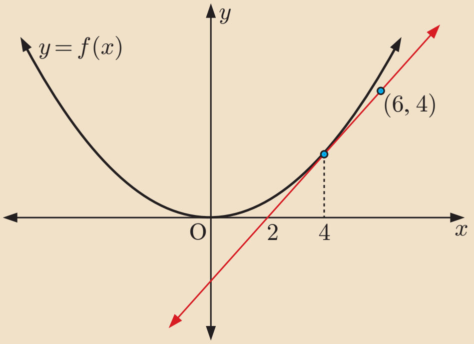
We read the derivative function as "eff dashed x".

The value of $f'(a)$ is the gradient of the tangent to $y = f(x)$ at the point where $x = a$.

Example 2

Self Tutor

For the given graph, find $f'(4)$ and $f(4)$.



The graph shows the tangent to the curve $y = f(x)$ at the point where $x = 4$.

The tangent passes through $(2, 0)$ and $(6, 4)$, so its gradient is $f'(4) = \dfrac{4-0}{6-2} = 1$.

The equation of the tangent is $y - 0 = 1(x - 2)$

$\therefore \quad y = x - 2$

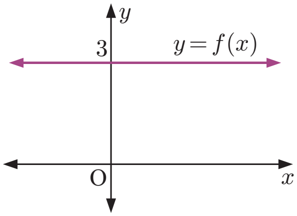
When $x = 4$, $y = 2$, so the point of contact between the tangent and the curve is $(4, 2)$.

$\therefore \quad f(4) = 2$

EXERCISE 13C

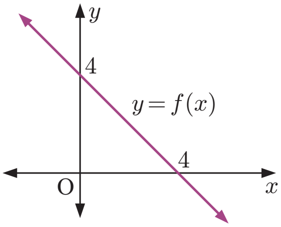
1 Using the graph below, find:

a $f(2)$ **b** $f'(2)$

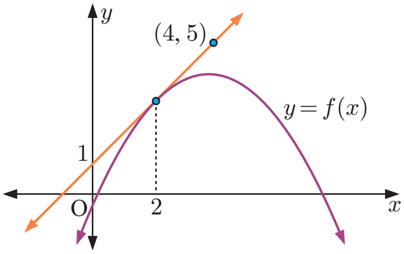


2 Using the graph below, find:

a $f(0)$ **b** $f'(0)$



3 Consider the graph alongside. Find $f(2)$ and $f'(2)$.



Discovery 3 Gradient functions

The software on the CD can be used to find the gradient of the tangent to a function $f(x)$ at any point. By sliding the point along the graph we can observe the changing gradient of the tangent. We can hence generate the gradient function $f'(x)$.

GRADIENT FUNCTIONS

What to do:

1 Consider the functions $f(x) = 0$, $f(x) = 2$, and $f(x) = 4$.
  - **a** For each of these functions, what is the gradient?
  - **b** Is the gradient constant for all values of x?

2 Consider the function $f(x) = mx + c$.
  - **a** State the gradient of the function.
  - **b** Is the gradient constant for all values of x?
  - **c** Use the CD software to graph the following functions and observe the gradient function $f'(x)$. Hence verify that your answer in **b** is correct.
    - **i** $f(x) = x - 1$
    - **ii** $f(x) = 3x + 2$
    - **iii** $f(x) = -2x + 1$

3
  - **a** Observe the function $f(x) = x^2$ using the CD software. What *type* of function is the gradient function $f'(x)$?
  - **b** Observe the following quadratic functions using the CD software:
    - **i** $f(x) = x^2 + x - 2$
    - **ii** $f(x) = 2x^2 - 3$
    - **iii** $f(x) = -x^2 + 2x - 1$
    - **iv** $f(x) = -3x^2 - 3x + 6$
  - **c** What *type* of function is each of the gradient functions $f'(x)$ in **b**?

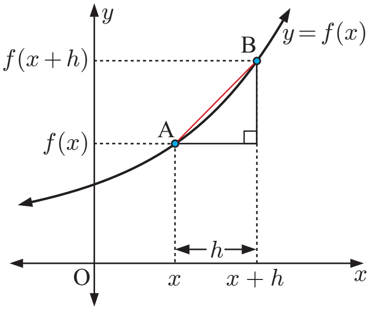
4
  - **a** Observe the function $f(x) = \ln x$ using the CD software.
  - **b** What *type* of function is the gradient function $f'(x)$?
  - **c** What is the *domain* of the gradient function $f'(x)$?

5 **a** Observe the function $f(x) = e^x$ using the CD software.

b What is the gradient function $f'(x)$?

D DIFFERENTIATION FROM FIRST PRINCIPLES

Consider a general function $y = f(x)$ where A is the point $(x, f(x))$ and B is the point $(x+h, f(x+h))$.



The chord AB has gradient $= \dfrac{f(x+h)-f(x)}{x+h-x}$

$= \dfrac{f(x+h)-f(x)}{h}$

If we let B approach A, then the gradient of AB approaches the gradient of the tangent at A.

So, the gradient of the tangent at the variable point $(x, f(x))$ is $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$.

This formula gives the gradient of the tangent to the curve $y = f(x)$ at the point $(x, f(x))$ for any value of x for which this limit exists. Since there is at most one value of the gradient for each value of x, the formula is actually a function.

The **derivative function** or simply **derivative** of $y = f(x)$ is defined as

$$\boldsymbol{f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}}$$

When we evaluate this limit to find a derivative function, we say we are **differentiating from first principles**.

Example 3

Self Tutor

Use the definition of $f'(x)$ to find the gradient function of $f(x) = x^2$.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}_1} \\
&= \lim_{h\to 0} (2x+h) \qquad \{\text{as } h \neq 0\} \\
&= 2x
\end{aligned}$$

ALTERNATIVE NOTATION

If we are given a function $f(x)$ then $f'(x)$ represents the derivative function.

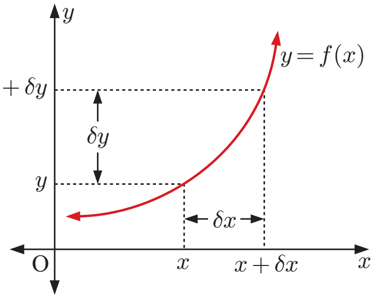
If we are given y in terms of x then y' or $\dfrac{dy}{dx}$ are commonly used to represent the derivative.

$\dfrac{dy}{dx}$ reads "dee y by dee x" or "the derivative of y with respect to x".

$\dfrac{dy}{dx}$ is **not a fraction**. However, the notation $\dfrac{dy}{dx}$ is a result of taking the limit of a fraction. If we replace h by δx and $f(x+h)-f(x)$ by δy, then

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \quad \text{becomes}$$

$$\begin{aligned} f'(x) &= \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \\ &= \frac{dy}{dx}. \end{aligned}$$



THE DERIVATIVE WHEN $x = a$

The gradient of the tangent to $y = f(x)$ at the point where $x = a$ is denoted $f'(a)$, where

$$\boldsymbol{f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}$$

Example 4

Self Tutor

Use the first principles formula $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h)-f(a)}{h}$ to find the instantaneous rate of change in $f(x) = x^2 + 2x$ at the point where $x = 5$.

$$f(5) = 5^2 + 2(5) = 35$$

$$\begin{aligned} \therefore \quad f'(5) &= \lim_{h \to 0} \frac{f(5+h)-f(5)}{h} \\ &= \lim_{h \to 0} \frac{(5+h)^2 + 2(5+h) - 35}{h} \\ \therefore \quad f'(5) &= \lim_{h \to 0} \frac{\cancel{25} + 10h + h^2 + \cancel{10} + 2h - \cancel{35}}{h} \\ &= \lim_{h \to 0} \frac{h^2 + 12h}{h} \\ &= \lim_{h \to 0} \frac{\cancel{h}(h+12)}{\cancel{h}_1} \qquad \{\text{as } h \neq 0\} \\ &= 12 \end{aligned}$$

$\therefore$ the instantaneous rate of change in $f(x)$ at $x = 5$ is 12.

EXERCISE 13D

1 Find, from first principles, the gradient function of:

a $f(x) = x$ **b** $f(x) = 5$ **c** $f(x) = 2x + 5$

2 Find $\dfrac{dy}{dx}$ from first principles given:

a $y = 4 - x$ **b** $y = x^2 - 3x$ **c** $y = 2x^2 + x - 1$

3 Use the first principles formula $f'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ to find the gradient of the tangent to:

a $f(x) = 3x + 5$ at $x = -2$ **b** $f(x) = 5 - 2x^2$ at $x = 3$

c $f(x) = x^2 + 3x - 4$ at $x = 3$ **d** $f(x) = 5 - 2x - 3x^2$ at $x = -2$

E SIMPLE RULES OF DIFFERENTIATION

Differentiation is the process of finding a derivative or gradient function.

Given a function $f(x)$, we obtain $f'(x)$ by **differentiating with respect to** the variable x.

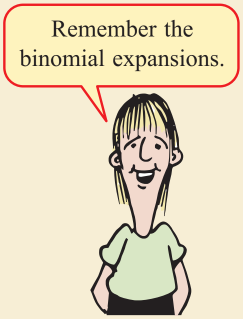
There are a number of rules associated with differentiation. These rules can be used to differentiate more complicated functions without having to use first principles.

Discovery 4 Simple rules of differentiation

In this Discovery we attempt to differentiate functions of the form x^n, cx^n where c is a constant, and functions which are a sum or difference of polynomial terms of the form cx^n.

What to do:

1 Differentiate from first principles: **a** x^2 **b** x^3 **c** x^4



2 Consider the binomial expansion:

$$(x+h)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \ldots. + \binom{n}{n} h^n$$
$$= x^n + nx^{n-1}h + \binom{n}{2} x^{n-2}h^2 + \ldots. + h^n$$

Use the first principles formula $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

to find the derivative of $f(x) = x^n$ for $x \in \mathbb{Z}^+$.

3 **a** Find, from first principles, the derivatives of: **i** $4x^2$ **ii** $2x^3$

b By comparison with **1**, copy and complete: "If $f(x) = cx^n$, then $f'(x) = \ldots\ldots$"

4 **a** Use first principles to find $f'(x)$ for:

i $f(x) = x^2 + 3x$ **ii** $f(x) = x^3 - 2x^2$

b Copy and complete: "If $f(x) = u(x) + v(x)$ then $f'(x) = \ldots\ldots$"

The rules you found in the **Discovery** are much more general than the cases you just considered.

For example, if $f(x) = x^n$ then $f'(x) = nx^{n-1}$ is true not just for all $n \in \mathbb{Z}^+$, but actually for all $n \in \mathbb{R}$.

We can summarise the following rules:

$f(x)$	$f'(x)$	*Name of rule*
c (a constant)	0	**differentiating a constant**
x^n	nx^{n-1}	**differentiating x^n**
$c\,u(x)$	$c\,u'(x)$	**constant times a function**
$u(x) + v(x)$	$u'(x) + v'(x)$	**addition rule**

The last two rules can be proved using the first principles definition of $f'(x)$.

- If $f(x) = c\,u(x)$ where c is a constant, then $f'(x) = c\,u'(x)$.

Proof:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{c\,u(x+h) - c\,u(x)}{h} \\ &= \lim_{h \to 0} c \left[\frac{u(x+h) - u(x)}{h} \right] \\ &= c \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \\ &= c\,u'(x) \end{aligned}$$

- If $f(x) = u(x) + v(x)$ then $f'(x) = u'(x) + v'(x)$

Proof:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \left(\frac{u(x+h) + v(x+h) - [u(x) + v(x)]}{h} \right) \\ &= \lim_{h \to 0} \left(\frac{u(x+h) - u(x) + v(x+h) - v(x)}{h} \right) \\ &= \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \to 0} \frac{v(x+h) - v(x)}{h} \\ &= u'(x) + v'(x) \end{aligned}$$

Using the rules we have now developed we can differentiate sums of powers of x.

For example, if $f(x) = 3x^4 + 2x^3 - 5x^2 + 7x + 6$ then

$$\begin{aligned} f'(x) &= 3(4x^3) + 2(3x^2) - 5(2x) + 7(1) + 0 \\ &= 12x^3 + 6x^2 - 10x + 7 \end{aligned}$$

Example 5

Self Tutor

If $y = 3x^2 - 4x$, find $\frac{dy}{dx}$ and interpret its meaning.

As $y = 3x^2 - 4x$, $\frac{dy}{dx} = 6x - 4$.

$\frac{dy}{dx}$ is:

- the gradient function or derivative of $y = 3x^2 - 4x$ from which the gradient of the tangent at any point on the curve can be found
- the instantaneous rate of change of y with respect to x.

Example 6

Self Tutor

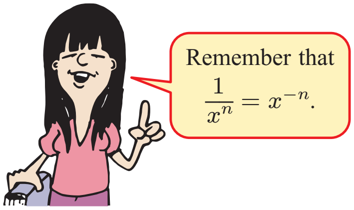
Find $f'(x)$ for $f(x)$ equal to:

a $5x^3 + 6x^2 - 3x + 2$

b $7x - \frac{4}{x} + \frac{3}{x^3}$

a
$$\begin{aligned} f(x) &= 5x^3 + 6x^2 - 3x + 2 \\ \therefore\ f'(x) &= 5(3x^2) + 6(2x) - 3(1) \\ &= 15x^2 + 12x - 3 \end{aligned}$$

b
$$\begin{aligned} f(x) &= 7x - \frac{4}{x} + \frac{3}{x^3} \\ &= 7x - 4x^{-1} + 3x^{-3} \\ \therefore\ f'(x) &= 7(1) - 4(-1x^{-2}) + 3(-3x^{-4}) \\ &= 7 + 4x^{-2} - 9x^{-4} \\ &= 7 + \frac{4}{x^2} - \frac{9}{x^4} \end{aligned}$$



Example 7

Self Tutor

Find the gradient function of $y = x^2 - \frac{4}{x}$ and hence find the gradient of the tangent to the function at the point where $x = 2$.

$$\begin{aligned} y &= x^2 - \frac{4}{x} \\ &= x^2 - 4x^{-1} \end{aligned}$$

$$\begin{aligned} \therefore\ \frac{dy}{dx} &= 2x - 4(-1x^{-2}) \\ &= 2x + 4x^{-2} \\ &= 2x + \frac{4}{x^2} \end{aligned}$$

When $x = 2$, $\frac{dy}{dx} = 4 + 1 = 5$.

So, the tangent has gradient 5.

Example 8

Self Tutor

Find the gradient function for each of the following:

a $f(x)=3\sqrt{x}+\dfrac{2}{x}$ **b** $g(x)=x^2-\dfrac{4}{\sqrt{x}}$

a
$$\begin{aligned} f(x)&=3\sqrt{x}+\frac{2}{x}\\ &=3x^{\frac{1}{2}}+2x^{-1}\\ \therefore\ f'(x)&=3(\tfrac{1}{2}x^{-\frac{1}{2}})+2(-1x^{-2})\\ &=\tfrac{3}{2}x^{-\frac{1}{2}}-2x^{-2}\\ &=\frac{3}{2\sqrt{x}}-\frac{2}{x^2}\end{aligned}$$

b
$$\begin{aligned} g(x)&=x^2-\frac{4}{\sqrt{x}}\\ &=x^2-4x^{-\frac{1}{2}}\\ \therefore\ g'(x)&=2x-4(-\tfrac{1}{2}x^{-\frac{3}{2}})\\ &=2x+2x^{-\frac{3}{2}}\\ &=2x+\frac{2}{x\sqrt{x}}\end{aligned}$$

EXERCISE 13E

1 Find $f'(x)$ given that $f(x)$ is:

a x^3 **b** $2x^3$ **c** $7x^2$ **d** $6\sqrt{x}$

e $3\sqrt[3]{x}$ **f** x^2+x **g** $4-2x^2$ **h** x^2+3x-5

i $\frac{1}{2}x^4-6x^2$ **j** $\dfrac{3x-6}{x}$ **k** $\dfrac{2x-3}{x^2}$ **l** $\dfrac{x^3+5}{x}$

m $\dfrac{x^3+x-3}{x}$ **n** $\dfrac{1}{\sqrt{x}}$ **o** $(2x-1)^2$ **p** $(x+2)^3$

2 Find $\dfrac{dy}{dx}$ for:

a $y=2.5x^3-1.4x^2-1.3$ **b** $y=\pi x^2$ **c** $y=\dfrac{1}{5x^2}$

d $y=100x$ **e** $y=10(x+1)$ **f** $y=4\pi x^3$

3 Differentiate with respect to x:

a $6x+2$ **b** $x\sqrt{x}$ **c** $(5-x)^2$ **d** $\dfrac{6x^2-9x^4}{3x}$

e $(x+1)(x-2)$ **f** $\dfrac{1}{x^2}+6\sqrt{x}$ **g** $4x-\dfrac{1}{4x}$ **h** $x(x+1)(2x-5)$

4 Find the gradient of the tangent to:

a $y=x^2$ at $x=2$ **b** $y=\dfrac{8}{x^2}$ at the point $(9,\ \frac{8}{81})$

c $y=2x^2-3x+7$ at $x=-1$ **d** $y=\dfrac{2x^2-5}{x}$ at the point $(2,\ \frac{3}{2})$

e $y=\dfrac{x^2-4}{x^2}$ at the point $(4,\ \frac{3}{4})$ **f** $y=\dfrac{x^3-4x-8}{x^2}$ at $x=-1$.

5 Suppose $f(x)=x^2+(b+1)x+2c$, $f(2)=4$, and $f'(-1)=2$.
Find the constants b and c.

6 Find the gradient function of:

a $f(x)=4\sqrt{x}+x$ **b** $f(x)=\sqrt[3]{x}$ **c** $f(x)=-\dfrac{2}{\sqrt{x}}$

d $f(x)=2x-\sqrt{x}$ **e** $f(x)=\dfrac{4}{\sqrt{x}}-5$ **f** $f(x)=3x^2-x\sqrt{x}$

g $f(x)=\dfrac{5}{x^2\sqrt{x}}$ **h** $f(x)=2x-\dfrac{3}{x\sqrt{x}}$

7 **a** If $y=4x-\dfrac{3}{x}$, find $\dfrac{dy}{dx}$ and interpret its meaning.

b The position of a car moving along a straight road is given by $S=2t^2+4t$ metres where t is the time in seconds. Find $\dfrac{dS}{dt}$ and interpret its meaning.

c The cost of producing x toasters each week is given by $C=1785+3x+0.002x^2$ dollars. Find $\dfrac{dC}{dx}$ and interpret its meaning.

F THE CHAIN RULE

In **Chapter 2** we defined the **composite** of two functions g and f as $(g\circ f)(x)$ or $gf(x)$.

We can often write complicated functions as the composite of two or more simpler functions.

For example $y=(x^2+3x)^4$ could be rewritten as $y=u^4$ where $u=x^2+3x$, or as $y=gf(x)$ where $g(x)=x^4$ and $f(x)=x^2+3x$.

Example 9 Self Tutor

Find: **a** $gf(x)$ if $g(x)=\sqrt{x}$ and $f(x)=2-3x$

b $g(x)$ and $f(x)$ such that $gf(x)=\dfrac{1}{x-x^2}$.

a $gf(x)$
$=g(2-3x)$
$=\sqrt{2-3x}$

b $gf(x)=\dfrac{1}{x-x^2}=\dfrac{1}{f(x)}$

$\therefore\ g(x)=\dfrac{1}{x}$ and $f(x)=x-x^2$

There are several possible answers for **b**.

EXERCISE 13F.1

1 Find $gf(x)$ if:

a $g(x)=x^2$ and $f(x)=2x+7$ **b** $g(x)=2x+7$ and $f(x)=x^2$

c $g(x)=\sqrt{x}$ and $f(x)=3-4x$ **d** $g(x)=3-4x$ and $f(x)=\sqrt{x}$

e $g(x)=\dfrac{2}{x}$ and $f(x)=x^2+3$ **f** $g(x)=x^2+3$ and $f(x)=\dfrac{2}{x}$

2 Find $g(x)$ and $f(x)$ such that $gf(x)$ is:

a $(3x+10)^3$ **b** $\dfrac{1}{2x+4}$ **c** $\sqrt{x^2-3x}$ **d** $\dfrac{10}{(3x-x^2)^3}$

DERIVATIVES OF COMPOSITE FUNCTIONS

The reason we are interested in writing complicated functions as composite functions is to make finding derivatives easier.

Discovery 5 — Differentiating composite functions

The purpose of this Discovery is to learn how to differentiate composite functions.

Based on the rule "if $y = x^n$ then $\frac{dy}{dx} = nx^{n-1}$", we might suspect that if $y = (2x+1)^2$ then $\frac{dy}{dx} = 2(2x+1)^1$. But is this so?

What to do:

1 Expand $y = (2x+1)^2$ and hence find $\frac{dy}{dx}$. How does this compare with $2(2x+1)^1$?

2 Expand $y = (3x+1)^2$ and hence find $\frac{dy}{dx}$. How does this compare with $2(3x+1)^1$?

3 Expand $y = (ax+1)^2$ where a is a constant, and hence find $\frac{dy}{dx}$. How does this compare with $2(ax+1)^1$?

4 Suppose $y = u^2$.

- **a** Find $\frac{dy}{du}$.
- **b** Now suppose $u = ax+1$, so $y = (ax+1)^2$.
  - **i** Find $\frac{du}{dx}$.
  - **ii** Write $\frac{dy}{du}$ from **a** in terms of x.
  - **iii** Hence find $\frac{dy}{du} \times \frac{du}{dx}$.
  - **iv** Compare your answer to the result in **3**.
- **c** If $y = u^2$ where u is a function of x, what do you suspect $\frac{dy}{dx}$ will be equal to?

5 Expand $y = (x^2+3x)^2$ and hence find $\frac{dy}{dx}$.

Does your answer agree with the rule you suggested in **4 c**?

6 Consider $y = (2x+1)^3$.

- **a** Expand the brackets and hence find $\frac{dy}{dx}$.
- **b** If we let $u = 2x+1$, then $y = u^3$.
  - **i** Find $\frac{du}{dx}$.
  - **ii** Find $\frac{dy}{du}$, and write it in terms of x.
  - **iii** Hence find $\frac{dy}{du} \times \frac{du}{dx}$.
  - **iv** Compare your answer to the result in **a**.

7 Copy and complete: "If y is a function of u, and u is a function of x, then $\frac{dy}{dx} = ...$"

THE CHAIN RULE

If $y = g(u)$ where $u = f(x)$ then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$.

This rule is extremely important and enables us to differentiate complicated functions much faster.

For example, for any function $f(x)$:

If $y = [f(x)]^n$ then $\dfrac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$.

Example 10

Self Tutor

Find $\dfrac{dy}{dx}$ if:

a $y = (x^2 - 2x)^4$

b $y = \dfrac{4}{\sqrt{1-2x}}$

a

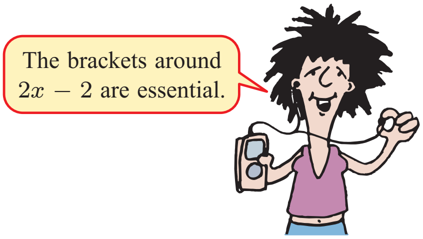
$$y = (x^2 - 2x)^4$$

$$\therefore\ y = u^4 \quad \text{where} \quad u = x^2 - 2x$$

$$\text{Now}\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \{\text{chain rule}\}$$

$$= 4u^3(2x-2)$$

$$= 4(x^2-2x)^3(2x-2)$$



b

$$y = \frac{4}{\sqrt{1-2x}}$$

$$\therefore\ y = 4u^{-\frac{1}{2}} \quad \text{where} \quad u = 1 - 2x$$

$$\text{Now}\ \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \{\text{chain rule}\}$$

$$= 4 \times (-\tfrac{1}{2}u^{-\frac{3}{2}}) \times (-2)$$

$$= 4u^{-\frac{3}{2}}$$

$$= 4(1-2x)^{-\frac{3}{2}}$$

EXERCISE 13F.2

1 Write in the form au^n, clearly stating what u is:

a $\dfrac{1}{(2x-1)^2}$

b $\sqrt{x^2 - 3x}$

c $\dfrac{2}{\sqrt{2-x^2}}$

d $\sqrt[3]{x^3 - x^2}$

e $\dfrac{4}{(3-x)^3}$

f $\dfrac{10}{x^2 - 3}$

2 Find the gradient function $\dfrac{dy}{dx}$ for:

a $y = (4x-5)^2$

b $y = \dfrac{1}{5-2x}$

c $y = \sqrt{3x - x^2}$

d $y = (1-3x)^4$

e $y = 6(5-x)^3$

f $y = \sqrt[3]{2x^3 - x^2}$

g $y = \dfrac{6}{(5x-4)^2}$

h $y = \dfrac{4}{3x - x^2}$

i $y = 2\left(x^2 - \dfrac{2}{x}\right)^3$

3 Find the gradient of the tangent to:

a $y = \sqrt{1-x^2}$ at $x = \frac{1}{2}$

b $y = (3x+2)^6$ at $x = -1$

c $y = \dfrac{1}{(2x-1)^4}$ at $x = 1$

d $y = 6 \times \sqrt[3]{1-2x}$ at $x = 0$

e $y = \dfrac{4}{x + 2\sqrt{x}}$ at $x = 4$

f $y = \left(x + \dfrac{1}{x}\right)^3$ at $x = 1$.

4 The gradient function of $f(x) = (2x-b)^a$ is $f'(x) = 24x^2 - 24x + 6$.
Find the constants a and b.

5 Suppose $y = \dfrac{a}{\sqrt{1+bx}}$ where a and b are constants. When $x = 3$, $y = 1$ and $\dfrac{dy}{dx} = -\frac{1}{8}$.
Find a and b.

6 If $y = x^3$ then $x = y^{\frac{1}{3}}$.

a Find $\dfrac{dy}{dx}$ and $\dfrac{dx}{dy}$, and hence show that $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$.

b Explain why $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = 1$ whenever these derivatives exist for any general function $y = f(x)$.

G THE PRODUCT RULE

We have seen the addition rule:

$$\text{If} \quad f(x) = u(x) + v(x) \quad \text{then} \quad f'(x) = u'(x) + v'(x).$$

We now consider the case $f(x) = u(x)\,v(x)$. Is $f'(x) = u'(x)\,v'(x)$?

In other words, does the derivative of a product of two functions equal the product of the derivatives of the two functions?

Discovery 6 The product rule

Suppose $u(x)$ and $v(x)$ are two functions of x, and that $f(x) = u(x)\,v(x)$ is the product of these functions.

The purpose of this Discovery is to find a rule for determining $f'(x)$.

What to do:

1 Suppose $u(x) = x$ and $v(x) = x$, so $f(x) = x^2$.

a Find $f'(x)$ by direct differentiation.

b Find $u'(x)$ and $v'(x)$.

c Does $f'(x) = u'(x)\,v'(x)$?

2 Suppose $u(x) = x$ and $v(x) = \sqrt{x}$, so $f(x) = x\sqrt{x} = x^{\frac{3}{2}}$.

a Find $f'(x)$ by direct differentiation.

b Find $u'(x)$ and $v'(x)$.

c Does $f'(x) = u'(x)\,v'(x)$?

3 Copy and complete the following table, finding $f'(x)$ by direct differentiation.

$f(x)$	$f'(x)$	$u(x)$	$v(x)$	$u'(x)$	$v'(x)$	$u'(x)\,v(x)+u(x)\,v'(x)$
x^2		x	x			
$x^{\frac{3}{2}}$		x	$\sqrt{x}$			
$x(x+1)$		x	$x+1$			
$(x-1)(2-x^2)$		$x-1$	$2-x^2$			

4 Copy and complete: "If $f(x)=u(x)\,v(x)$ then $f'(x)=$ ......"

THE PRODUCT RULE

If $\boldsymbol{f(x)=u(x)\,v(x)}$ then $\boldsymbol{f'(x)=u'(x)\,v(x)+u(x)\,v'(x)}$.

Alternatively, if $\boldsymbol{y=uv}$ where u and v are functions of x, then $\boldsymbol{\dfrac{dy}{dx}=\dfrac{du}{dx}v+u\dfrac{dv}{dx}}$.

Example 11

Self Tutor

Find $\dfrac{dy}{dx}$ if:

a $y=\sqrt{x}(2x+1)^3$

b $y=x^2(x^2-2x)^4$

a $y=\sqrt{x}(2x+1)^3$ is the product of $u=x^{\frac{1}{2}}$ and $v=(2x+1)^3$

$\therefore\quad u'=\frac{1}{2}x^{-\frac{1}{2}}$ and $v'=3(2x+1)^2\times 2$ {chain rule}

$= 6(2x+1)^2$

$$\text{Now}\quad \frac{dy}{dx}=u'v+uv' \qquad \{\text{product rule}\}$$
$$=\tfrac{1}{2}x^{-\frac{1}{2}}(2x+1)^3+x^{\frac{1}{2}}\times 6(2x+1)^2$$
$$=\tfrac{1}{2}x^{-\frac{1}{2}}(2x+1)^3+6x^{\frac{1}{2}}(2x+1)^2$$

b $y=x^2(x^2-2x)^4$ is the product of $u=x^2$ and $v=(x^2-2x)^4$

$\therefore\quad u'=2x$ and $v'=4(x^2-2x)^3(2x-2)$ {chain rule}

$$\text{Now}\quad \frac{dy}{dx}=u'v+uv' \qquad \{\text{product rule}\}$$
$$=2x(x^2-2x)^4+x^2\times 4(x^2-2x)^3(2x-2)$$
$$=2x(x^2-2x)^4+4x^2(x^2-2x)^3(2x-2)$$

EXERCISE 13G

1 Use the product rule to differentiate:

a $f(x)=x(x-1)$

b $f(x)=2x(x+1)$

c $f(x)=x^2\sqrt{x+1}$

2 Find $\frac{dy}{dx}$ using the product rule:

a $y = x^2(2x - 1)$
b $y = 4x(2x + 1)^3$
c $y = x^2\sqrt{3 - x}$
d $y = \sqrt{x}(x - 3)^2$
e $y = 5x^2(3x^2 - 1)^2$
f $y = \sqrt{x}(x - x^2)^3$

3 Find the gradient of the tangent to:

a $y = x^4(1 - 2x)^2$ at $x = -1$
b $y = \sqrt{x}(x^2 - x + 1)^2$ at $x = 4$
c $y = x\sqrt{1 - 2x}$ at $x = -4$
d $y = x^3\sqrt{5 - x^2}$ at $x = 1$.

4 Consider $y = \sqrt{x}(3 - x)^2$.

a Show that $\frac{dy}{dx} = \frac{(3 - x)(3 - 5x)}{2\sqrt{x}}$.
b Find the x-coordinates of all points on $y = \sqrt{x}(3 - x)^2$ where the tangent is horizontal.
c For what values of x is $\frac{dy}{dx}$ undefined?

5 Suppose $y = -2x^2(x + 4)$. For what values of x does $\frac{dy}{dx} = 10$?

H THE QUOTIENT RULE

Expressions like $\frac{x^2 + 1}{2x - 5}$, $\frac{\sqrt{x}}{1 - 3x}$, and $\frac{x^3}{(x - x^2)^4}$ are called **quotients** because they represent the division of one function by another.

Quotient functions have the form $Q(x) = \frac{u(x)}{v(x)}$.

$$\begin{aligned} \text{Notice that} \quad u(x) &= Q(x)\,v(x) \\ \therefore \quad u'(x) &= Q'(x)\,v(x) + Q(x)\,v'(x) \qquad \{\text{product rule}\} \\ \therefore \quad u'(x) - Q(x)\,v'(x) &= Q'(x)\,v(x) \\ \therefore \quad Q'(x)\,v(x) &= u'(x) - \frac{u(x)}{v(x)}\,v'(x) \\ \therefore \quad Q'(x)\,v(x) &= \frac{u'(x)\,v(x) - u(x)\,v'(x)}{v(x)} \\ \therefore \quad Q'(x) &= \frac{u'(x)\,v(x) - u(x)\,v'(x)}{[v(x)]^2} \qquad \text{when this exists.} \end{aligned}$$

THE QUOTIENT RULE

If $Q(x) = \frac{u(x)}{v(x)}$ then $Q'(x) = \frac{u'(x)\,v(x) - u(x)\,v'(x)}{[v(x)]^2}$.

Alternatively, if $y = \frac{u}{v}$ where u and v are functions of x, then $\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$.

Example 12

Self Tutor

Use the quotient rule to find $\frac{dy}{dx}$ if:

a $y = \frac{1+3x}{x^2+1}$

b $y = \frac{\sqrt{x}}{(1-2x)^2}$

a $y = \frac{1+3x}{x^2+1}$ is a quotient with $u = 1+3x$ and $v = x^2+1$

$\therefore$ $u' = 3$ and $v' = 2x$

Now
$$\begin{aligned}\frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \qquad \{\text{quotient rule}\}\\ &= \frac{3(x^2+1) - (1+3x)2x}{(x^2+1)^2}\\ &= \frac{3x^2+3-2x-6x^2}{(x^2+1)^2}\\ &= \frac{3-2x-3x^2}{(x^2+1)^2}\end{aligned}$$

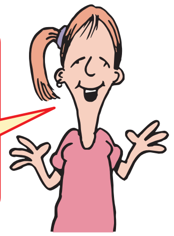
b $y = \frac{\sqrt{x}}{(1-2x)^2}$ is a quotient with $u = x^{\frac{1}{2}}$ and $v = (1-2x)^2$

$\therefore$ $u' = \frac{1}{2}x^{-\frac{1}{2}}$ and $v' = 2(1-2x)^1 \times (-2)$ {chain rule}

$= -4(1-2x)$

Now
$$\begin{aligned}\frac{dy}{dx} &= \frac{u'v - uv'}{v^2} \qquad \{\text{quotient rule}\}\\ &= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 - x^{\frac{1}{2}} \times (-4(1-2x))}{(1-2x)^4}\\ &= \frac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x)^2 + 4x^{\frac{1}{2}}(1-2x)}{(1-2x)^4}\\ &= \frac{\cancel{(1-2x)}\left[\frac{1-2x}{2\sqrt{x}} + 4\sqrt{x}\left(\frac{2\sqrt{x}}{2\sqrt{x}}\right)\right]}{(1-2x)^{\cancel{4}3}} \qquad \{\text{look for common factors}\}\\ &= \frac{1-2x+8x}{2\sqrt{x}(1-2x)^3}\\ &= \frac{6x+1}{2\sqrt{x}(1-2x)^3}\end{aligned}$$

Simplification of $\frac{dy}{dx}$ is often unnecessary, especially if you simply want the gradient of a tangent at a given point. In such cases, substitute a value for x without simplifying the derivative function first.



EXERCISE 13H

1 Use the quotient rule to find $\dfrac{dy}{dx}$ if:

a $y = \dfrac{1+3x}{2-x}$ **b** $y = \dfrac{x^2}{2x+1}$ **c** $y = \dfrac{x}{x^2-3}$

d $y = \dfrac{\sqrt{x}}{1-2x}$ **e** $y = \dfrac{x^2-3}{3x-x^2}$ **f** $y = \dfrac{x}{\sqrt{1-3x}}$

2 Find the gradient of the tangent to:

a $y = \dfrac{x}{1-2x}$ at $x = 1$ **b** $y = \dfrac{x^3}{x^2+1}$ at $x = -1$

c $y = \dfrac{\sqrt{x}}{2x+1}$ at $x = 4$ **d** $y = \dfrac{x^2}{\sqrt{x^2+5}}$ at $x = -2$.

3 **a** If $y = \dfrac{2\sqrt{x}}{1-x}$, show that $\dfrac{dy}{dx} = \dfrac{x+1}{\sqrt{x}(1-x)^2}$.

b For what values of x is $\dfrac{dy}{dx}$ **i** zero **ii** undefined?

4 **a** If $y = \dfrac{x^2-3x+1}{x+2}$, show that $\dfrac{dy}{dx} = \dfrac{x^2+4x-7}{(x+2)^2}$.

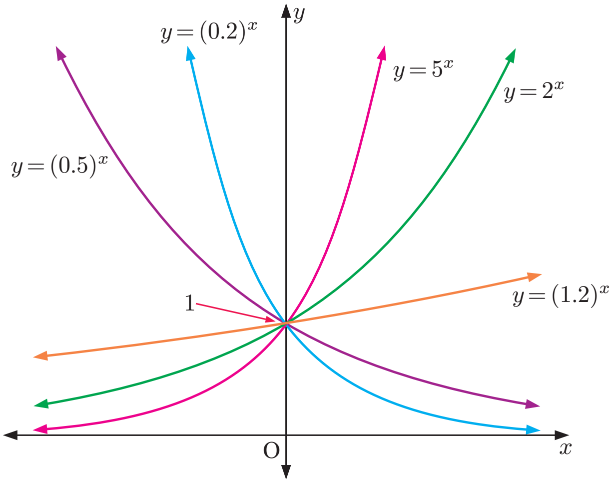
b For what values of x is $\dfrac{dy}{dx}$ **i** zero **ii** undefined?

I DERIVATIVES OF EXPONENTIAL FUNCTIONS

In **Chapter 4** we saw that the simplest **exponential functions** have the form $f(x) = b^x$ where b is any positive constant, $b \neq 1$.

The graphs of all members of the exponential family $f(x) = b^x$ have the following properties:

- pass through the point $(0, 1)$
- asymptotic to the x-axis at one end
- lie above the x-axis for all x.



Discovery 7 — The derivative of $y = b^x$

The purpose of this Discovery is to observe the nature of the derivatives of $f(x) = b^x$ for various values of b.

What to do:

CALCULUS DEMO

1 Use the software provided to help fill in the table for $y = 2^x$:

x	y	$\frac{dy}{dx}$	$\frac{dy}{dx} \div y$
0			
0.5			
1			
1.5			
2			

2 Repeat **1** for the following functions:

a $y = 3^x$ **b** $y = 5^x$ **c** $y = (0.5)^x$

3 Use your observations from **1** and **2** to write a statement about the derivative of the general exponential $y = b^x$ for $b > 0$, $b \neq 1$.

From the **Discovery** you should have found that:

If $f(x) = b^x$ then $f'(x) = f'(0) \times b^x$.

Proof:

If $f(x) = b^x$,

$$\text{then}\quad f'(x) = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} \qquad \{\text{first principles definition of the derivative}\}$$

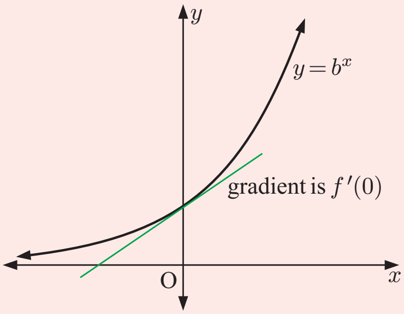
$$= \lim_{h \to 0} \frac{b^x(b^h - 1)}{h}$$

$$= b^x \times \left(\lim_{h \to 0} \frac{b^h - 1}{h} \right) \qquad \{\text{as } b^x \text{ is independent of } h\}$$

$$\text{But}\quad f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{b^h - 1}{h}$$

$$\therefore\quad f'(x) = b^x \times f'(0)$$



Given this result, if we can find a value of b such that $f'(0) = 1$, then we will have found *a function which is its own derivative*!

We have already shown that if $f(x) = b^x$ then $f'(x) = b^x \left(\lim\limits_{h \to 0} \dfrac{b^h - 1}{h} \right)$.

So if $f'(x) = b^x$ then we require $\lim\limits_{h \to 0} \dfrac{b^h - 1}{h} = 1$.

$$\therefore \lim_{h \to 0} b^h = \lim_{h \to 0} (1 + h)$$

Letting $h = \dfrac{1}{n}$, we notice that $\dfrac{1}{n} \to 0$ if $n \to \infty$

$$\therefore \lim_{n \to \infty} b^{\frac{1}{n}} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)$$

$$\therefore b = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \quad \text{if this limit exists}$$

We have in fact already seen this limit in **Chapter 4 Discovery 2** on page **123**.

We found that as $n \to \infty$,

$$\left(1 + \frac{1}{n}\right)^n \to 2.718\,281\,828\,459\,045\,235\ldots.$$

and this irrational number is the natural exponential e.

e^x is sometimes written as $\exp(x)$. For example, $\exp(1 - x) = e^{1-x}$.

We now have: If $f(x) = e^x$ then $f'(x) = e^x$.

THE DERIVATIVE OF $e^{f(x)}$

The functions e^{-x}, e^{2x+3}, and e^{-x^2} all have the form $e^{f(x)}$.

Since $e^x > 0$ for all x, $e^{f(x)} > 0$ for all x, no matter what the function $f(x)$.

Suppose $y = e^{f(x)} = e^u$ where $u = f(x)$.

$$\begin{aligned} \text{Now} \quad \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\} \\ &= e^u \frac{du}{dx} \\ &= e^{f(x)} \times f'(x) \end{aligned}$$

Function	*Derivative*
e^x	e^x
$e^{f(x)}$	$e^{f(x)} \times f'(x)$

Example 13

Self Tutor

Find the gradient function for y equal to:

a $2e^x + e^{-3x}$

b x^2e^{-x}

c $\dfrac{e^{2x}}{x}$

a If $y = 2e^x + e^{-3x}$ then

$$\begin{aligned} \frac{dy}{dx} &= 2e^x + e^{-3x}(-3) \\ &= 2e^x - 3e^{-3x} \end{aligned}$$

b If $y = x^2e^{-x}$ then

$$\begin{aligned} \frac{dy}{dx} &= 2xe^{-x} + x^2e^{-x}(-1) \qquad \{\text{product rule}\} \\ &= 2xe^{-x} - x^2e^{-x} \end{aligned}$$

c If $y = \dfrac{e^{2x}}{x}$ then $\dfrac{dy}{dx} = \dfrac{e^{2x}(2)x - e^{2x}(1)}{x^2}$ {quotient rule}

$$= \frac{e^{2x}(2x-1)}{x^2}$$

Example 14

Self Tutor

Find the gradient function for y equal to: **a** $(e^x - 1)^3$ **b** $\dfrac{1}{\sqrt{2e^{-x}+1}}$

a
$$\begin{aligned} y &= (e^x - 1)^3 \\ &= u^3 \quad \text{where} \quad u = e^x - 1 \\ \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\} \\ &= 3u^2\frac{du}{dx} \\ &= 3(e^x-1)^2 \times e^x \\ &= 3e^x(e^x-1)^2 \end{aligned}$$

b
$$\begin{aligned} y &= (2e^{-x}+1)^{-\frac{1}{2}} \\ &= u^{-\frac{1}{2}} \quad \text{where} \quad u = 2e^{-x}+1 \\ \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\} \\ &= -\tfrac{1}{2}u^{-\frac{3}{2}}\frac{du}{dx} \\ &= -\tfrac{1}{2}(2e^{-x}+1)^{-\frac{3}{2}} \times 2e^{-x}(-1) \\ &= e^{-x}(2e^{-x}+1)^{-\frac{3}{2}} \end{aligned}$$

EXERCISE 13I

1 Find the gradient function for $f(x)$ equal to:

a e^{4x} **b** $e^x + 3$ **c** $\exp(-2x)$ **d** $e^{\frac{x}{2}}$

e $2e^{-\frac{x}{2}}$ **f** $1 - 2e^{-x}$ **g** $4e^{\frac{x}{2}} - 3e^{-x}$ **h** $\dfrac{e^x + e^{-x}}{2}$

i e^{-x^2} **j** $e^{\frac{1}{x}}$ **k** $10(1 + e^{2x})$ **l** $20(1 - e^{-2x})$

m e^{2x+1} **n** $e^{\frac{x}{4}}$ **o** e^{1-2x^2} **p** $e^{-0.02x}$

2 Find the derivative of:

a xe^x **b** x^3e^{-x} **c** $\dfrac{e^x}{x}$ **d** $\dfrac{x}{e^x}$

e x^2e^{3x} **f** $\dfrac{e^x}{\sqrt{x}}$ **g** $\sqrt{x}e^{-x}$ **h** $\dfrac{e^x+2}{e^{-x}+1}$

3 Find the gradient of the tangent to:

a $y = (e^x + 2)^4$ at $x = 0$ **b** $y = \dfrac{1}{2 - e^{-x}}$ at $x = 0$

c $y = \sqrt{e^{2x} + 10}$ at $x = \ln 3$.

4 Given $f(x) = e^{kx} + x$ and $f'(0) = -8$, find k.

5 **a** By substituting $e^{\ln 2}$ for 2 in $y = 2^x$, find $\dfrac{dy}{dx}$.

b Show that if $y = b^x$ where $b > 0$, $b \neq 1$, then $\dfrac{dy}{dx} = b^x \times \ln b$.

6 The tangent to $f(x) = x^2e^{-x}$ at point P is horizontal. Find the possible coordinates of P.

J DERIVATIVES OF LOGARITHMIC FUNCTIONS

Discovery 8 The derivative of $\ln x$

If $y = \ln x$, what is the gradient function?

CALCULUS DEMO

What to do:

1. Click on the icon to see the graph of $y = \ln x$. Observe the gradient function being drawn as the point moves from left to right along the graph.
2. Predict a formula for the gradient function of $y = \ln x$.
3. Find the gradient of the tangent to $y = \ln x$ for $x = 0.25, 0.5, 1, 2, 3, 4,$ and 5. Do your results confirm your prediction in **2**?

From the **Discovery** you should have observed:

If $y = \ln x$ then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

The proof of this result is beyond the scope of this course.

THE DERIVATIVE OF $\ln f(x)$

Suppose $y = \ln f(x)$

$\therefore$ $y = \ln u$ where $u = f(x)$.

$$\text{Now } \frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \quad \{\text{chain rule}\}$$

$$\therefore \frac{dy}{dx} = \frac{1}{u}\frac{du}{dx}$$

$$= \frac{f'(x)}{f(x)}$$

Function	Derivative
$\ln x$	$\dfrac{1}{x}$
$\ln f(x)$	$\dfrac{f'(x)}{f(x)}$

Example 15 Self Tutor

Find the gradient function of:

a $y = \ln(kx)$, k a constant **b** $y = \ln(1 - 3x)$ **c** $y = x^3 \ln x$

a

$$y = \ln(kx)$$

$$\therefore \frac{dy}{dx} = \frac{k}{kx} = \frac{1}{x}$$

b

$$y = \ln(1 - 3x)$$

$$\therefore \frac{dy}{dx} = \frac{-3}{1 - 3x} = \frac{3}{3x - 1}$$

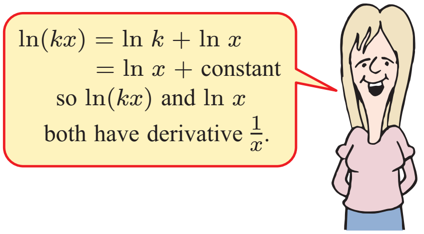
c

$$y = x^3 \ln x$$

$$\therefore \frac{dy}{dx} = 3x^2 \ln x + x^3\left(\frac{1}{x}\right) \quad \{\text{product rule}\}$$

$$= 3x^2 \ln x + x^2$$

$$= x^2(3 \ln x + 1)$$



The laws of logarithms can help us to differentiate some logarithmic functions more easily.

For $a > 0,\ b > 0,\ n \in \mathbb{R}$:

$$\ln(ab) = \ln a + \ln b$$
$$\ln\left(\frac{a}{b}\right) = \ln a - \ln b$$
$$\ln(a^n) = n \ln a$$

Example 16

Self Tutor

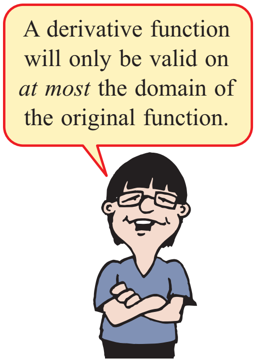
Differentiate with respect to x:

a $y = \ln(xe^{-x})$

b $y = \ln\left[\dfrac{x^2}{(x+2)(x-3)}\right]$

a
$$\begin{aligned} y &= \ln(xe^{-x}) \\ &= \ln x + \ln e^{-x} \quad \{\ln(ab) = \ln a + \ln b\} \\ &= \ln x - x \quad \{\ln e^a = a\} \\ \therefore\ \frac{dy}{dx} &= \frac{1}{x} - 1 \end{aligned}$$

b
$$\begin{aligned} y &= \ln\left[\frac{x^2}{(x+2)(x-3)}\right] \\ &= \ln x^2 - \ln[(x+2)(x-3)] \quad \{\ln\left(\tfrac{a}{b}\right) = \ln a - \ln b\} \\ &= 2\ln x - [\ln(x+2) + \ln(x-3)] \\ &= 2\ln x - \ln(x+2) - \ln(x-3) \\ \therefore\ \frac{dy}{dx} &= \frac{2}{x} - \frac{1}{x+2} - \frac{1}{x-3} \end{aligned}$$



EXERCISE 13J

1 Find the gradient function of:

a $y = \ln(7x)$
b $y = \ln(2x+1)$
c $y = \ln(x - x^2)$
d $y = 3 - 2\ln x$
e $y = x^2 \ln x$
f $y = \dfrac{\ln x}{2x}$
g $y = e^x \ln x$
h $y = (\ln x)^2$
i $y = \sqrt{\ln x}$
j $y = e^{-x} \ln x$
k $y = \sqrt{x}\ln(2x)$
l $y = \dfrac{2\sqrt{x}}{\ln x}$
m $y = 3 - 4\ln(1-x)$
n $y = x\ln(x^2+1)$

2 Find $\dfrac{dy}{dx}$ for:

a $y = x \ln 5$
b $y = \ln(x^3)$
c $y = \ln(x^4 + x)$
d $y = \ln(10 - 5x)$
e $y = [\ln(2x+1)]^3$
f $y = \dfrac{\ln(4x)}{x}$
g $y = \ln\left(\dfrac{1}{x}\right)$
h $y = \ln(\ln x)$
i $y = \dfrac{1}{\ln x}$

3 Use the laws of logarithms to help differentiate with respect to x:

a $y = \ln\sqrt{1-2x}$

b $y = \ln\left(\frac{1}{2x+3}\right)$

c $y = \ln\left(e^x\sqrt{x}\right)$

d $y = \ln\left(x\sqrt{2-x}\right)$

e $y = \ln\left(\frac{x+3}{x-1}\right)$

f $y = \ln\left(\frac{x^2}{3-x}\right)$

g $f(x) = \ln\left((3x-4)^3\right)$

h $f(x) = \ln\left(x(x^2+1)\right)$

i $f(x) = \ln\left(\frac{x^2+2x}{x-5}\right)$

4 Find the gradient of the tangent to:

a $y = x\ln x$ at the point where $x = e$

b $y = \ln\left(\frac{x+2}{x^2}\right)$ at the point where $x = 1$.

5 Suppose $f(x) = a\ln(2x+b)$ where $f(e) = 3$ and $f'(e) = \frac{6}{e}$. Find the constants a and b.

K DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

In **Chapter 9** we saw that sine and cosine curves arise naturally from motion in a circle.

Click on the icon to observe the motion of point P around the unit circle. Observe the graphs of P's height relative to the x-axis, and then P's horizontal displacement from the y-axis. The resulting graphs are those of $y = \sin t$ and $y = \cos t$.

DEMO

Discovery 9 — Derivatives of $\sin x$ and $\cos x$

Our aim is to use a computer demonstration to investigate the derivatives of $\sin x$ and $\cos x$.

What to do:

DERIVATIVES DEMO

1. Click on the icon to observe the graph of $y = \sin x$. A tangent with x-step of length 1 unit moves across the curve, and its y-step is translated onto the gradient graph. Predict the derivative of the function $y = \sin x$.
2. Repeat the process in **1** for the graph of $y = \cos x$. Hence predict the derivative of the function $y = \cos x$.

From the **Discovery** you should have deduced that:

For x in radians: If $f(x) = \sin x$ then $f'(x) = \cos x$.
If $f(x) = \cos x$ then $f'(x) = -\sin x$.

THE DERIVATIVE OF $\tan x$

DERIVATIVE DEMO

Consider $y = \tan x = \dfrac{\sin x}{\cos x}$

We let $u = \sin x$ and $v = \cos x$

$\therefore \dfrac{du}{dx} = \cos x$ and $\dfrac{dv}{dx} = -\sin x$

$$\therefore \frac{dy}{dx} = \frac{u'v - uv'}{v^2} \qquad \{\text{quotient rule}\}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{[\cos x]^2}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} \qquad \{\text{since } \sin^2 x + \cos^2 x = 1\}$$

$$= \sec^2 x$$

Function	*Derivative*
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

THE DERIVATIVES OF $\sin[f(x)]$, $\cos[f(x)]$, AND $\tan[f(x)]$

Suppose $y = \sin[f(x)]$

If we let $u = f(x)$, then $y = \sin u$.

But $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$ {chain rule}

$$\therefore \frac{dy}{dx} = \cos u \times f'(x)$$

$$= \cos[f(x)] \times f'(x)$$

We can perform the same procedure for $\cos[f(x)]$ and $\tan[f(x)]$, giving the following results:

Function	*Derivative*
$\sin[f(x)]$	$\cos[f(x)]\, f'(x)$
$\cos[f(x)]$	$-\sin[f(x)]\, f'(x)$
$\tan[f(x)]$	$\sec^2[f(x)] f'(x)$

Example 17

Self Tutor

Differentiate with respect to x:

a $x \sin x$

b $4\tan^2(3x)$

a If $y = x \sin x$

then by the product rule

$$\frac{dy}{dx} = (1)\sin x + (x)\cos x$$

$$= \sin x + x\cos x$$

b If $y = 4\tan^2(3x)$

$$= 4[\tan(3x)]^2$$

$$= 4u^2 \quad \text{where } u = \tan(3x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \qquad \{\text{chain rule}\}$$

$$\therefore \frac{dy}{dx} = 8u \times \frac{du}{dx}$$

$$= 8\tan(3x) \times 3\sec^2(3x)$$

$$= 24\sin(3x)\sec^3(3x)$$

EXERCISE 13K

1 Find $\frac{dy}{dx}$ for:

a $y=\sin(2x)$ **b** $y=\sin x+\cos x$ **c** $y=\cos(3x)-\sin x$

d $y=\sin(x+1)$ **e** $y=\cos(3-2x)$ **f** $y=\tan(5x)$

g $y=\sin(\frac{x}{2})-3\cos x$ **h** $y=3\tan(\pi x)$ **i** $y=4\sin x-\cos(2x)$

2 Differentiate with respect to x:

a $x^2+\cos x$ **b** $\tan x-3\sin x$ **c** $e^x\cos x$ **d** $e^{-x}\sin x$

e $\ln(\sin x)$ **f** $e^{2x}\tan x$ **g** $\sin(3x)$ **h** $\cos(\frac{x}{2})$

i $3\tan(2x)$ **j** $x\cos x$ **k** $\frac{\sin x}{x}$ **l** $x\tan x$

3 Differentiate with respect to x:

a $\sin(x^2)$ **b** $\cos(\sqrt{x})$ **c** $\sqrt{\cos x}$ **d** $\sin^2 x$

e $\cos^3 x$ **f** $\cos x\sin(2x)$ **g** $\cos(\cos x)$ **h** $\cos^3(4x)$

i $\frac{1}{\sin x}$ **j** $\frac{1}{\cos(2x)}$ **k** $\frac{2}{\sin^2(2x)}$ **l** $\frac{8}{\tan^3(\frac{x}{2})}$

4 Find the gradient of the tangent to:

a $f(x)=\sin^3 x$ at the point where $x=\frac{2\pi}{3}$

b $f(x)=\cos x\sin x$ at the point where $x=\frac{\pi}{4}$.

L SECOND DERIVATIVES

Given a function $f(x)$, the derivative $f'(x)$ is known as the **first derivative**.

The **second derivative** of $f(x)$ is the derivative of $f'(x)$, or **the derivative of the first derivative**.

We use $f''(x)$ or y'' or $\frac{d^2y}{dx^2}$ to represent the second derivative.

$f''(x)$ reads "*f double dashed x*".

$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$ reads "*dee two y by dee x squared*".

Example 18 Self Tutor

Find $f''(x)$ given that $f(x)=x^3-\frac{3}{x}$.

$$\begin{aligned}\text{Now}\quad f(x)&=x^3-3x^{-1}\\ \therefore\quad f'(x)&=3x^2+3x^{-2}\\ \therefore\quad f''(x)&=6x-6x^{-3}\\ &=6x-\frac{6}{x^3}\end{aligned}$$

EXERCISE 13L

1 Find $f''(x)$ given that:

a $f(x) = 3x^2 - 6x + 2$ **b** $f(x) = \frac{2}{\sqrt{x}} - 1$ **c** $f(x) = 2x^3 - 3x^2 - x + 5$

d $f(x) = \frac{2-3x}{x^2}$ **e** $f(x) = (1-2x)^3$ **f** $f(x) = \frac{x+2}{2x-1}$

2 Find $\frac{d^2y}{dx^2}$ given that:

a $y = x - x^3$ **b** $y = x^2 - \frac{5}{x^2}$ **c** $y = 2 - \frac{3}{\sqrt{x}}$

d $y = \frac{4-x}{x}$ **e** $y = (x^2 - 3x)^3$ **f** $y = x^2 - x + \frac{1}{1-x}$

3 Given $f(x) = x^3 - 2x + 5$, find:

a $f(2)$ **b** $f'(2)$ **c** $f''(2)$

4 Suppose $y = Ae^{kx}$ where A and k are constants. Show that:

a $\frac{dy}{dx} = ky$ **b** $\frac{d^2y}{dx^2} = k^2y$

5 Find the value(s) of x such that $f''(x) = 0$, given:

a $f(x) = 2x^3 - 6x^2 + 5x + 1$ **b** $f(x) = \frac{x}{x^2+2}$

6 Consider the function $f(x) = 2x^3 - x$.
Complete the following table by indicating whether $f(x)$, $f'(x)$, and $f''(x)$ are positive (+), negative (–), or zero (0) at the given values of x.

x	-1	0	1
$f(x)$	–		
$f'(x)$			
$f''(x)$			

7 Suppose $f(x) = 2\sin^3 x - 3\sin x$.

a Show that $f'(x) = -3\cos x \cos 2x$. **b** Find $f''(x)$.

8 Find $\frac{d^2y}{dx^2}$ given:

a $y = -\ln x$ **b** $y = x \ln x$ **c** $y = (\ln x)^2$

9 Given $f(x) = x^2 - \frac{1}{x}$, find:

a $f(1)$ **b** $f'(1)$ **c** $f''(1)$

10 If $y = 2e^{3x} + 5e^{4x}$, show that $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$.

11 If $y = \sin(2x+3)$, show that $\frac{d^2y}{dx^2} + 4y = 0$.

12 If $y = 2\sin x + 3\cos x$, show that $y'' + y = 0$ where y'' represents $\frac{d^2y}{dx^2}$.

Review set 13A

1 Evaluate:

a $\lim\limits_{x\to 1}(6x-7)$ **b** $\lim\limits_{h\to 0}\dfrac{2h^2-h}{h}$ **c** $\lim\limits_{x\to 4}\dfrac{x^2-16}{x-4}$

2 Find, from first principles, the derivative of:

a $f(x)=x^2+2x$ **b** $y=4-3x^2$

3 In the **Opening Problem** on page **334**, the altitude of the jumper is given by $f(t)=452-4.8t^2$ metres, where $0\leqslant t\leqslant 3$ seconds.

a Find $f'(t)=\lim\limits_{h\to 0}\dfrac{f(t+h)-f(t)}{h}$.

b *Hence* find the speed of the jumper when $t=2$ seconds.

4 If $f(x)=7+x-3x^2$, find: **a** $f(3)$ **b** $f'(3)$ **c** $f''(3)$.

5 Find $\dfrac{dy}{dx}$ for: **a** $y=3x^2-x^4$ **b** $y=\dfrac{x^3-x}{x^2}$

6 At what point on the curve $f(x)=\dfrac{x}{\sqrt{x^2+1}}$ does the tangent have gradient 1?

7 Find $\dfrac{dy}{dx}$ if: **a** $y=e^{x^3+2}$ **b** $y=\ln\left(\dfrac{x+3}{x^2}\right)$

8 Given $y=3e^x-e^{-x}$, show that $\dfrac{d^2y}{dx^2}=y$.

9 Differentiate with respect to x:

a $5x-3x^{-1}$ **b** $(3x^2+x)^4$ **c** $(x^2+1)(1-x^2)^3$

10 Find all points on the curve $y=2x^3+3x^2-10x+3$ where the gradient of the tangent is 2.

11 If $y=\sqrt{5-4x}$, find: **a** $\dfrac{dy}{dx}$ **b** $\dfrac{d^2y}{dx^2}$

12 Differentiate with respect to x:

a $\sin(5x)\ln(x)$ **b** $\sin(x)\cos(2x)$ **c** $e^{-2x}\tan x$

13 Find the gradient of the tangent to $y=\sin^2 x$ at the point where $x=\frac{\pi}{3}$.

14 Find the derivative with respect to x of:

a $f(x)=(x^2+3)^4$ **b** $g(x)=\dfrac{\sqrt{x+5}}{x^2}$

15 Find $f''(2)$ for:

a $f(x)=3x^2-\dfrac{1}{x}$ **b** $f(x)=\sqrt{x}$

16 Differentiate with respect to x:

a $10x-\sin(10x)$ **b** $\ln\left(\dfrac{1}{\cos x}\right)$ **c** $\sin(5x)\ln(2x)$

Review set 13B

1 Evaluate the limits:

a $\lim\limits_{h\to 0} \dfrac{h^3-3h}{h}$ **b** $\lim\limits_{x\to 1} \dfrac{3x^2-3x}{x-1}$ **c** $\lim\limits_{x\to 2} \dfrac{x^2-3x+2}{2-x}$

2 Given $f(x)=5x-x^2$, find $f'(1)$ from first principles.

3 **a** Given $y=2x^2-1$, find $\dfrac{dy}{dx}$ from first principles.

b Hence state the gradient of the tangent to $y=2x^2-1$ at the point where $x=4$.

c For what value of x is the gradient of the tangent to $y=2x^2-1$ equal to -12?

4 Differentiate with respect to x: **a** $y=x^3\sqrt{1-x^2}$ **b** $y=\dfrac{x^2-3x}{\sqrt{x+1}}$

5 Find $\dfrac{d^2y}{dx^2}$ for: **a** $y=3x^4-\dfrac{2}{x}$ **b** $y=x^3-x+\dfrac{1}{\sqrt{x}}$

6 Find all points on the curve $y=xe^x$ where the gradient of the tangent is $2e$.

7 Differentiate with respect to x: **a** $f(x)=\ln(e^x+3)$ **b** $f(x)=\ln\left[\dfrac{(x+2)^3}{x}\right]$

8 Suppose $y=\left(x-\dfrac{1}{x}\right)^4$. Find $\dfrac{dy}{dx}$ when $x=1$.

9 Find $\dfrac{dy}{dx}$ if: **a** $y=\ln(x^3-3x)$ **b** $y=\dfrac{e^x}{x^2}$

10 Find x if $f''(x)=0$ and $f(x)=2x^4-4x^3-9x^2+4x+7$.

11 If $f(x)=x-\cos x$, find

a $f(\pi)$ **b** $f'(\frac{\pi}{2})$ **c** $f''(\frac{3\pi}{4})$

12 **a** Find $f'(x)$ and $f''(x)$ for $f(x)=\sqrt{x}\cos(4x)$.

b Hence find $f'(\frac{\pi}{16})$ and $f''(\frac{\pi}{8})$.

13 Suppose $y=3\sin 2x+2\cos 2x$. Show that $4y+\dfrac{d^2y}{dx^2}=0$.

14 Consider $f(x)=\dfrac{6x}{3+x^2}$. Find the value(s) of x when:

a $f(x)=-\frac{1}{2}$ **b** $f'(x)=0$ **c** $f''(x)=0$

15 The function f is defined by $f: x\mapsto -10\sin 2x\cos 2x$, $0\leqslant x\leqslant \pi$.

a Write down an expression for $f(x)$ in the form $k\sin 4x$.

b Solve $f'(x)=0$, giving exact answers.

16 Given that a and b are constants, differentiate $y=3\sin bx-a\cos 2x$ with respect to x.

Find a and b if $y+\dfrac{d^2y}{dx^2}=6\cos 2x$.

14

Applications of differential calculus

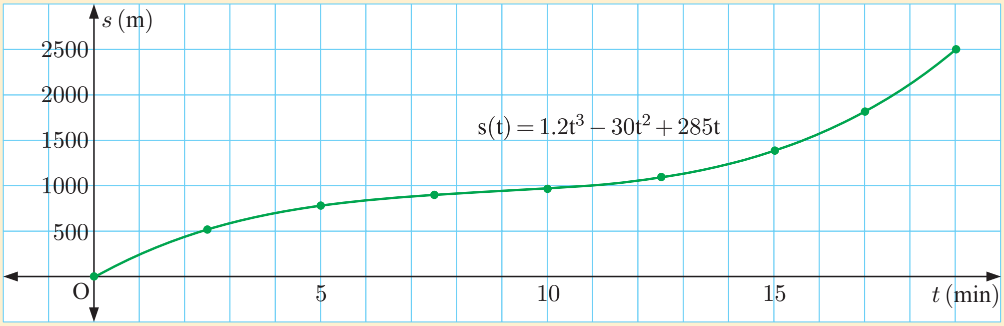
Contents:

Opening problem

Michael rides up a hill and down the other side to his friend's house. The dots on the graph show Michael's position at various times t.

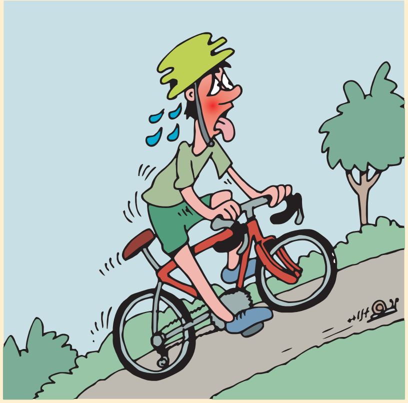
DEMO

The distance Michael has travelled at various times is given by the function $s(t) = 1.2t^3 - 30t^2 + 285t$ metres for $0 \leqslant t \leqslant 19$ minutes.



Things to think about:

- **a** Can you find a function for Michael's *speed* at any time t?
- **b** Michael's *acceleration* is the rate at which his speed is changing with respect to time. How can we interpret $s''(t)$?
- **c** Can you find Michael's speed and acceleration at the time $t = 15$ minutes?
- **d** At what point do you think the hill was steepest? How far had Michael travelled to this point?



In the previous chapter we saw how to differentiate many types of functions.

In this chapter we will use derivatives to find:

- tangents and normals to curves
- turning points which are local minima and maxima.

We will then look at applying these techniques to real world problems including:

- kinematics (motion problems of displacement, velocity, and acceleration)
- rates of change
- optimisation (maxima and minima).

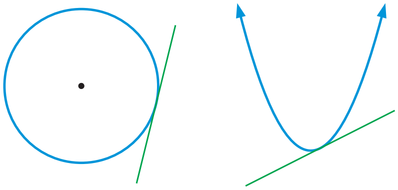
A TANGENTS AND NORMALS

TANGENTS

The **tangent** to a curve at point A is the best approximating straight line to the curve at A.

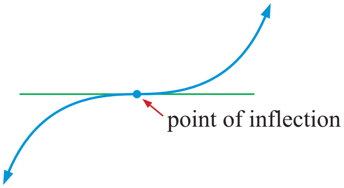
In cases we have seen already, the tangent *touches* the curve.

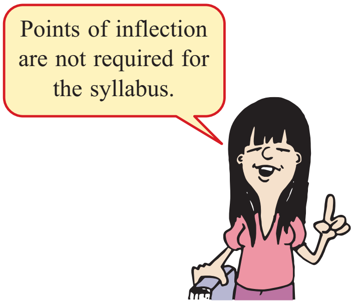
For example, consider tangents to a circle or a quadratic.



However, we note that for some functions:

- The tangent may intersect the curve again somewhere else.
- It is possible for the tangent to pass through the curve at the point of tangency. If this happens, we call it a **point of inflection**.





Points of inflection are not required for the syllabus.

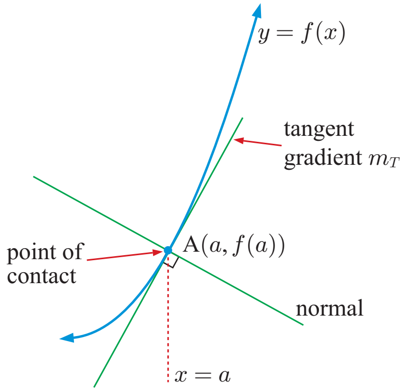
Consider a curve $y = f(x)$.

If A is the point with x-coordinate a, then the gradient of the tangent to the curve at this point is $f'(a) = m_T$.

The equation of the tangent is

$$y - f(a) = f'(a)(x - a)$$

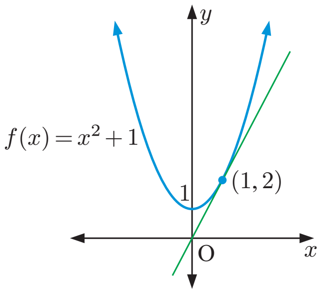
or $\boldsymbol{y = f(a) + f'(a)(x - a)}$.



Example 1

Self Tutor

Find the equation of the tangent to $f(x) = x^2 + 1$ at the point where $x = 1$.



Since $f(1) = 1 + 1 = 2$, the point of contact is $(1, 2)$.

Now $f'(x) = 2x$, so $m_T = f'(1) = 2$

$\therefore$ the tangent has equation $y = 2 + 2(x - 1)$

which is $y = 2x$.

NORMALS

A **normal** to a curve is a line which is perpendicular to the tangent at the point of contact.

The gradients of perpendicular lines are negative reciprocals of each other, so:

The gradient of the normal to the curve at $x = a$ is $m_N = -\dfrac{1}{f'(a)}$.

The equation of the normal to the curve at $x = a$ is $\boldsymbol{y = f(a) - \dfrac{1}{f'(a)}(x - a)}$.

Reminder: If a line has gradient $\frac{4}{5}$ and passes through $(2, -3)$, another quick way to write down its equation is $4x - 5y = 4(2) - 5(-3)$ or $4x - 5y = 23$.

If the gradient was $-\frac{4}{5}$, we would have:

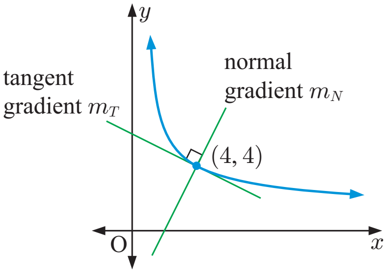
$$4x + 5y = 4(2) + 5(-3) \quad \text{or} \quad 4x + 5y = -7.$$

Example 2

Self Tutor

Find the equation of the normal to $y = \dfrac{8}{\sqrt{x}}$ at the point where $x = 4$.

When $x = 4$, $y = \dfrac{8}{\sqrt{4}} = \dfrac{8}{2} = 4$. So, the point of contact is $(4, 4)$.



Now as $y = 8x^{-\frac{1}{2}}$, $\dfrac{dy}{dx} = -4x^{-\frac{3}{2}}$

$\therefore$ when $x = 4$, $m_T = -4 \times 4^{-\frac{3}{2}} = -\frac{1}{2}$

$\therefore$ the normal at $(4, 4)$ has gradient $m_N = \frac{2}{1}$.

$\therefore$ the equation of the normal is

$$2x - 1y = 2(4) - 1(4)$$

or $2x - y = 4$

EXERCISE 14A

1 Find the equation of the tangent to:

a $y = x - 2x^2 + 3$ at $x = 2$

b $y = \sqrt{x} + 1$ at $x = 4$

c $y = x^3 - 5x$ at $x = 1$

d $y = \dfrac{4}{\sqrt{x}}$ at $(1, 4)$

e $y = \dfrac{3}{x} - \dfrac{1}{x^2}$ at $(-1, -4)$

f $y = 3x^2 - \dfrac{1}{x}$ at $x = -1$.

2 Find the equation of the normal to:

a $y = x^2$ at the point $(3, 9)$

b $y = x^3 - 5x + 2$ at $x = -2$

c $y = \dfrac{5}{\sqrt{x}} - \sqrt{x}$ at the point $(1, 4)$

d $y = 8\sqrt{x} - \dfrac{1}{x^2}$ at $x = 1$.

Example 3

Self Tutor

Find the equations of any horizontal tangents to $y = x^3 - 12x + 2$.

Since $y = x^3 - 12x + 2$, $\frac{dy}{dx} = 3x^2 - 12$

Horizontal tangents have gradient 0, so $3x^2 - 12 = 0$

$$\therefore\ 3(x^2 - 4) = 0$$
$$\therefore\ 3(x + 2)(x - 2) = 0$$
$$\therefore\ x = -2 \text{ or } 2$$

When $x = 2$, $y = 8 - 24 + 2 = -14$

When $x = -2$, $y = -8 + 24 + 2 = 18$

$\therefore$ the points of contact are $(2, -14)$ and $(-2, 18)$

$\therefore$ the tangents are $y = -14$ and $y = 18$.

3 Find the equations of any horizontal tangents to $y = 2x^3 + 3x^2 - 12x + 1$.

4 Find the points of contact where horizontal tangents meet the curve $y = 2\sqrt{x} + \frac{1}{\sqrt{x}}$.

5 Find k if the tangent to $y = 2x^3 + kx^2 - 3$ at the point where $x = 2$ has gradient 4.

6 Find the equation of another tangent to $y = 1 - 3x + 12x^2 - 8x^3$ which is parallel to the tangent at $(1, 2)$.

7 Consider the curve $y = x^2 + ax + b$ where a and b are constants. The tangent to this curve at the point where $x = 1$ is $2x + y = 6$. Find the values of a and b.

8 Consider the curve $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$ where a and b are constants. The normal to this curve at the point where $x = 4$ is $4x + y = 22$. Find the values of a and b.

9 Show that the equation of the tangent to $y = 2x^2 - 1$ at the point where $x = a$, is $4ax - y = 2a^2 + 1$.

10 Find the equation of the tangent to:

a $y = \sqrt{2x + 1}$ at $x = 4$

b $y = \frac{1}{2 - x}$ at $x = -1$

c $f(x) = \frac{x}{1 - 3x}$ at $(-1, -\frac{1}{4})$

d $f(x) = \frac{x^2}{1 - x}$ at $(2, -4)$.

11 Find the equation of the normal to:

a $y = \frac{1}{(x^2 + 1)^2}$ at $(1, \frac{1}{4})$

b $y = \frac{1}{\sqrt{3 - 2x}}$ at $x = -3$

c $f(x) = \sqrt{x}(1 - x)^2$ at $x = 4$

d $f(x) = \frac{x^2 - 1}{2x + 3}$ at $x = -1$.

12 Consider the curve $y = a\sqrt{1 - bx}$ where a and b are constants. The tangent to this curve at the point where $x = -1$ is $3x + y = 5$. Find the values of a and b.

Example 4

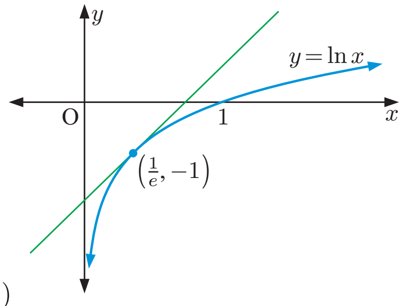
Self Tutor

Show that the equation of the tangent to $y = \ln x$ at the point where $y = -1$ is $y = ex - 2$.

When $y = -1$, $\ln x = -1$

$\therefore$ $x = e^{-1} = \frac{1}{e}$

$\therefore$ the point of contact is $\left(\frac{1}{e}, -1\right)$.



Now $f(x) = \ln x$ has derivative $f'(x) = \dfrac{1}{x}$

$\therefore$ the tangent at $\left(\frac{1}{e}, -1\right)$ has gradient $\dfrac{1}{\frac{1}{e}} = e$

$\therefore$ the tangent has equation $y = -1 + e(x - \frac{1}{e})$

which is $y = ex - 2$

13 Find the equation of:

a the tangent to $f : x \mapsto e^{-x}$ at the point where $x = 1$

b the tangent to $y = \ln(2 - x)$ at the point where $x = -1$

c the normal to $y = \ln\sqrt{x}$ at the point where $y = -1$.

Example 5

Self Tutor

Find the equation of the tangent to $y = \tan x$ at the point where $x = \frac{\pi}{4}$.

When $x = \frac{\pi}{4}$, $y = \tan\frac{\pi}{4} = 1$

$\therefore$ the point of contact is $(\frac{\pi}{4}, 1)$.

Now $f(x) = \tan x$ has derivative $f'(x) = \sec^2 x$

$\therefore$ the tangent at $(\frac{\pi}{4}, 1)$ has gradient $\sec^2\frac{\pi}{4} = (\sqrt{2})^2 = 2$

$\therefore$ the tangent has equation $y = 1 + 2(x - \frac{\pi}{4})$

which is $y = 2x + (1 - \frac{\pi}{2})$

14 Show that the curve with equation $y = \dfrac{\cos x}{1 + \sin x}$ does not have any horizontal tangents.

15 Find the equation of:

a the tangent to $y = \sin x$ at the origin

b the tangent to $y = \tan x$ at the origin

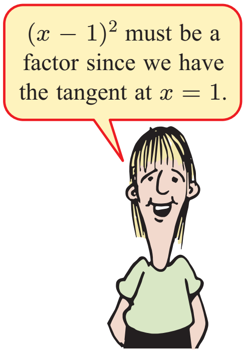
c the normal to $y = \cos x$ at the point where $x = \frac{\pi}{6}$

d the normal to $y = \dfrac{1}{\sin(2x)}$ at the point where $x = \frac{\pi}{4}$.

Example 6

Self Tutor

Find where the tangent to $y = x^3 + x + 2$ at $(1, 4)$ meets the curve again.



Let $f(x) = x^3 + x + 2$

$\therefore\ f'(x) = 3x^2 + 1$ and $\therefore\ f'(1) = 3 + 1 = 4$

$\therefore$ the equation of the tangent at $(1, 4)$ is $4x - y = 4(1) - 4$

or $y = 4x$.

The curve meets the tangent again when $x^3 + x + 2 = 4x$

$$\therefore\ x^3 - 3x + 2 = 0$$

$$\therefore\ (x - 1)^2(x + 2) = 0$$

When $x = -2$, $y = (-2)^3 + (-2) + 2 = -8$

$\therefore$ the tangent meets the curve again at $(-2, -8)$.

16 **a** Find where the tangent to the curve $y = x^3$ at the point where $x = 2$, meets the curve again.

b Find where the tangent to the curve $y = -x^3 + 2x^2 + 1$ at the point where $x = -1$, meets the curve again.

17 Consider the function $f(x) = x^2 + \dfrac{4}{x^2}$.

a Find $f'(x)$. **b** Find the values of x at which the tangent to the curve is horizontal.

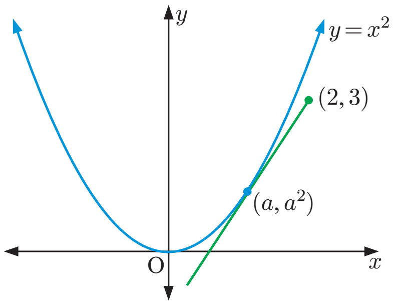
c Show that the tangents at these points are the same line.

18 The tangent to $y = x^2e^x$ at $x = 1$ cuts the x and y-axes at A and B respectively. Find the coordinates of A and B.

Example 7

Self Tutor

Find the equations of the tangents to $y = x^2$ from the external point $(2, 3)$.



Let (a, a^2) be a general point on $f(x) = x^2$.

Now $f'(x) = 2x$, so $f'(a) = 2a$

$\therefore$ the equation of the tangent at (a, a^2) is

$$y = a^2 + 2a(x - a)$$

which is $y = 2ax - a^2$

Thus the tangents which pass through $(2, 3)$ satisfy

$$3 = 2a(2) - a^2$$

$$\therefore\ a^2 - 4a + 3 = 0$$

$$\therefore\ (a - 1)(a - 3) = 0$$

$$\therefore\ a = 1 \text{ or } 3$$

$\therefore$ exactly two tangents pass through the external point $(2, 3)$.

If $a = 1$, the tangent has equation $y = 2x - 1$ with point of contact $(1, 1)$.

If $a = 3$, the tangent has equation $y = 6x - 9$ with point of contact $(3, 9)$.

19 **a** Find the equation of the tangent to $y = x^2 - x + 9$ at the point where $x = a$.

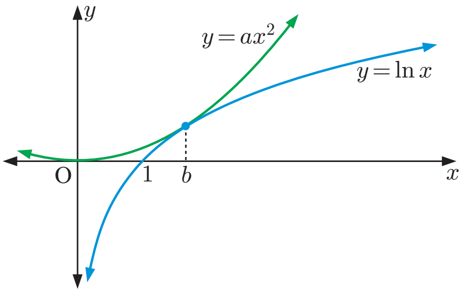
b Hence, find the equations of the two tangents from $(0, 0)$ to the curve. State the coordinates of the points of contact.

20 Find the equations of the tangents to $y = x^3$ from the external point $(-2, 0)$.

21 Find the equation of the normal to $y = \sqrt{x}$ from the external point $(4, 0)$.
Hint: There is no normal at the point where $x = 0$, as this is the endpoint of the function.

22 Find the equation of the tangent to $y = e^x$ at the point where $x = a$.
Hence, find the equation of the tangent to $y = e^x$ which passes through the origin.

23 A quadratic of the form $y = ax^2$, $a > 0$, touches the logarithmic function $y = \ln x$ as shown.

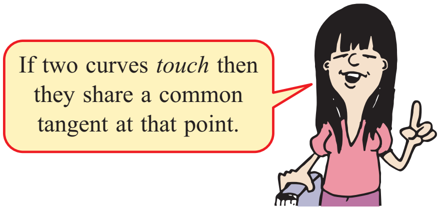


a If the x-coordinate of the point of contact is b, explain why $ab^2 = \ln b$ and $2ab = \frac{1}{b}$.

b Deduce that the point of contact is $(\sqrt{e}, \frac{1}{2})$.

c Find the value of a.

d Find the equation of the common tangent.



24 Find, correct to 2 decimal places, the angle between the tangents to $y = 3e^{-x}$ and $y = 2 + e^x$ at their point of intersection.

25 Consider the cubic function $f(x) = 2x^3 + 5x^2 - 4x - 3$.

a Show that the equation of the tangent to the curve at the point where $x = -1$ can be written in the form $y = 4 - 8(x + 1)$.

b Show that $f(x)$ can be written in the form $f(x) = 4 - 8(x + 1) - (x + 1)^2 + 2(x + 1)^3$.

c Hence explain why the tangent is the best approximating straight line to the curve at the point where $x = -1$.

26 A cubic has three real roots. Prove that the tangent line at the average of any two roots of the cubic, passes through the third root.
Hint: Let $f(x) = a(x - \alpha)(x - \beta)(x - \gamma)$.

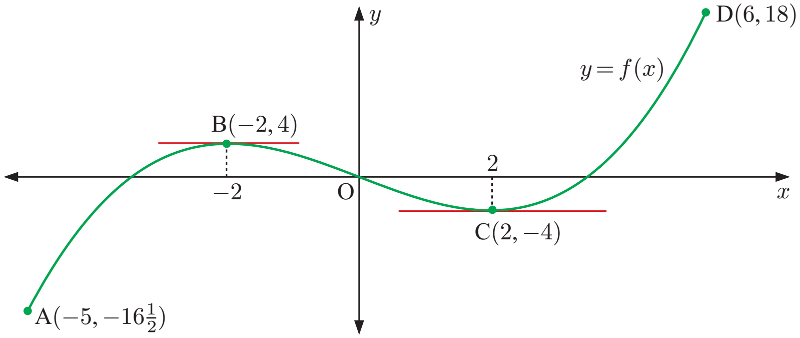
B STATIONARY POINTS

A **stationary point** of a function is a point where $f'(x) = 0$.

It could be a local maximum, local minimum, or stationary inflection.

TURNING POINTS (MAXIMA AND MINIMA)

Consider the following graph which has a restricted domain of $-5 \leqslant x \leqslant 6$.



A is a **global minimum** as it has the minimum value of y on the entire domain.

B is a **local maximum** as it is a turning point where $f'(x) = 0$ and the curve has shape  .

C is a **local minimum** as it is a turning point where $f'(x) = 0$ and the curve has shape  .

D is a **global maximum** as it is the maximum value of y on the entire domain.

Use of the words "local" and "global" is not required for the syllabus, but is useful for understanding.

For many functions, a local maximum or minimum is also the global maximum or minimum.

For example, for $y = x^2$ the point $(0, 0)$ is a local minimum and is also the global minimum.

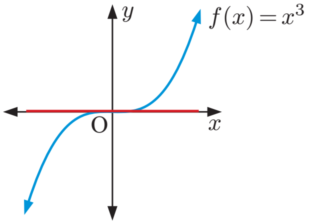
STATIONARY POINTS OF INFLECTION

It is not always true that whenever we find a value of x where $f'(x) = 0$, we have a local maximum or minimum.

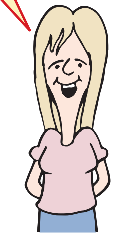
For example,

$f(x) = x^3$ has $f'(x) = 3x^2$,

so $f'(x) = 0$ when $x = 0$.



Points of inflection are not required for the syllabus.



The x-axis is a tangent to the curve which actually crosses over the curve at O$(0, 0)$. This tangent is horizontal, but O$(0, 0)$ is neither a local maximum nor a local minimum. It is called a **stationary inflection** as the curve changes its curvature or shape.

SIGN DIAGRAMS

A **sign diagram** is used to display the intervals on which a function is positive and negative.

In calculus we commonly use sign diagrams of the *derivative function* $f'(x)$ so we can determine the nature of a stationary point.

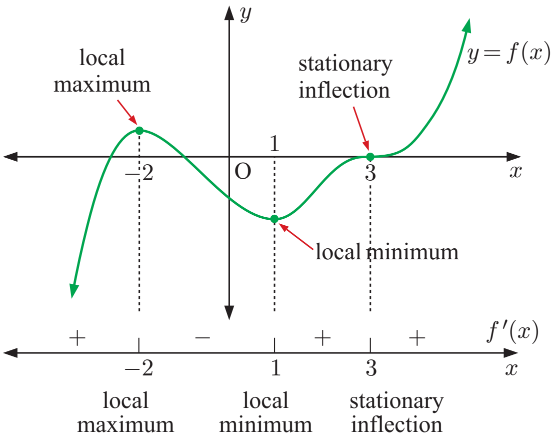
Consider the graph alongside.

The sign diagram of its gradient function is shown directly beneath it.

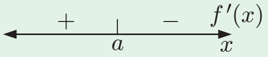
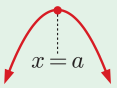
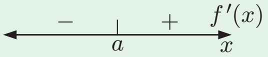
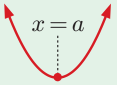
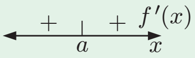
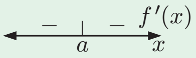
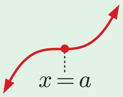
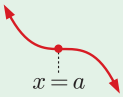
We can use the sign diagram to describe the stationary points of the function.

The signs on the sign diagram of $f'(x)$ indicate whether the gradient of $y = f(x)$ is positive or negative in that interval.

DEMO



We observe the following properties:

Stationary point where $f'(a) = 0$	*Sign diagram of* $f'(x)$ *near* $x = a$	*Shape of curve near* $x = a$
local maximum		
local minimum		
stationary inflection	 or 	 or 

Example 8

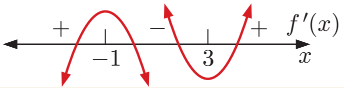
Self Tutor

Consider the function $f(x) = x^3 - 3x^2 - 9x + 5$.

- **a** Find the y-intercept.
- **b** Find and classify all stationary points.
- **c** Hence sketch the curve $y = f(x)$.

a $f(0) = 5$, so the y-intercept is 5.

b
$$\begin{aligned} f(x) &= x^3 - 3x^2 - 9x + 5 \\ \therefore\quad f'(x) &= 3x^2 - 6x - 9 \\ &= 3(x^2 - 2x - 3) \\ &= 3(x-3)(x+1) \end{aligned}$$
which has sign diagram:



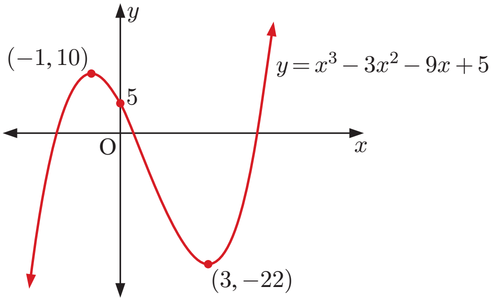
So, we have a local maximum at $x = -1$ and a local minimum at $x = 3$.

$$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 5 = 10$$

$$f(3) = 3^3 - 3 \times 3^2 - 9 \times 3 + 5 = -22$$

$\therefore$ there is a local maximum at $(-1,\ 10)$ and a local minimum at $(3,\ -22)$.

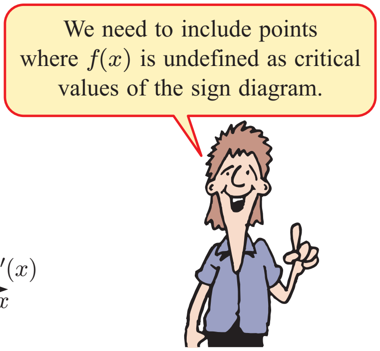
c



Example 9

Self Tutor

Find and classify all stationary points of $f(x) = \dfrac{x^2+1}{x}$.



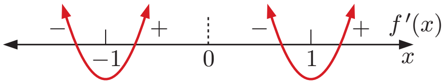
$$= \frac{x^2+1}{x}$$

$$\therefore\ f'(x) = \frac{2x(x) - (x^2+1)}{x^2}$$

$$= \frac{x^2-1}{x^2}$$

$$= \frac{(x+1)(x-1)}{x}$$

$f'(x)$ has sign diagram:



So, we have local minima when $x = \pm 1$.

$$f(-1) = \frac{(-1)^2+1}{(-1)} = -2 \quad \text{and} \quad f(1) = \frac{1^2+1}{1} = 2$$

$\therefore$ there are local minima at $(-1,\ -2)$ and $(1,\ 2)$.

SECOND DERIVATIVES AND STATIONARY POINTS

The second derivative of a function can be used to determine the nature of its stationary points.

For a function $f(x)$ with a stationary point at $x = a$:

- If $f''(a) > 0$, then it is a **local minimum**.
- If $f''(a) < 0$, then it is a **local maximum**.
- If $f''(a) = 0$, then it could be a **local maximum**, a **local minimum**, or a **stationary inflection point**.

Example 10

Self Tutor

Find and classify all stationary points of $f(x)=2x^3+3x^2-12$.

$$\begin{aligned} f(x)&=2x^3+3x^2-12\\ \therefore\quad f'(x)&=6x^2+6x\\ &=6x(x+1)\end{aligned}$$

$$\therefore\quad f'(x)=0 \quad\text{when}\quad 6x=0 \quad\text{or}\quad x+1=0$$
$$\therefore\quad x=0 \quad\text{or}\quad x=-1$$

Also, $f''(x)=12x+6$

$\therefore\quad f''(0)=12(0)+6=6$ which is >0

and $f''(-1)=12(-1)+6=-6$ which is <0

So, we have a local minimum at $x=0$ and a local maximum at $x=-1$.

Now $f(0)=2(0)^3+3(0)^2-12=-12$

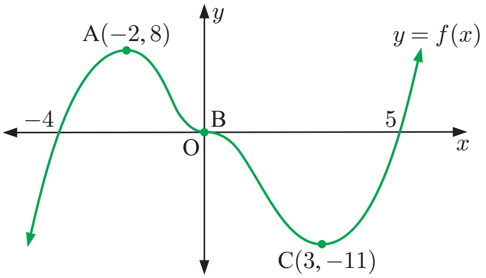
$f(-1)=2(-1)^3+3(-1)^2-12=-11$

$\therefore$ there is a local minimum at $(0,\,-12)$ and a local maximum at $(-1,\,-11)$.

EXERCISE 14B

1 The tangents at points A, B, and C are horizontal.

- **a** Classify points A, B, and C.
- **b** Draw a sign diagram for:
  - **i** $f(x)$
  - **ii** $f'(x)$



2 For each of the following functions, find and classify any stationary points. Sketch the function, showing all important features.

- **a** $f(x)=x^2-2$
- **b** $f(x)=x^3+1$
- **c** $f(x)=x^3-3x+2$
- **d** $f(x)=x^4-2x^2$
- **e** $f(x)=x^3-6x^2+12x-7$
- **f** $f(x)=\sqrt{x}+2$
- **g** $f(x)=x-\sqrt{x}$
- **h** $f(x)=x^4-6x^2+8x-3$
- **i** $f(x)=1-x\sqrt{x}$
- **j** $f(x)=x^4-2x^2-8$

GRAPHING PACKAGE

3 At what value of x does the quadratic function $f(x)=ax^2+bx+c$, $a\neq0$, have a stationary point? Under what conditions is the stationary point a local maximum or a local minimum?

4 $f(x)=2x^3+ax^2-24x+1$ has a local maximum at $x=-4$. Find a.

5 $f(x)=x^3+ax+b$ has a stationary point at $(-2,\,3)$.

- **a** Find the values of a and b.
- **b** Find the position and nature of all stationary points.

Example 11

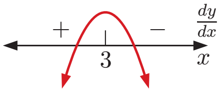
Self Tutor

Find the exact position and nature of the stationary point of $y = (x-2)e^{-x}$.

$$\begin{aligned}\frac{dy}{dx} &= (1)e^{-x} + (x-2)e^{-x}(-1) \qquad \{\text{product rule}\}\\ &= e^{-x}(1-(x-2))\\ &= \frac{3-x}{e^x} \quad \text{where } e^x \text{ is positive for all } x\end{aligned}$$

So, $\frac{dy}{dx} = 0$ when $x = 3$.

The sign diagram of $\frac{dy}{dx}$ is:



$\therefore$ at $x = 3$ we have a local maximum.

But when $x = 3$, $y = (1)e^{-3} = \frac{1}{e^3}$

$\therefore$ the local maximum is at $(3, \frac{1}{e^3})$.

To determine the nature of a stationary point, we can use a sign diagram or the second derivative.

6 Find the position and nature of the stationary point(s) of:

a $y = xe^{-x}$ **b** $y = x^2e^x$ **c** $y = \frac{e^x}{x}$ **d** $y = e^{-x}(x+2)$

7 Consider $f(x) = x\ln x$.

a For what values of x is $f(x)$ defined? **b** Show that the global minimum value of $f(x)$ is $-\frac{1}{e}$.

8 Find the greatest and least value of:

a $x^3 - 12x - 2$ for $-3 \leqslant x \leqslant 5$ **b** $4 - 3x^2 + x^3$ for $-2 \leqslant x \leqslant 3$

9 The cubic polynomial $P(x) = ax^3 + bx^2 + cx + d$ touches the line with equation $y = 9x + 2$ at the point $(0, 2)$, and has a stationary point at $(-1, -7)$. Find $P(x)$.

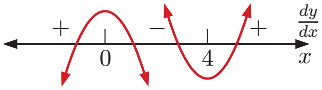
Example 12

Self Tutor

Find the greatest and least value of $y = x^3 - 6x^2 + 5$ on the interval $-2 \leqslant x \leqslant 5$.

$$\begin{aligned}\text{Now } \frac{dy}{dx} &= 3x^2 - 12x\\ &= 3x(x-4)\\ \therefore \frac{dy}{dx} &= 0 \text{ when } x = 0 \text{ or } 4\end{aligned}$$

The sign diagram of $\frac{dy}{dx}$ is:



If the domain is restricted, we need to check the value of the function at the endpoints of the domain.

$\therefore$ there is a local maximum at $x = 0$, and a local minimum at $x = 4$.

Critical value (x)	$f(x)$
-2 (endpoint)	-27
0 (local max)	5
4 (local min)	-27
5 (endpoint)	-20

The greatest of these values is 5 when $x = 0$.

The least of these values is -27 when $x = -2$ and when $x = 4$.

10 For each of the following, determine the position and nature of the stationary points on the interval $0 \leqslant x \leqslant 2\pi$, then show them on a graph of the function.

GRAPHING PACKAGE

- **a** $f(x) = \sin x$
- **b** $f(x) = \cos(2x)$
- **c** $f(x) = \sin^2 x$
- **d** $f(x) = e^{\sin x}$
- **e** $f(x) = \sin(2x) + 2\cos x$

11 Show that $y = 4e^{-x}\sin x$ has a local maximum when $x = \frac{\pi}{4}$.

12 Prove that $\dfrac{\ln x}{x} \leqslant \dfrac{1}{e}$ for all $x > 0$. **Hint:** Let $f(x) = \dfrac{\ln x}{x}$ and find its greatest value.

13 Consider the function $f(x) = x - \ln x$.

- **a** Show that the graph of $y = f(x)$ has a local minimum and that this is the only turning point.
- **b** Hence prove that $\ln x \leqslant x - 1$ for all $x > 0$.

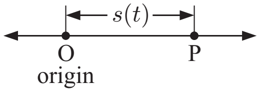
C KINEMATICS

In the **Opening Problem** we are dealing with the movement of Michael riding his bicycle. We do not know the direction Michael is travelling, so we talk simply about the *distance* he has travelled and his *speed*.

For problems of **motion in a straight line**, we can include the direction the object is travelling along the line. We therefore can talk about *displacement* and *velocity*.

DISPLACEMENT

Suppose an object P moves along a straight line so that its position s from an origin O is given as some function of time t. We write $s = s(t)$ where $t \geqslant 0$.



$s(t)$ is a **displacement function** and for any value of t it gives the displacement from O.

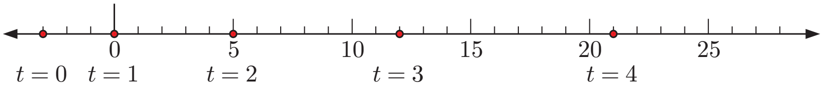
$s(t)$ is a vector quantity. Its magnitude is the distance from O, and its sign indicates the direction from O.

For example, consider $s(t) = t^2 + 2t - 3$ cm.

$s(0) = -3$ cm, $s(1) = 0$ cm, $s(2) = 5$ cm, $s(3) = 12$ cm, $s(4) = 21$ cm.

To appreciate the motion of P we draw a **motion graph**. You can also view the motion by clicking on the icon.

DEMO



VELOCITY

The **average velocity** of an object moving in a straight line in the time interval from $t = t_1$ to $t = t_2$ is the ratio of the change in displacement to the time taken.

If $s(t)$ is the displacement function then **average velocity** $= \dfrac{s(t_2) - s(t_1)}{t_2 - t_1}$.

On a graph of $s(t)$ against t for the time interval from $t = t_1$ to $t = t_2$, the average velocity is the gradient of a chord through the points $(t_1, s(t_1))$ and $(t_2, s(t_2))$.

In **Chapter 13** we established that the instantaneous rate of change of a quantity is given by its derivative.

If $s(t)$ is the displacement function of an object moving in a straight line, then $v(t) = s'(t) = \lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$ is the **instantaneous velocity** or **velocity function** of the object at time t.

On a graph of $s(t)$ against t, the instantaneous velocity at a particular time is the gradient of the tangent to the graph at that point.

ACCELERATION

If an object moves in a straight line with velocity function $v(t)$ then:

- the **average acceleration** for the time interval from $t = t_1$ to $t = t_2$ is the ratio of the change in velocity to the time taken

$$\textbf{average acceleration} = \frac{v(t_2) - v(t_1)}{t_2 - t_1}$$

- the **instantaneous acceleration** at time t is $a(t) = v'(t) = \lim\limits_{h \to 0} \dfrac{v(t+h) - v(t)}{h}$.

UNITS

Each time we differentiate with respect to time t, we calculate a rate per unit of time. So, for a displacement in metres and time in seconds:

- the units of velocity are m s^{-1}
- the units of acceleration are m s^{-2}.

Discussion

- What is the relationship between the displacement function $s(t)$ and the acceleration function $a(t)$?
- How are the units of velocity and acceleration related to their formulae? You may wish to research “dimensional analysis”.

Example 13

Self Tutor

A particle moves in a straight line with displacement from O given by $s(t) = 3t - t^2$ metres at time t seconds. Find:

a the average velocity for the time interval from $t = 2$ to $t = 5$ seconds

b the average velocity for the time interval from $t = 2$ to $t = 2 + h$ seconds

c $\lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$ and comment on its significance.

a average velocity

$$= \frac{s(5) - s(2)}{5 - 2}$$
$$= \frac{(15 - 25) - (6 - 4)}{3}$$
$$= \frac{-10 - 2}{3}$$
$$= -4 \text{ m s}^{-1}$$

b average velocity

$$= \frac{s(2+h) - s(2)}{2 + h - 2}$$
$$= \frac{3(2+h) - (2+h)^2 - 2}{h}$$
$$= \frac{6 + 3h - 4 - 4h - h^2 - 2}{h}$$
$$= \frac{-h - h^2}{h}$$
$$= -1 - h \text{ m s}^{-1} \quad \text{provided } h \neq 0$$

c

$$\lim_{h \to 0} \frac{s(2+h) - s(2)}{h}$$
$$= \lim_{h \to 0} (-1 - h) \qquad \{\text{since } h \neq 0\}$$
$$= -1 \text{ m s}^{-1}$$

This is the instantaneous velocity of the particle at time $t = 2$ seconds.

EXERCISE 14C.1

1 A particle P moves in a straight line with displacement function $s(t) = t^2 + 3t - 2$ metres, where $t \geqslant 0$, t in seconds.

- **a** Find the average velocity from $t = 1$ to $t = 3$ seconds.
- **b** Find the average velocity from $t = 1$ to $t = 1 + h$ seconds.
- **c** Find the value of $\lim\limits_{h \to 0} \dfrac{s(1+h) - s(1)}{h}$ and comment on its significance.
- **d** Find the average velocity from time t to time $t + h$ seconds and interpret $\lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$.

2 A particle P moves in a straight line with displacement function $s(t) = 5 - 2t^2$ cm, where $t \geqslant 0$, t in seconds.

- **a** Find the average velocity from $t = 2$ to $t = 5$ seconds.
- **b** Find the average velocity from $t = 2$ to $t = 2 + h$ seconds.
- **c** Find the value of $\lim\limits_{h \to 0} \dfrac{s(2+h) - s(2)}{h}$ and state the meaning of this value.
- **d** Interpret $\lim\limits_{h \to 0} \dfrac{s(t+h) - s(t)}{h}$.

3 A particle moves in a straight line with velocity function $v(t) = 2\sqrt{t} + 3$ cm s^{-1}, $t \geqslant 0$.

- **a** Find the average acceleration from $t = 1$ to $t = 4$ seconds.
- **b** Find the average acceleration from $t = 1$ to $t = 1 + h$ seconds.
- **c** Find the value of $\lim\limits_{h \to 0} \dfrac{v(1+h) - v(1)}{h}$. Interpret this value.
- **d** Interpret $\lim\limits_{h \to 0} \dfrac{v(t+h) - v(t)}{h}$.

4 An object moves in a straight line with displacement function $s(t)$ and velocity function $v(t)$, $t \geqslant 0$. State the meaning of:

- **a** $\lim\limits_{h \to 0} \dfrac{s(4+h) - s(4)}{h}$
- **b** $\lim\limits_{h \to 0} \dfrac{v(4+h) - v(4)}{h}$

VELOCITY AND ACCELERATION FUNCTIONS

If a particle P moves in a straight line and its position is given by the displacement function $s(t)$, $t \geqslant 0$, then:

- the **velocity** of P at time t is given by $\boldsymbol{v(t) = s'(t)}$
- the **acceleration** of P at time t is given by $\boldsymbol{a(t) = v'(t) = s''(t)}$
- $s(0)$, $v(0)$, and $a(0)$ give us the position, velocity, and acceleration of the particle at time $t = 0$, and these are called the **initial conditions**.

SIGN INTERPRETATION

Suppose a particle P moves in a straight line with displacement function $s(t)$ relative to an origin O. Its velocity function is $v(t)$ and its acceleration function is $a(t)$.

We can use **sign diagrams** to interpret:

- where the particle is located relative to O
- the direction of motion and where a change of direction occurs
- when the particle's velocity is increasing or decreasing.

SIGNS OF $s(t)$:

$s(t)$	*Interpretation*
$= 0$	P is at O
> 0	P is located to the right of O
< 0	P is located to the left of O

SIGNS OF $v(t)$:

$v(t)$	*Interpretation*
$= 0$	P is instantaneously at rest
> 0	P is moving to the right
< 0	P is moving to the left

SIGNS OF $a(t)$:

$a(t)$	*Interpretation*
> 0	velocity is increasing
< 0	velocity is decreasing
$= 0$	velocity may be a maximum or minimum or possibly constant

ZEROS:

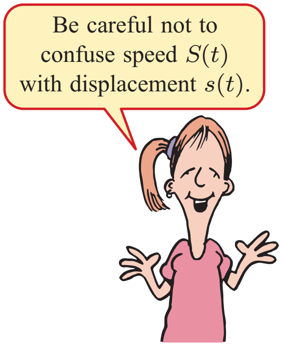
Phrase used in a question	t	s	v	a
initial conditions	0			
at the origin		0		
stationary			0	
reverses			0	
maximum or minimum displacement			0	
constant velocity				0
maximum or minimum velocity				0

When a particle reverses direction, its velocity must change sign.
This corresponds to a local maximum or local minimum distance from the origin O.

SPEED

As we have seen, velocities have size (magnitude) and sign (direction). In contrast, speed simply measures *how fast* something is travelling, regardless of the direction of travel. Speed is a *scalar* quantity which has size but no sign. Speed cannot be negative.

The **speed** at any instant is the magnitude of the object's velocity.
If $S(t)$ represents speed then $S = |v|$.



To determine when the speed $S(t)$ of an object P with displacement $s(t)$ is increasing or decreasing, we use a **sign test**.

- If the signs of $v(t)$ and $a(t)$ are the same (both positive or both negative), then the speed of P is increasing.
- If the signs of $v(t)$ and $a(t)$ are opposite, then the speed of P is decreasing.

Discovery *Displacement, velocity, and acceleration graphs*

In this Discovery we examine the motion of a projectile which is fired in a vertical direction. The projectile is affected by gravity, which is responsible for the projectile's constant acceleration.

We then extend the Discovery to consider other cases of motion in a straight line.

MOTION DEMO

What to do:

1 Click on the icon to examine vertical projectile motion.
Observe first the displacement along the line, then look at the velocity which is the rate of change in displacement. When is the velocity positive and when is it negative?

2 Examine the following graphs and comment on their shapes:
- *displacement* v *time*
- *velocity* v *time*
- *acceleration* v *time*

3 Pick from the menu or construct functions of your own choosing to investigate the relationship between displacement, velocity, and acceleration.

Example 14

Self Tutor

A particle moves in a straight line with position relative to O given by $s(t) = t^3 - 3t + 1$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find expressions for the particle's velocity and acceleration, and draw sign diagrams for each of them.

b Find the initial conditions and hence describe the motion at this instant.

c Describe the motion of the particle at $t = 2$ seconds.

d Find the position of the particle when the changes in direction occur.

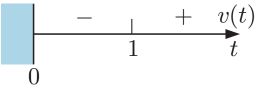
e Draw a motion diagram for the particle.

f For what time interval is the particle's speed increasing?

g What is the total distance travelled in the time from $t = 0$ to $t = 2$ seconds?

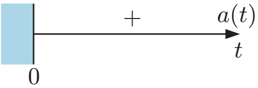
a
$$\begin{aligned} s(t) &= t^3 - 3t + 1 \text{ cm} \\ \therefore\ v(t) &= 3t^2 - 3 \qquad \{\text{as } v(t) = s'(t)\} \\ &= 3(t^2 - 1) \\ &= 3(t+1)(t-1) \text{ cm s}^{-1} \end{aligned}$$

which has sign diagram:



and $a(t) = 6t \text{ cm s}^{-2}$ $\{\text{as } a(t) = v'(t)\}$

which has sign diagram:



Since $t \geqslant 0$, the stationary point at $t = -1$ is not required.

b When $t = 0$, $s(0) = 1$ cm

$v(0) = -3 \text{ cm s}^{-1}$

$a(0) = 0 \text{ cm s}^{-2}$

$\therefore$ the particle is 1 cm to the right of O, moving to the left at a speed of 3 cm s^{-1}.

c When $t = 2$, $s(2) = 8 - 6 + 1 = 3$ cm

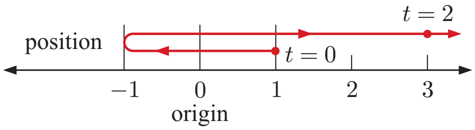
$v(2) = 12 - 3 = 9 \text{ cm s}^{-1}$

$a(2) = 12 \text{ cm s}^{-2}$

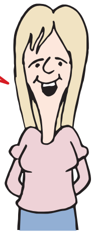
$\therefore$ the particle is 3 cm to the right of O, moving to the right at a speed of 9 cm s^{-1}.
Since a and v have the same sign, the speed of the particle is increasing.

d Since $v(t)$ changes sign when $t = 1$, a change of direction occurs at this instant.
$s(1) = 1 - 3 + 1 = -1$, so the particle changes direction when it is 1 cm to the left of O.

e



The motion is actually **on the line**, not above it as shown.



f Speed is increasing when $v(t)$ and $a(t)$ have the same sign. This is for $t \geqslant 1$.

g Total distance travelled $= 2 + 4 = 6$ cm.

In later chapters on integral calculus we will see another technique for finding the distances travelled and displacement over time.

EXERCISE 14C.2

1 An object moves in a straight line with position given by $s(t) = t^2 - 4t + 3$ cm from O, where t is in seconds, $t \geqslant 0$.

- **a** Find expressions for the object's velocity and acceleration, and draw sign diagrams for each function.
- **b** Find the initial conditions and explain what is happening to the object at that instant.
- **c** Describe the motion of the object at time $t = 2$ seconds.
- **d** At what time does the object reverse direction? Find the position of the object at this instant.
- **e** Draw a motion diagram for the object.
- **f** For what time intervals is the speed of the object decreasing?

2 A stone is projected vertically so that its position above ground level after t seconds is given by $s(t) = 98t - 4.9t^2$ metres, $t \geqslant 0$.

- **a** Find the velocity and acceleration functions for the stone, and draw sign diagrams for each function.
- **b** Find the initial position and velocity of the stone.
- **c** Describe the stone's motion at times $t = 5$ and $t = 12$ seconds.
- **d** Find the maximum height reached by the stone.
- **e** Find the time taken for the stone to hit the ground.

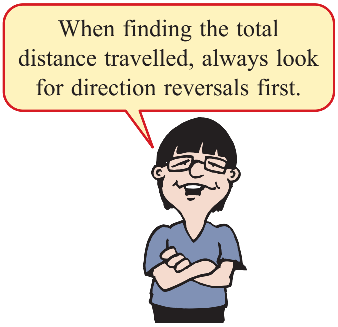
3 When a ball is thrown, its height above the ground is given by $s(t) = 1.2 + 28.1t - 4.9t^2$ metres where t is the time in seconds.

- **a** From what distance above the ground was the ball released?
- **b** Find $s'(t)$ and state what it represents.
- **c** Find t when $s'(t) = 0$. What is the significance of this result?
- **d** What is the maximum height reached by the ball?
- **e** Find the ball's speed:
  - **i** when released
  - **ii** at $t = 2$ s
  - **iii** at $t = 5$ s.

  State the significance of the sign of the derivative $s'(t)$.
- **f** How long will it take for the ball to hit the ground?
- **g** What is the significance of $s''(t)$?

4 The position of a particle moving along the x-axis is given by $x(t) = t^3 - 9t^2 + 24t$ metres where t is in seconds, $t \geqslant 0$.

- **a** Draw sign diagrams for the particle's velocity and acceleration functions.
- **b** Find the position of the particle at the times when it reverses direction, and hence draw a motion diagram for the particle.
- **c** At what times is the particle's:
  - **i** speed decreasing
  - **ii** velocity decreasing?
- **d** Find the total distance travelled by the particle in the first 5 seconds of motion.



5 A particle P moves in a straight line with displacement function $s(t) = 100t + 200e^{-\frac{t}{5}}$ cm, where t is the time in seconds, $t \geqslant 0$.

- **a** Find the velocity and acceleration functions.
- **b** Find the initial position, velocity, and acceleration of P.
- **c** Sketch the graph of the velocity function.
- **d** Find when the velocity of P is 80 cm per second.

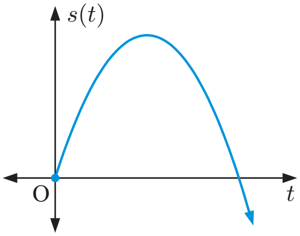
6 A particle P moves along the x-axis with position given by $x(t) = 1 - 2\cos t$ cm where t is the time in seconds.

- **a** State the initial position, velocity, and acceleration of P.
- **b** Describe the motion when $t = \frac{\pi}{4}$ seconds.
- **c** Find the times when the particle reverses direction on $0 < t < 2\pi$, and find the position of the particle at these instants.
- **d** When is the particle's speed increasing on $0 \leqslant t \leqslant 2\pi$?

7 In an experiment, an object is fired vertically from the earth's surface. From the results, a two-dimensional graph of the position $s(t)$ metres above the earth's surface is plotted, where t is the time in seconds. It is noted that the graph is *parabolic*.

Assuming a constant gravitational acceleration g and an initial velocity of $v(0)$, show that:

- **a** $v(t) = v(0) + gt$
- **b** $s(t) = v(0) \times t + \frac{1}{2}gt^2$.

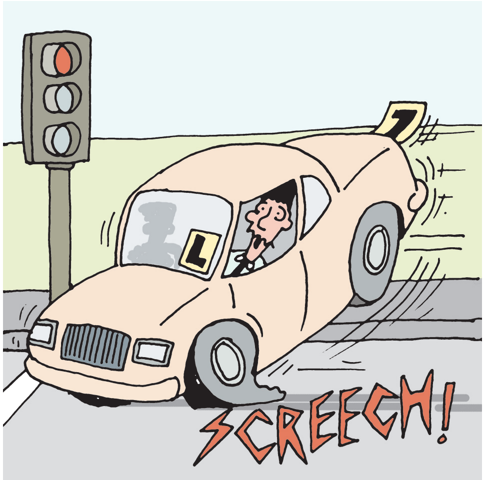


8 The table alongside shows data from a driving test in the United Kingdom.

A driver is travelling with constant speed. In response to a red light they must first react and press the brake. During this time the car travels a *thinking distance*. Once the brake is applied, the car travels a further *braking distance* before it comes to rest.

Speed (km h^{-1})	*Thinking distance* (m)	*Braking distance* (m)
32	6	6
48	9	14
64	12	24
80	15	38
96	18	55
112	21	75

- **a** Using the data from the driving test, find the reaction time for the driver at 96 km h^{-1}.
- **b** The distance $S(t)$ travelled by an object moving initially at speed u m s^{-1}, subject to constant acceleration a m s^{-2}, is $S(t) = ut + \frac{1}{2}at^2$ m.
  - **i** Differentiate this formula with respect to time.
  - **ii** Hence calculate the time taken for the object to be at rest.
  - **iii** Using the data from the driving test, find the braking acceleration for the driver at 96 km h^{-1}.
  - **iv** Show that in general, an object starting at speed u comes to rest in a distance $-\frac{1}{2}\frac{u^2}{a}$ m.
  - **v** If a driver doubles their speed, what happens to their braking distance?



D RATES OF CHANGE

There are countless examples in the real world where quantities vary with time, or with respect to some other variable.

For example:
- temperature varies continuously
- the height of a tree varies as it grows
- the prices of stocks and shares vary with each day's trading.

We have already seen that if $y = f(x)$ then $f'(x)$ or $\frac{dy}{dx}$ is the gradient of the tangent to $y = f(x)$ at the given point.

$\frac{dy}{dx}$ gives the **rate of change in y with respect to x.**

We can therefore use the derivative of a function to tell us the **rate** at which something is happening.

For example:

- $\frac{dH}{dt}$ or $H'(t)$ could be the instantaneous rate of ascent of a person in a Ferris wheel.
  It might have units metres per second or m s^{-1}.
- $\frac{dC}{dt}$ or $C'(t)$ could be a person's instantaneous rate of change in lung capacity.
  It might have units litres per second or L s^{-1}.

Example 15

Self Tutor

According to a psychologist, the ability of a person to understand spatial concepts is given by $A = \frac{1}{3}\sqrt{t}$ where t is the age in years, $5 \leqslant t \leqslant 18$.

a Find the rate of improvement in ability to understand spatial concepts when a person is:
  i 9 years old **ii** 16 years old.

b Show that $\frac{dA}{dt} > 0$ for $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

c Show that $\frac{d^2A}{dt^2} < 0$ for $5 \leqslant t \leqslant 18$. Comment on the significance of this result.

a $A = \frac{1}{3}\sqrt{t} = \frac{1}{3}t^{\frac{1}{2}}$

$\therefore \frac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}} = \frac{1}{6\sqrt{t}}$

i When $t = 9$, $\frac{dA}{dt} = \frac{1}{18}$

$\therefore$ the rate of improvement is $\frac{1}{18}$ units per year for a 9 year old.

ii When $t = 16$, $\frac{dA}{dt} = \frac{1}{24}$

$\therefore$ the rate of improvement is $\frac{1}{24}$ units per year for a 16 year old.

b Since $\sqrt{t}$ is never negative, $\frac{1}{6\sqrt{t}}$ is never negative

$\therefore \frac{dA}{dt} > 0$ for all $5 \leqslant t \leqslant 18$.

This means that the ability to understand spatial concepts increases with age.

c $\frac{dA}{dt} = \frac{1}{6}t^{-\frac{1}{2}}$

$\therefore \frac{d^2A}{dt^2} = -\frac{1}{12}t^{-\frac{3}{2}} = -\frac{1}{12t\sqrt{t}}$

$\therefore \frac{d^2A}{dt^2} < 0$ for all $5 \leqslant t \leqslant 18$.

This means that while the ability to understand spatial concepts increases with age, the rate of increase slows down with age.

You are encouraged to use technology to graph each function you need to consider. This is often useful in interpreting results.

GRAPHING PACKAGE

EXERCISE 14D

1 The estimated future profits of a small business are given by $P(t) = 2t^2 - 12t + 118$ thousand dollars, where t is the time in years from now.

- **a** What is the current annual profit?
- **b** Find $\frac{dP}{dt}$ and state its units.
- **c** Explain the significance of $\frac{dP}{dt}$.
- **d** For what values of t will the profit:
  - **i** decrease **ii** increase on the previous year?
- **e** What is the minimum profit and when does it occur?
- **f** Find $\frac{dP}{dt}$ when $t = 4$, 10, and 25. What do these figures represent?

2 The quantity of a chemical in human skin which is responsible for its 'elasticity' is given by $Q = 100 - 10\sqrt{t}$ where t is the age of a person in years.

- **a** Find Q at:
  - **i** $t = 0$ **ii** $t = 25$ **iii** $t = 100$ years.
- **b** At what rate is the quantity of the chemical changing at the age of:
  - **i** 25 years **ii** 50 years?
- **c** Show that the quantity of the chemical is decreasing for all $t > 0$.

3 The height of *pinus radiata*, grown in ideal conditions, is given by $H = 20 - \frac{97.5}{t+5}$ metres, where t is the number of years after the tree was planted from an established seedling.

- **a** How high was the tree at the time of its planting?
- **b** Find the height of the tree after 4, 8, and 12 years.
- **c** Find the rate at which the tree is growing after 0, 5, and 10 years.
- **d** Show that $\frac{dH}{dt} > 0$ for all $t \geqslant 0$.

  What is the significance of this result?

Example 16

Self Tutor

The cost in dollars of producing x items in a factory each day is given by

$$C(x) = \underbrace{0.000\,13x^3 + 0.002x^2}_{\text{labour}} + \underbrace{5x}_{\text{raw materials}} + \underbrace{2200}_{\text{fixed costs}}$$

a Find $C'(x)$, which is called the marginal cost function.

b Find the marginal cost when 150 items are produced. Interpret this result.

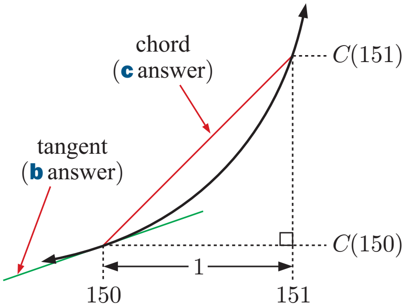
c Find $C(151) - C(150)$. Compare this with the answer in **b**.

a The marginal cost function is
$C'(x) = 0.000\,39x^2 + 0.004x + 5$ dollars per item.

b $C'(150) = \$14.38$

This is the rate at which the costs are increasing with respect to the production level x when 150 items are made per day.

It gives an estimate of the cost of making the 151st item each day.



c $C(151) - C(150) \approx \$3448.19 - \$3433.75$
$\approx \$14.44$

This is the actual cost of making the 151st item each day, so the answer in **b** gives a good estimate.

4 Seablue make denim jeans. The cost model for making x pairs per day is

$$C(x) = 0.0003x^3 + 0.02x^2 + 4x + 2250 \text{ dollars.}$$

a Find the marginal cost function $C'(x)$.

b Find $C'(220)$. What does it estimate?

c Find $C(221) - C(220)$. What does this represent?

d Find $C''(x)$ and the value of x when $C''(x) = 0$. What is the significance of this point?

5

The total cost of running a train from Paris to Marseille is given by $C(v) = \frac{1}{5}v^2 + \dfrac{200\,000}{v}$ euros where v is the average speed of the train in km h^{-1}.

a Find the total cost of the journey if the average speed is:

i 50 km h^{-1} **ii** 100 km h^{-1}.

b Find the rate of change in the cost of running the train at speeds of:

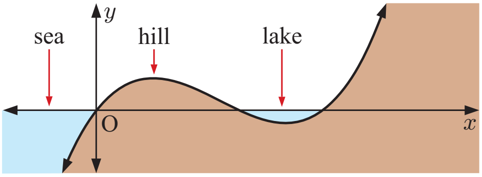
i 30 km h^{-1} **ii** 90 km h^{-1}.

c At what speed will the cost be a minimum?

6 A tank contains 50 000 litres of water. The tap is left fully on and all the water drains from the tank in 80 minutes. The volume of water remaining in the tank after t minutes is given by $V = 50\,000\left(1 - \frac{t}{80}\right)^2$ litres where $0 \leqslant t \leqslant 80$.

- **a** Find $\frac{dV}{dt}$ and draw the graph of $\frac{dV}{dt}$ against t.
- **b** At what time was the outflow fastest?
- **c** Show that $\frac{d^2V}{dt^2}$ is always constant and positive.

  Interpret this result.

7 

Alongside is a land and sea profile where the x-axis is sea level. The function $y = \frac{1}{10}x(x-2)(x-3)$ km gives the height of the land or sea bed relative to sea level at distance x km from the shore line.

- **a** Find where the lake is located relative to the shore line of the sea.
- **b** Find $\frac{dy}{dx}$ and interpret its value when $x = \frac{1}{2}$ km and when $x = 1\frac{1}{2}$ km.
- **c** Find the deepest point of the lake, and the depth at this point.

8 A radioactive substance decays according to the formula $W = 20e^{-kt}$ grams where t is the time in hours.

- **a** Find k given that after 50 hours the weight is 10 grams.
- **b** Find the weight of radioactive substance present:
  - **i** initially
  - **ii** after 24 hours
  - **iii** after 1 week.
- **c** How long will it take for the weight to reach 1 gram?
- **d** Find the rate of radioactive decay at:
  - **i** $t = 100$ hours
  - **ii** $t = 1000$ hours.
- **e** Show that $\frac{dW}{dt}$ is proportional to the weight of substance remaining.

9 The temperature of a liquid after being placed in a refrigerator is given by $T = 5 + 95e^{-kt}$ °C where k is a positive constant and t is the time in minutes.

- **a** Find k if the temperature of the liquid is 20°C after 15 minutes.
- **b** What was the temperature of the liquid when it was first placed in the refrigerator?
- **c** Show that $\frac{dT}{dt} = c(T - 5)$ for some constant c. Find the value of c.
- **d** At what rate is the temperature changing at:
  - **i** $t = 0$ mins
  - **ii** $t = 10$ mins
  - **iii** $t = 20$ mins?

10 The height of a shrub t years after it is planted is given by $H(t) = 20\ln(3t + 2) + 30$ cm, $t \geqslant 0$.

- **a** How high was the shrub when it was planted?
- **b** How long will it take for the shrub to reach a height of 1 m?
- **c** At what rate is the shrub's height changing:
  - **i** 3 years after being planted
  - **ii** 10 years after being planted?

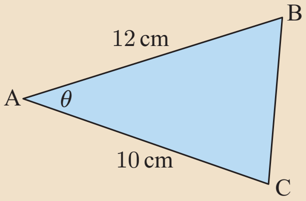
11 In the conversion of sugar solution to alcohol, the chemical reaction obeys the law $A = s(1 - e^{-kt})$, $t \geqslant 0$ where t is the number of hours after the reaction commences, s is the original sugar concentration (%), and A is the alcohol produced, in litres.

- **a** Find A when $t = 0$.
- **b** Suppose $s = 10$ and $A = 5$ after 3 hours.
  - **i** Find k.
  - **ii** Find the speed of the reaction at time 5 hours.
- **c** Show that the speed of the reaction is proportional to $A - s$.

Example 17

Self Tutor

Find the rate of change in the area of triangle ABC as θ changes, at the time when $\theta = 60°$.



Area $A = \frac{1}{2} \times 10 \times 12 \times \sin\theta$ $\{\text{Area} = \frac{1}{2}bc\sin A\}$

$\therefore A = 60\sin\theta$ cm^2

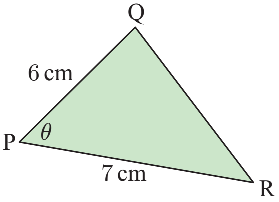
$$\therefore \frac{dA}{d\theta} = 60\cos\theta$$

When $\theta = \frac{\pi}{3}$, $\cos\theta = \frac{1}{2}$

$$\therefore \frac{dA}{d\theta} = 30 \text{ cm}^2 \text{ per radian}$$

θ must be in **radians** so the dimensions are correct.

12 Find the rate of change in the area of triangle PQR as θ changes, at the time when $\theta = 45°$.

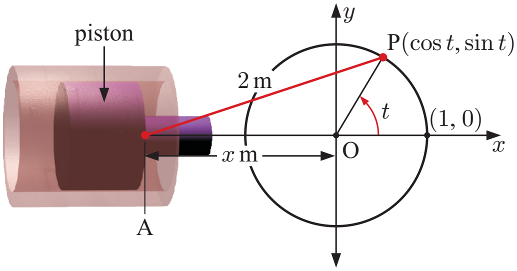


13 On the Indonesian coast, the depth of water at time t hours after midnight is given by $d = 9.3 + 6.8\cos(0.507t)$ metres.

- **a** Find the rate of change in the depth of water at 8:00 am.
- **b** Is the tide rising or falling at this time?

14 A piston is operated by rod [AP] attached to a flywheel of radius 1 m. AP = 2 m. P has coordinates $(\cos t, \sin t)$ and point A is $(-x, 0)$.

- **a** Show that $x = \sqrt{4 - \sin^2 t} - \cos t$.
- **b** Find the rate at which x is changing at the instant when:
  - **i** $t = 0$
  - **ii** $t = \frac{\pi}{2}$
  - **iii** $t = \frac{2\pi}{3}$



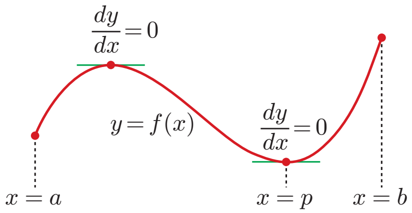
E OPTIMISATION

There are many problems for which we need to find the **maximum** or **minimum** value of a function. The solution is often referred to as the **optimum** solution and the process is called **optimisation**.

The maximum or minimum value does not always occur when the first derivative is zero.

It is essential to also examine the values of the function at the endpoint(s) of the interval under consideration for global maxima and minima.

For example:



The maximum value of y occurs at the endpoint $x=b$.

The minimum value of y occurs at the local minimum $x=p$.

OPTIMISATION PROBLEM SOLVING METHOD

Step 1: Draw a large, clear diagram of the situation.

Step 2: Construct a formula with the variable to be **optimised** as the subject. It should be written in terms of **one** convenient variable, for example x. You should write down what domain restrictions there are on x.

Step 3: Find the **first derivative** and find the values of x which make the first derivative **zero**.

Step 4: For each stationary point, use a sign diagram to determine if you have a local maximum or local minimum.

Step 5: Identify the optimum solution, also considering endpoints where appropriate.

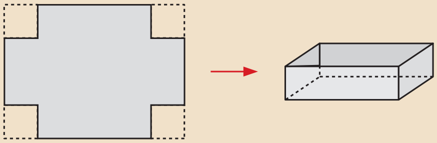
Step 6: Write your answer in a sentence, making sure you specifically answer the question.

Example 18

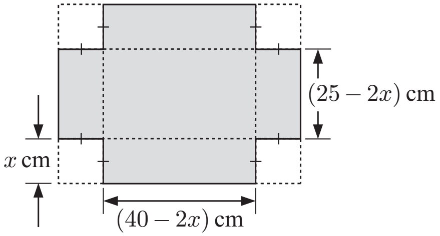
Self Tutor

A rectangular cake dish is made by cutting out squares from the corners of a 25 cm by 40 cm rectangle of tin-plate, and then folding the metal to form the container.

What size squares must be cut out to produce the cake dish of maximum volume?



Step 1: Let x cm be the side lengths of the squares that are cut out.



Step 2: Volume = length × width × depth

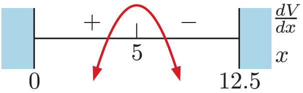
$$\begin{aligned}&= (40-2x)(25-2x)x\\ &= (1000-80x-50x+4x^2)x\\ &= 1000x-130x^2+4x^3 \quad \text{cm}^3\end{aligned}$$

Since the side lengths must be positive, $x>0$ and $25-2x>0$.

$\therefore \quad 0<x<12.5$

Step 3:
$$\frac{dV}{dx} = 12x^2 - 260x + 1000$$
$$= 4(3x^2 - 65x + 250)$$
$$= 4(3x - 50)(x - 5)$$
$$\therefore \quad \frac{dV}{dx} = 0 \quad \text{when} \quad x = \tfrac{50}{3} = 16\tfrac{2}{3} \quad \text{or} \quad x = 5$$

Step 4: $\frac{dV}{dx}$ has sign diagram:



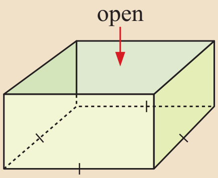
Step 5: There is a local maximum when $x = 5$. This is the global maximum for the given domain.

Step 6: The maximum volume is obtained when $x = 5$, which is when 5 cm squares are cut from the corners.

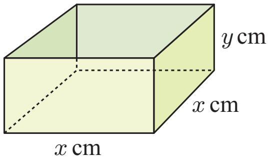
Example 19

Self Tutor

A 4 litre container must have a square base, vertical sides, and an open top. Find the most economical shape which minimises the surface area of material needed.



Step 1:



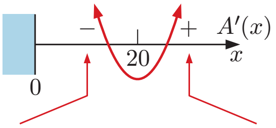
Let the base lengths be x cm and the depth be y cm.

The volume $V = \text{length} \times \text{width} \times \text{depth}$
$$\therefore \quad V = x^2 y$$
$$\therefore \quad 4000 = x^2 y \quad .... \ (1) \qquad \{1 \text{ litre} \equiv 1000 \text{ cm}^3\}$$

Step 2: The total surface area
$$A = \text{area of base} + 4(\text{area of one side})$$
$$= x^2 + 4xy$$
$$= x^2 + 4x\left(\frac{4000}{x^2}\right) \qquad \{\text{using } (1)\}$$
$$\therefore \quad A(x) = x^2 + 16\,000x^{-1} \quad \text{where} \quad x > 0$$

Step 3:
$$A'(x) = 2x - 16\,000x^{-2}$$
$$\therefore \quad A'(x) = 0 \quad \text{when} \quad 2x = \frac{16\,000}{x^2}$$
$$\therefore \quad 2x^3 = 16\,000$$
$$\therefore \quad x = \sqrt[3]{8000} = 20$$

Step 4: $A'(x)$ has sign diagram:

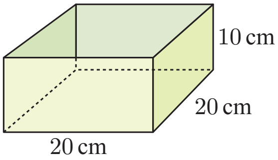


If $x = 10$,
$$A'(10) = 20 - \frac{16\,000}{100}$$
$$= 20 - 160$$
$$= -140$$

If $x = 30$,
$$A'(30) = 60 - \frac{16\,000}{900}$$
$$\approx 60 - 17.8$$
$$\approx 42.2$$

Step 5: The minimum material is used to make the container when $x = 20$ and $y = \frac{4000}{20^2} = 10$.

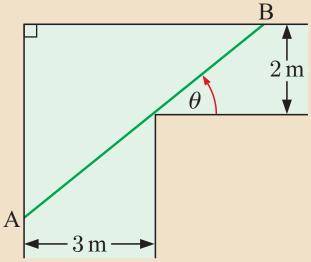
Step 6: The most economical shape has a square base $20 \text{ cm} \times 20 \text{ cm}$, and height 10 cm.



Example 20

Self Tutor

Two corridors meet at right angles and are 2 m and 3 m wide respectively. θ is the angle marked on the given figure. [AB] is a thin metal tube which must be kept horizontal and cannot be bent as it moves around the corner from one corridor to the other.



a Show that the length AB is given by $L = \frac{3}{\cos\theta} + \frac{2}{\sin\theta}$.

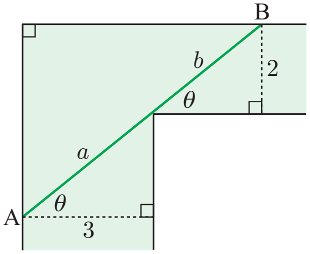
b Show that $\frac{dL}{d\theta} = 0$ when $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right) \approx 41.1°$.

c Find L when $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right)$ and comment on the significance of this value.

a $\cos\theta = \frac{3}{a}$ and $\sin\theta = \frac{2}{b}$

$\therefore\ a = \frac{3}{\cos\theta}$ and $b = \frac{2}{\sin\theta}$

$\therefore\ L = a + b = \frac{3}{\cos\theta} + \frac{2}{\sin\theta}$



DEMO

b $L = 3[\cos\theta]^{-1} + 2[\sin\theta]^{-1}$

$$\therefore\ \frac{dL}{d\theta} = -3[\cos\theta]^{-2}(-\sin\theta) - 2[\sin\theta]^{-2}\cos\theta$$

$$= \frac{3\sin\theta}{\cos^2\theta} - \frac{2\cos\theta}{\sin^2\theta}$$

$$= \frac{3\sin^3\theta - 2\cos^3\theta}{\cos^2\theta\sin^2\theta}$$

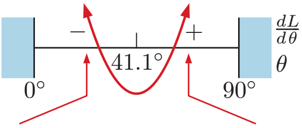
Thus $\frac{dL}{d\theta} = 0$ when $3\sin^3\theta = 2\cos^3\theta$

$\therefore\ \tan^3\theta = \frac{2}{3}$

$\therefore\ \tan\theta = \sqrt[3]{\frac{2}{3}}$

$\therefore\ \theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right) \approx 41.1°$

c Sign diagram of $\frac{dL}{d\theta}$:



When $\theta = 30°$, $\frac{dL}{d\theta} \approx -4.93 < 0$

When $\theta = 60°$, $\frac{dL}{d\theta} \approx 9.06 > 0$

Thus, AB is minimised when $\theta \approx 41.1°$. At this time $L \approx 7.02$ metres. Ignoring the width of the rod, the greatest length of rod able to be horizontally carried around the corner is 7.02 m.

GRAPHING PACKAGE

Use **calculus techniques** to answer the following problems.

In cases where finding the zeros of the derivatives is difficult you may use the **graphing package** to help you.

EXERCISE 14E

1 When a manufacturer makes x items per day, the cost function is $C(x) = 720 + 4x + 0.02x^2$ dollars, and the price function is $p(x) = 15 - 0.002x$ dollars per item. Find the production level that will maximise profits.

2 A duck farmer wishes to build a rectangular enclosure of area 100 m^2. The farmer must purchase wire netting for three of the sides, as the fourth side is an existing fence. Naturally, the farmer wishes to minimise the length (and therefore cost) of fencing required to complete the job.

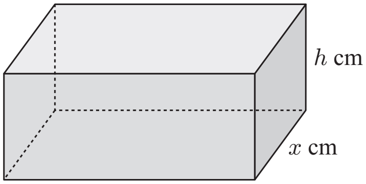
- **a** If the shorter sides have length x m, show that the required length of wire netting to be purchased is $L = 2x + \dfrac{100}{x}$.
- **b** Find the minimum value of L and the corresponding value of x when this occurs.
- **c** Sketch the optimum situation, showing all dimensions.

3 The total cost of producing x blankets per day is $\frac{1}{4}x^2 + 8x + 20$ dollars, and for this production level each blanket may be sold for $(23 - \frac{1}{2}x)$ dollars.
How many blankets should be produced per day to maximise the total profit?

4 The cost of running a boat is $\left(\dfrac{v^2}{10} + 22\right)$ dollars per hour, where v km h^{-1} is the speed of the boat.
Find the speed which will minimise the total cost per kilometre.

5 A psychologist claims that the ability A to memorise simple facts during infancy years can be calculated using the formula $A(t) = t\ln t + 1$ where $0 < t \leqslant 5$, t being the age of the child in years. At what age is the child's memorising ability a minimum?

6 Radioactive waste is to be disposed of in fully enclosed lead boxes of inner volume 200 cm^3. The base of the box has dimensions in the ratio $2 : 1$.



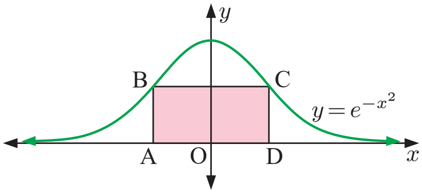
- **a** Show that $x^2h = 100$.
- **b** Show that the inner surface area of the box is given by $A(x) = 4x^2 + \dfrac{600}{x}$ cm^2.
- **c** Find the minimum inner surface area of the box and the corresponding value of x.
- **d** Sketch the optimum box shape, showing all dimensions.

7 A manufacturer of electric kettles performs a cost control study. They discover that to produce x kettles per day, the cost per kettle is given by

$$C(x) = 4\ln x + \left(\frac{30 - x}{10}\right)^2 \quad \text{dollars}$$

with a minimum production capacity of 10 kettles per day.
How many kettles should be manufactured to keep the cost per kettle to a minimum?

8 Infinitely many rectangles which sit on the x-axis can be inscribed under the curve $y = e^{-x^2}$.
Determine the coordinates of C such that rectangle ABCD has maximum area.

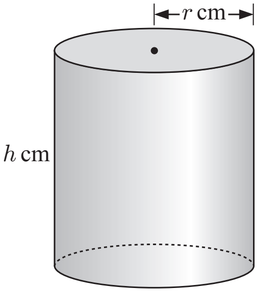


9 Consider the manufacture of cylindrical tin cans of 1 L capacity, where the cost of the metal used is to be minimised.

a Explain why the height h is given by $h = \dfrac{1000}{\pi r^2}$ cm.

b Show that the total surface area A is given by
$A = 2\pi r^2 + \dfrac{2000}{r}$ cm^2.

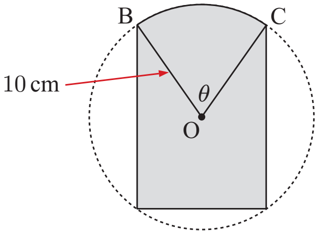
c Find the dimensions of the can which make A as small as possible.



10 A circular piece of tinplate of radius 10 cm has 3 segments removed as illustrated. The angle θ is measured in radians.

a Show that the remaining area is given by
$A = 50(\theta + 3\sin\theta)$ cm^2.

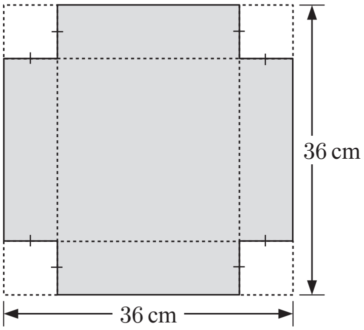
b Find θ such that the area A is a maximum, and find the area A in this case.



11 Sam has sheets of metal which are 36 cm by 36 cm square. He wants to cut out identical squares which are x cm by x cm from the corners of each sheet. He will then bend the sheets along the dashed lines to form an open container.

a Show that the volume of the container is given by
$V(x) = x(36 - 2x)^2$ cm^3.

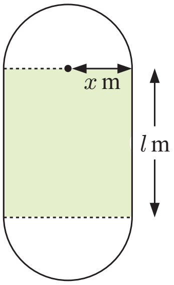
b What sized squares should be cut out to produce the container of greatest capacity?



12 An athletics track has two 'straights' of length l m, and two semicircular ends of radius x m. The perimeter of the track is 400 m.

a Show that $l = 200 - \pi x$ and write down the possible values that x may have.

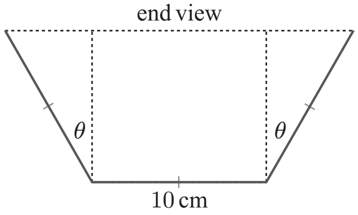
b What values of l and x maximise the shaded rectangle inside the track? What is this maximum area?



13 A small population of wasps is observed. After t weeks the population is modelled by $P(t) = \dfrac{50\,000}{1 + 1000e^{-0.5t}}$ wasps, where $0 \leqslant t \leqslant 25$.
Find when the wasp population is growing fastest.

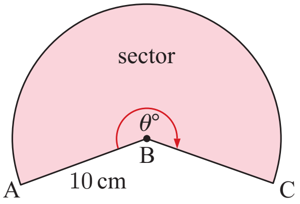
14 When a new pain killing injection is administered, the effect is modelled by $E(t) = 750te^{-1.5t}$ units, where $t \geqslant 0$ is the time in hours after the injection.
At what time is the drug most effective?

15 A symmetrical gutter is made from a sheet of metal 30 cm wide by bending it twice as shown.

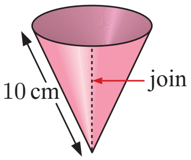


- **a** Deduce that the cross-sectional area of the gutter is given by $A = 100\cos\theta(1 + \sin\theta)$.
- **b** Show that $\dfrac{dA}{d\theta} = 0$ when $\sin\theta = \frac{1}{2}$ or -1.
- **c** For what value of θ does the gutter have maximum carrying capacity? Find the cross-sectional area for this value of θ.

16 A sector of radius 10 cm and angle $\theta°$ is bent to form a conical cup as shown.



becomes

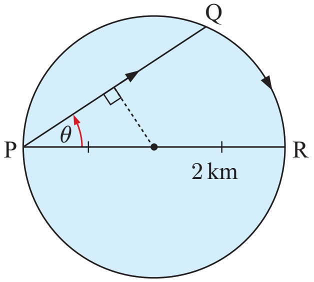


when edges [AB] and [CB] are joined with tape.

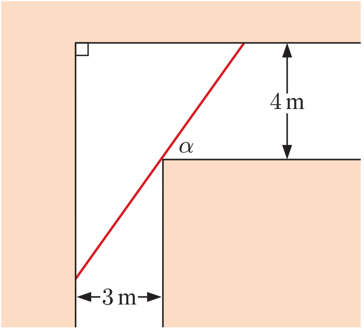
Suppose the resulting cone has base radius r cm and height h cm.

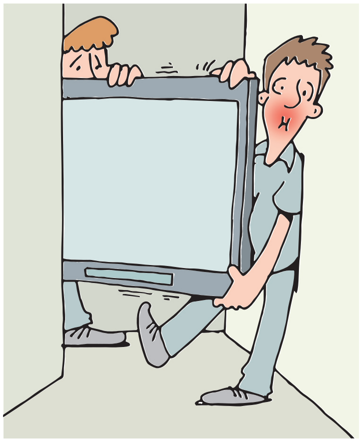
- **a** Show using the sector that arc AC $= \dfrac{\theta\pi}{18}$.
- **b** Explain why $r = \dfrac{\theta}{36}$.
- **c** Show that $h = \sqrt{100 - \left(\frac{\theta}{36}\right)^2}$.
- **d** Find the cone's capacity V in terms of θ only.
- **e** Find θ when $V(\theta)$ is a maximum.

17 Hieu can row a boat across a circular lake of radius 2 km at 3 km h^{-1}. He can walk around the edge of the lake at 5 km h^{-1}.
What is the longest possible time Hieu could take to get from P to R by rowing from P to Q and then walking from Q to R?



18 In a hospital, two corridors 4 m wide and 3 m wide meet at right angles. What is the maximum possible length of an X-ray screen which can be carried upright around the corner?





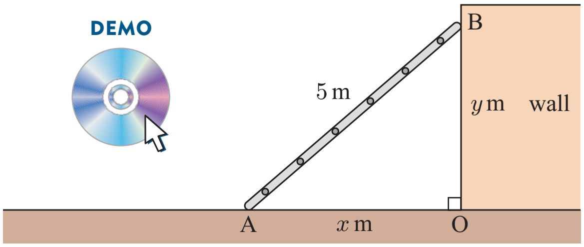
F RELATED RATES

A 5 m ladder rests against a vertical wall at point B. Its feet are at point A on horizontal ground.

The ladder slips and slides down the wall.

Click on the icon to view the motion of the sliding ladder.

DEMO



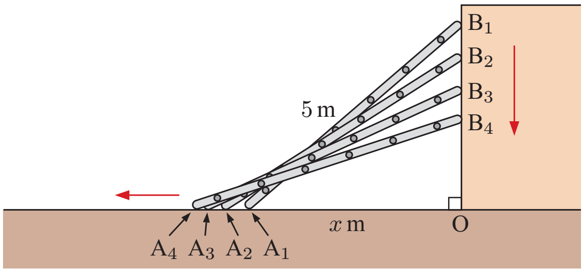
The following diagram shows the positions of the ladder at certain instances.

If AO $= x$ m and OB $= y$ m,
then $x^2 + y^2 = 5^2$. {Pythagoras}

Differentiating this equation with respect to time t gives

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\text{or} \quad x\frac{dx}{dt} + y\frac{dy}{dt} = 0.$$



This equation is called a **differential equation** and describes the motion of the ladder at any instant.

$\frac{dx}{dt}$ is the rate of change in x with respect to time t, and is the speed of A relative to point O.

$\frac{dx}{dt}$ is *positive* as x is increasing.

$\frac{dy}{dt}$ is the rate of change in y with respect to time t, and is the speed at which B moves downwards.

$\frac{dy}{dt}$ is *negative* as y is decreasing.

Problems involving differential equations where one of the variables is time t are called **related rates** problems.

The method for solving related rates problems is:

Step 1: Draw a large, clear **diagram** of the situation. Sometimes two or more diagrams are necessary.

Step 2: Write down the information, label the diagram(s), and make sure you distinguish between the **variables** and the **constants**.

Step 3: Write an **equation** connecting the variables. You will often need to use:

- Pythagoras' theorem
- right angled triangle trigonometry
- similar triangles
- sine and cosine rules.

Step 4: **Differentiate** the equation with respect to t to obtain a **differential equation**.

Step 5: Solve for the **particular case** which is some instant in time.

Warning:

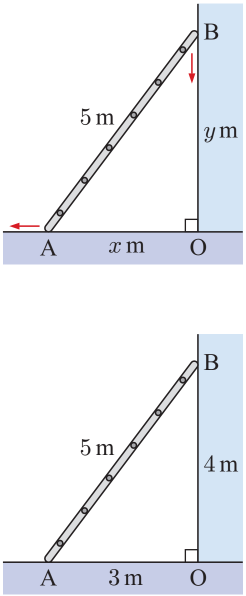
We **must not** substitute values for the particular case too early. Otherwise we will incorrectly treat variables as constants. The differential equation in fully generalised form must be established first.

Example 21

Self Tutor

A 5 m long ladder rests against a vertical wall with its feet on horizontal ground. The feet on the ground slip, and at the instant when they are 3 m from the wall, they are moving at 10 m s^{-1}.

At what speed is the other end of the ladder moving at this instant?



Let OA $= x$ m and OB $= y$ m

$\therefore\ x^2+y^2=5^2$ {Pythagoras}

Differentiating with respect to t gives

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$\therefore\ x\frac{dx}{dt}+y\frac{dy}{dt}=0$$

Particular case:

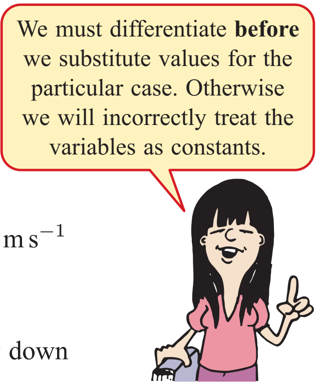
At the instant when $\dfrac{dx}{dt}=10\ \text{m s}^{-1}$,

$$\therefore\ 3(10)+4\frac{dy}{dt}=0$$

$$\therefore\ \frac{dy}{dt}=-\tfrac{15}{2}=-7.5\ \text{m s}^{-1}$$

Thus OB is decreasing at 7.5 m s^{-1}.

$\therefore$ the other end of the ladder is moving down the wall at 7.5 m s^{-1} at that instant.



Example 22

Self Tutor

A cube is expanding so its volume increases at a constant rate of 10 $\text{cm}^3\ \text{s}^{-1}$. Find the rate of change in its total surface area, at the instant when its sides are 20 cm long.

Let x cm be the lengths of the sides of the cube, so the surface area $A=6x^2\ \text{cm}^2$ and the volume $V=x^3\ \text{cm}^3$.

$$\therefore\ \frac{dA}{dt}=12x\frac{dx}{dt} \quad\text{and}\quad \frac{dV}{dt}=3x^2\frac{dx}{dt}$$

Particular case:

At the instant when $x=20$, $\dfrac{dV}{dt}=10$

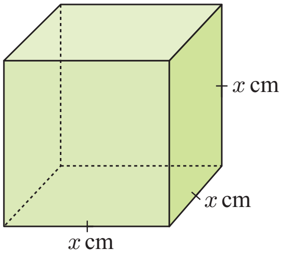
$$\therefore\ 10=3\times 20^2\times\frac{dx}{dt}$$

$$\therefore\ \frac{dx}{dt}=\frac{10}{1200}=\frac{1}{120}\ \text{cm s}^{-1}$$

$$\text{Thus}\quad \frac{dA}{dt}=12\times 20\times\tfrac{1}{120}\ \text{cm}^2\ \text{s}^{-1}$$

$$=2\ \text{cm}^2\ \text{s}^{-1}$$

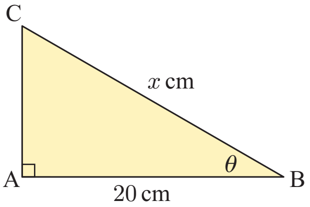
$\therefore$ the surface area is increasing at 2 $\text{cm}^2\ \text{s}^{-1}$.



Example 23

Self Tutor

Triangle ABC is right angled at A, and AB = 20 cm. $\widehat{ABC}$ increases at a constant rate of 1° per minute. At what rate is BC changing at the instant when $\widehat{ABC}$ measures 30°?



Let $\widehat{ABC} = \theta$ and $BC = x$ cm

$$\text{Now} \quad \cos\theta = \frac{20}{x} = 20x^{-1}$$

$$\therefore \quad -\sin\theta\,\frac{d\theta}{dt} = -20x^{-2}\,\frac{dx}{dt}$$

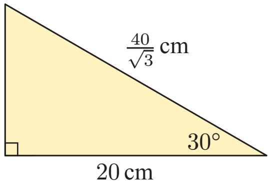
$\frac{d\theta}{dt}$ must be measured in **radians** per time unit.

Particular case:

$$\text{When} \quad \theta = 30°, \quad \cos 30° = \frac{20}{x}$$

$$\therefore \quad \frac{\sqrt{3}}{2} = \frac{20}{x}$$

$$\therefore \quad x = \frac{40}{\sqrt{3}}$$



$$\text{Also,} \quad \frac{d\theta}{dt} = 1° \text{ per min}$$

$$= \tfrac{\pi}{180} \text{ radians per min}$$

$$\text{Thus} \quad -\sin 30° \times \tfrac{\pi}{180} = -20 \times \frac{3}{1600} \times \frac{dx}{dt}$$

$$\therefore \quad -\tfrac{1}{2} \times \tfrac{\pi}{180} = -\tfrac{3}{80}\,\frac{dx}{dt}$$

$$\therefore \quad \frac{dx}{dt} = \tfrac{\pi}{360} \times \tfrac{80}{3} \text{ cm per min}$$

$$\approx 0.2327 \text{ cm per min}$$

$\therefore$ BC is increasing at approximately 0.233 cm per min.

EXERCISE 14F

1 a and b are variables related by the equation $ab^3 = 40$. At the instant when $a = 5$, b is increasing at 1 unit per second. What is happening to a at this instant?

2 The length of a rectangle is decreasing at 1 cm per minute. However, the area of the rectangle remains constant at 100 cm^2. At what rate is the breadth increasing at the instant when the rectangle is a square?

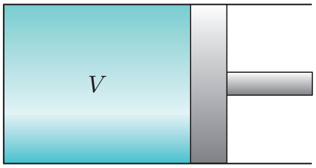
3 A stone is thrown into a lake and a circular ripple moves out at a constant speed of 1 m s^{-1}. Find the rate at which the circle's area is increasing at the instant when:

a $t = 2$ seconds **b** $t = 4$ seconds.

4 Air is pumped into a spherical weather balloon at a constant rate of 6π m^3 per minute. Find the rate of change in its surface area at the instant when the radius of the balloon is 2 m.

5 For a given mass of gas in a piston, $pV^{1.5} = 400$ where p is the pressure in N m^{-2}, and V is the volume in m^3.

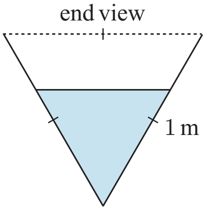
Suppose the pressure increases at a constant rate of 3 N m^{-2} per minute. Find the rate at which the volume is changing at the instant when the pressure is 50 N m^{-2}.



6 Wheat runs from a hole in a silo at a constant rate and forms a conical heap whose base radius is treble its height. After 1 minute, the height of the heap is 20 cm. Find the rate at which the height is rising at this instant.

7 A trough of length 6 m has a uniform cross-section which is an equilateral triangle with sides of length 1 m. Water leaks from the bottom of the trough at a constant rate of 0.1 m^3 per min.

Find the rate at which the water level is falling at the instant when the water is 20 cm deep.

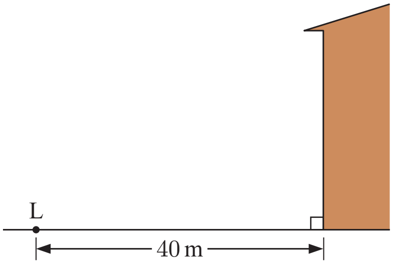


8 Two jet aeroplanes fly on parallel courses which are 12 km apart. Their air speeds are 200 m s^{-1} and 250 m s^{-1} respectively. How fast is the distance between them changing at the instant when the slower jet is 5 km ahead of the faster one?

9 A ground-level floodlight located 40 m from the foot of a building shines in the direction of the building.

A 2 m tall person walks directly from the floodlight towards the building at 1 m s^{-1}. How fast is the person's shadow on the building shortening at the instant when the person is:

- **a** 20 m from the building
- **b** 10 m from the building?



10 A right angled triangle ABC has a fixed hypotenuse [AC] of length 10 cm, and side [AB] increases in length at 0.1 cm s^{-1}. At what rate is $\text{C}\widehat{\text{A}}\text{B}$ decreasing at the instant when the triangle is isosceles?

11 Triangle PQR is right angled at Q, and [PQ] is 6 cm long. [QR] increases in length at 2 cm per minute. Find the rate of change in $\text{Q}\widehat{\text{P}}\text{R}$ at the instant when [QR] is 8 cm long.

Review set 14A

1 Find the equation of the tangent to:

- **a** $y = -2x^2$ at the point where $x = -1$
- **b** $f(x) = 4\ln(2x)$ at the point $(1, 4\ln 2)$
- **c** $f(x) = \dfrac{e^x}{x-1}$ at the point where $x = 2$.

2 The tangent to $y = \dfrac{ax+b}{\sqrt{x}}$ at $x = 1$ is $2x - y = 1$. Find a and b.

3 Suppose $f(x) = x^3 + ax$, $a < 0$ has a turning point when $x = \sqrt{2}$.

- **a** Find a.
- **b** Find the position and nature of all stationary points of $y = f(x)$.
- **c** Sketch the graph of $y = f(x)$.

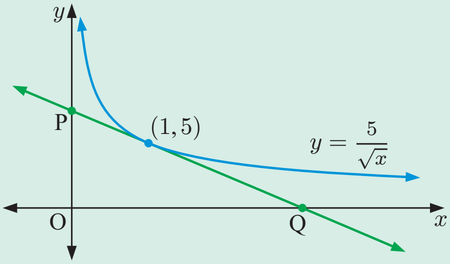
4 Find the equation of the normal to:

a $y = \dfrac{x+1}{x^2-2}$ at the point where $x = 1$ **b** $\sqrt{x+1}$ at the point where $x = 3$.

5 The tangent to $y = x^2\sqrt{1-x}$ at $x = -3$ cuts the axes at points A and B. Determine the area of triangle OAB.

6 The line through A(2, 4) and B(0, 8) is a tangent to $y = \dfrac{a}{(x+2)^2}$. Find a.

7 Find the coordinates of P and Q if PQ is the tangent to $y = \dfrac{5}{\sqrt{x}}$ at (1, 5).



8 Show that $y = 2 - \dfrac{7}{1+2x}$ has no horizontal tangents.

9 Show that the curves whose equations are $y = \sqrt{3x+1}$ and $y = \sqrt{5x - x^2}$ have a common tangent at their point of intersection. Find the equation of this common tangent.

10 Consider the function $f(x) = x + \ln x$.

a Find the values of x for which $f(x)$ is defined.

b Find the sign of $f'(x)$ and comment on its geometrical significance.

c Sketch the graph of $y = f(x)$.

d Find the equation of the normal at the point where $x = 1$.

11 **a** Sketch the graph of $x \mapsto \dfrac{4}{x}$ for $x > 0$.

b Find the equation of the tangent to the function at the point where $x = k$, $k > 0$.

c If the tangent in **b** cuts the x-axis at A and the y-axis at B, find the coordinates of A and B.

d What can be deduced about the area of triangle OAB?

e Find k if the normal to the curve at $x = k$ passes through the point (1, 1).

12 A particle P moves in a straight line with position relative to the origin O given by $s(t) = 2t^3 - 9t^2 + 12t - 5$ cm, where t is the time in seconds, $t \geqslant 0$.

a Find expressions for the particle's velocity and acceleration and draw sign diagrams for each of them.

b Find the initial conditions.

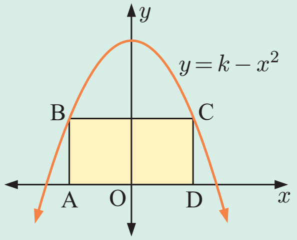
c Describe the motion of the particle at time $t = 2$ seconds.

d Find the times and positions where the particle changes direction.

e Draw a diagram to illustrate the motion of P.

f Determine the time intervals when the particle's speed is increasing.

13 Rectangle ABCD is inscribed within the parabola $y = k - x^2$ and the x-axis, as shown.

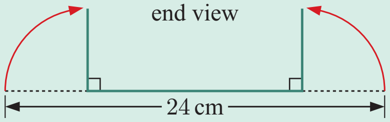


- **a** If $\text{OD} = x$, show that the rectangle ABCD has area function $A(x) = 2kx - 2x^3$.
- **b** If the area of ABCD is a maximum when $\text{AD} = 2\sqrt{3}$, find k.

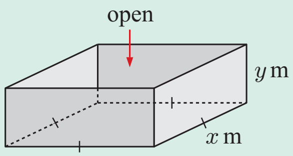
14 A particle moves in a straight line along the x-axis with position given by $x(t) = 3 + \sin(2t)$ cm after t seconds.

- **a** Find the initial position, velocity, and acceleration of the particle.
- **b** Find the times when the particle changes direction during $0 \leqslant t \leqslant \pi$ seconds.
- **c** Find the total distance travelled by the particle in the first π seconds.

15 A rectangular gutter is formed by bending a 24 cm wide sheet of metal as shown.
Where must the bends be made in order to maximise the capacity of the gutter?



16 A manufacturer of open steel boxes has to make one with a square base and a capacity of 1 m^3. The steel costs $2 per square metre.

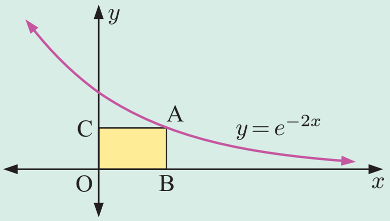


- **a** If the base measures x m by x m and the height is y m, find y in terms of x.
- **b** Hence, show that the total cost of the steel is $C(x) = 2x^2 + \dfrac{8}{x}$ dollars.
- **c** Find the dimensions of the box which would cost the least in steel to make.

17 A particle P moves in a straight line with position from O given by $s(t) = 15t - \dfrac{60}{(t+1)^2}$ cm, where t is the time in seconds, $t \geqslant 0$.

- **a** Find velocity and acceleration functions for P's motion.
- **b** Describe the motion of P at $t = 3$ seconds.
- **c** For what values of t is the particle's speed increasing?

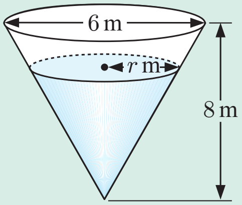
18 Infinitely many rectangles can be inscribed under the curve $y = e^{-2x}$ as shown.
Determine the coordinates of A such that the rectangle OBAC has maximum area.



19 A man on a jetty pulls a boat directly towards him so the rope is coming in at a rate of 20 metres per minute. The rope is attached to the boat 1 m above water level, and the man's hands are 6 m above the water level. How fast is the boat approaching the jetty at the instant when it is 15 m from the jetty?

20 Water exits a conical tank at a constant rate of 0.2 m^3 per minute. Suppose the surface of the water has radius r.

- **a** Find $V(r)$, the volume of the water remaining in the tank.
- **b** Find the rate at which the surface radius is changing at the instant when the water is 5 m deep.



Review set 14B

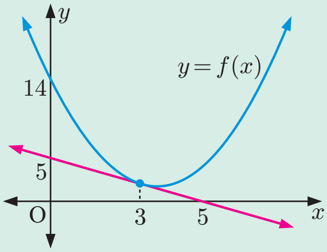
1 Find the equation of the normal to:

- **a** $y = \dfrac{1-2x}{x^2}$ at the point where $x = 1$
- **b** $y = e^{-x^2}$ at the point where $x = 1$
- **c** $y = \dfrac{1}{\sqrt{x}}$ at the point where $x = 4$.

2 The curve $y = 2x^3 + ax + b$ has a tangent with gradient 10 at the point $(-2, 33)$. Find the values of a and b.

3 $y = f(x)$ is the parabola shown.

- **a** Find $f(3)$ and $f'(3)$.
- **b** Hence find $f(x)$ in the form $f(x) = ax^2 + bx + c$.



4 Find the equation of:

- **a** the tangent to $y = \dfrac{1}{\sin x}$ at the point where $x = \frac{\pi}{3}$
- **b** the normal to $y = \cos(\frac{x}{2})$ at the point where $x = \frac{\pi}{2}$.

5 At the point where $x = 0$, the tangent to $f(x) = e^{4x} + px + q$ has equation $y = 5x - 7$. Find p and q.

6 Find where the tangent to $y = 2x^3 + 4x - 1$ at $(1, 5)$ cuts the curve again.

7 Find a given that the tangent to $y = \dfrac{4}{(ax+1)^2}$ at $x = 0$ passes through $(1, 0)$.

8 Consider the function $f(x) = e^x - x$.

- **a** Find and classify any stationary points of $y = f(x)$.
- **b** Show that $e^x \geqslant x + 1$ for all x.
- **c** Find $f''(x)$.

9 Find where the tangent to $y = \ln(x^4 + 3)$ at $x = 1$ cuts the y-axis.

10 Consider the function $f(x) = 2x^3 - 19x^2 + 52x - 35$.

- **a** Find the y-intercept of the graph $y = f(x)$.
- **b** Show that $x = 1$ is a root of the function, and hence find all roots.
- **c** Find and classify all stationary points.
- **d** Sketch the graph of $y = f(x)$, showing all important features.

11 If the normal to $f(x) = \dfrac{3x}{1+x}$ at $(2, 2)$ cuts the axes at B and C, determine the length BC.

12 The height of a tree t years after it was planted is given by $H(t) = 60 + 40\ln(2t+1)$ cm, $t \geqslant 0$.

- **a** How high was the tree when it was planted?
- **b** How long does it take for the tree to reach:
  - **i** 150 cm
  - **ii** 300 cm?
- **c** At what rate is the tree's height increasing after:
  - **i** 2 years
  - **ii** 20 years?

13 A particle P moves in a straight line with position given by $s(t) = 80e^{-\frac{t}{10}} - 40t$ m where t is the time in seconds, $t \geqslant 0$.

- **a** Find the velocity and acceleration functions.
- **b** Find the initial position, velocity, and acceleration of P.
- **c** Sketch the graph of the velocity function.
- **d** Find the exact time when the velocity is -44 m s^{-1}.

14 The cost per hour of running a freight train is given by $C(v) = \dfrac{v^2}{30} + \dfrac{9000}{v}$ dollars where v is the average speed of the train in km h^{-1}.

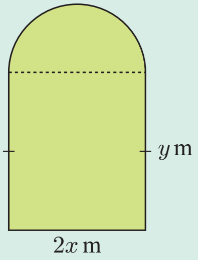
- **a** Find the cost of running the train for:
  - **i** two hours at 45 km h^{-1}
  - **ii** 5 hours at 64 km h^{-1}.
- **b** Find the rate of change in the hourly cost of running the train at speeds of:
  - **i** 50 km h^{-1}
  - **ii** 66 km h^{-1}.
- **c** At what speed will the cost per hour be a minimum?

15 A particle moves along the x-axis with position relative to origin O given by $x(t) = 3t - \sqrt{t+1}$ cm, where t is the time in seconds, $t \geqslant 0$.

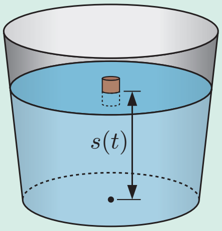
- **a** Find expressions for the particle's velocity and acceleration at any time t, and draw sign diagrams for each function.
- **b** Find the initial conditions, and hence describe the motion at that instant.
- **c** Describe the motion of the particle at $t = 8$ seconds.
- **d** Find the time and position when the particle reverses direction.
- **e** Determine the time interval when the particle's speed is decreasing.

16 A 200 m fence is placed around a lawn which has the shape of a rectangle with a semi-circle on one of its sides.

- **a** Using the dimensions shown on the figure, show that $y = 100 - x - \frac{\pi}{2}x$.
- **b** Find the area of the lawn A in terms of x only.
- **c** Find the dimensions of the lawn if it has the maximum possible area.



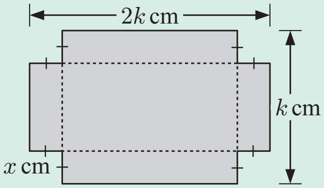
17 A cork bobs up and down in a bucket of water such that the distance from the centre of the cork to the bottom of the bucket is given by $s(t) = 30 + \cos(\pi t)$ cm, $t \geqslant 0$ seconds.



a Find the cork's velocity at times $t = 0, \frac{1}{2}, 1, 1\frac{1}{2}$, and 2 s.

b Find the time intervals when the cork is falling.

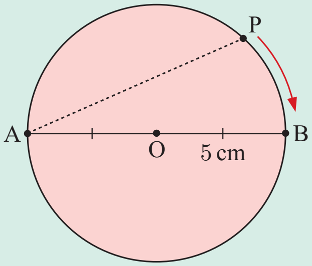
18 A rectangular sheet of tin-plate is $2k$ cm by k cm. Four squares, each with sides x cm, are cut from its corners. The remainder is bent into the shape of an open rectangular container. Find the value of x which will maximise the capacity of the container.



19 Two runners run in different directions, 60° apart. A runs at 5 m s^{-1} and B runs at 4 m s^{-1}. B passes through X 3 seconds after A passes through X.
At what rate is the distance between them increasing at the time when A is 20 metres past X?

20 Rectangle PQRS has PQ of fixed length 20 cm, and [QR] increases in length at a constant rate of 2 cm s^{-1}. At what rate is the acute angle between the diagonals of the rectangle changing at the instant when QR is 15 cm long?

21 AOB is a fixed diameter of a circle of radius 5 cm. Point P moves around the circle at a constant rate of 1 revolution in 10 seconds. Find the rate at which the distance AP is changing at the instant when:



a AP = 5 cm and increasing

b P is at B.

22 Consider the parabola $f(x) = \dfrac{1}{4b}x^2$ where $b > 0$.

a **i** Find the equation of the tangent to $y = f(x)$ at the point $\text{P}(a, f(a))$.

ii Show that this meets the x-axis at the point $\text{P}'(\frac{a}{2}, 0)$.

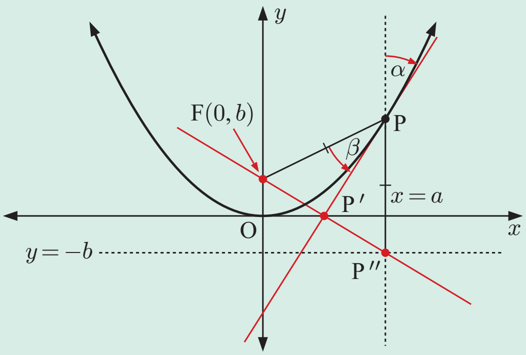
b **i** Find the equation of the line perpendicular to this tangent line, and which passes through P$'$.

ii Show that this line has y-intercept $\text{F}(0, b)$.

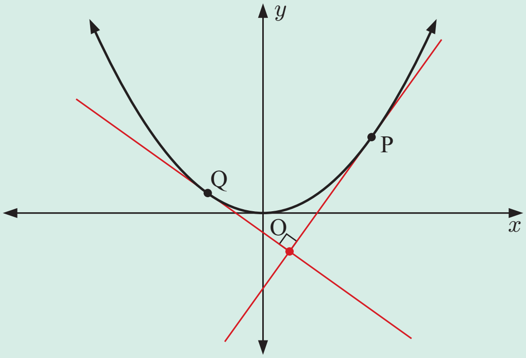
iii Show that the distance FP equals the distance from P to the line $y = -b$.

c The point $\text{F}(0, b)$ is *invariant* since it is independent of our choice of a. F is called the *focus* of the parabola. The line $y = -b$ is called the *directrix*.

i Prove the *reflective* property of the parabola, that any vertical ray will be reflected off the parabola into the focus.
Hint: Show that $\alpha = \beta$.



ii Suppose $a \neq 0$ and that Q$(c,\ f(c))$ is another point on the parabola such that the tangents from P and Q are perpendicular. Show that the intersection of the tangents occurs on the directrix.



15

Integration

Contents:

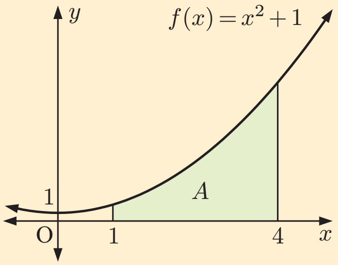
Opening problem

The function $f(x) = x^2 + 1$ lies above the x-axis for all $x \in \mathbb{R}$.

Things to think about:

a How can we calculate the shaded area A, which is the area under the curve for $1 \leqslant x \leqslant 4$?

b What function has $x^2 + 1$ as its derivative?



In the previous chapters we used differential calculus to find the derivatives of many types of functions. We also used it in problem solving, in particular to find the gradients of graphs and rates of changes, and to solve optimisation problems.

In this chapter we consider **integral calculus**. This involves **antidifferentiation** which is the reverse process of differentiation. Integral calculus also has many useful applications, including:

- finding areas of shapes with curved boundaries
- finding volumes of revolution
- finding distances travelled from velocity functions
- solving problems in economics, biology, and statistics
- solving differential equations.

A THE AREA UNDER A CURVE

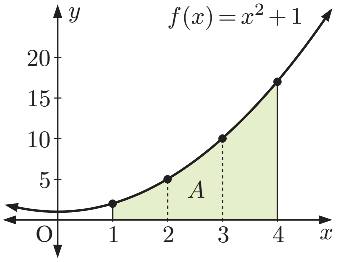
The task of finding the area under a curve has been important to mathematicians for thousands of years. In the history of mathematics it was fundamental to the development of integral calculus. We will therefore begin our study by calculating the area under a curve using the same methods as the ancient mathematicians.

UPPER AND LOWER RECTANGLES

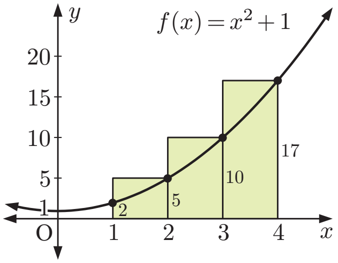
Consider the function $f(x) = x^2 + 1$.

We wish to estimate the area A enclosed by $y = f(x)$, the x-axis, and the vertical lines $x = 1$ and $x = 4$.

Suppose we divide the interval $1 \leqslant x \leqslant 4$ into three strips of width 1 unit as shown. We obtain three subintervals of equal width.



The diagram alongside shows **upper rectangles**, which are rectangles with top edges at the maximum value of the curve on that subinterval.



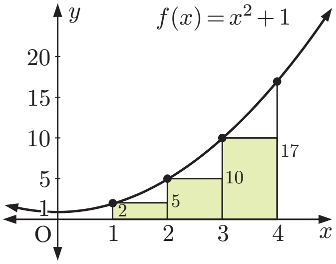
The area of the upper rectangles,

$$A_U = 1 \times f(2) + 1 \times f(3) + 1 \times f(4)$$
$$= 5 + 10 + 17$$
$$= 32 \text{ units}^2$$

The next diagram shows **lower rectangles**, which are rectangles with top edges at the minimum value of the curve on that subinterval.

The area of the lower rectangles,

$$\begin{aligned} A_L &= 1\times f(1) + 1\times f(2) + 1\times f(3) \\ &= 2+5+10 \\ &= 17 \text{ units}^2 \end{aligned}$$



Now clearly $A_L < A < A_U$, so the area A lies between 17 units^2 and 32 units^2.

If the interval $1 \leqslant x \leqslant 4$ was divided into 6 subintervals, each of width $\frac{1}{2}$, then

$$\begin{aligned} A_U &= \tfrac{1}{2}f(1\tfrac{1}{2}) + \tfrac{1}{2}f(2) + \tfrac{1}{2}f(2\tfrac{1}{2}) + \tfrac{1}{2}f(3) + \tfrac{1}{2}f(3\tfrac{1}{2}) + \tfrac{1}{2}f(4) \\ &= \tfrac{1}{2}(\tfrac{13}{4} + 5 + \tfrac{29}{4} + 10 + \tfrac{53}{4} + 17) \\ &= 27.875 \text{ units}^2 \end{aligned}$$

$$\begin{aligned} \text{and}\quad A_L &= \tfrac{1}{2}f(1) + \tfrac{1}{2}f(1\tfrac{1}{2}) + \tfrac{1}{2}f(2) + \tfrac{1}{2}f(2\tfrac{1}{2}) + \tfrac{1}{2}f(3) + \tfrac{1}{2}f(3\tfrac{1}{2}) \\ &= \tfrac{1}{2}(2 + \tfrac{13}{4} + 5 + \tfrac{29}{4} + 10 + \tfrac{53}{4}) \\ &= 20.375 \text{ units}^2 \end{aligned}$$

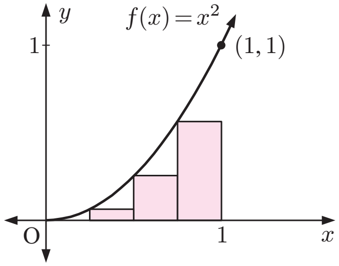
From this refinement we conclude that the area A lies between 20.375 and 27.875 units^2.

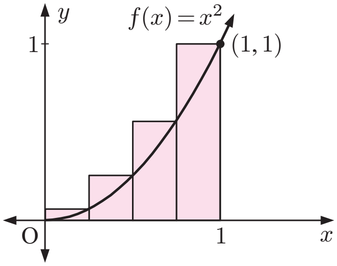
As we create more subintervals, the estimates A_L and A_U will become more and more accurate. In fact, as the subinterval width is reduced further and further, both A_L and A_U will **converge** to A.

We illustrate this process by estimating the area A between the graph of $y = x^2$ and the x-axis for $0 \leqslant x \leqslant 1$.

This example is of historical interest. **Archimedes** (287 - 212 BC) found the exact area. In an article that contains 24 propositions, he developed the essential theory for what is now known as integral calculus.

Consider $f(x) = x^2$ and divide the interval $0 \leqslant x \leqslant 1$ into 4 subintervals of equal width.





$$\begin{aligned} A_L &= \tfrac{1}{4}(0)^2 + \tfrac{1}{4}(\tfrac{1}{4})^2 + \tfrac{1}{4}(\tfrac{1}{2})^2 + \tfrac{1}{4}(\tfrac{3}{4})^2 \\ &\approx 0.219 \end{aligned} \qquad \text{and} \qquad \begin{aligned} A_U &= \tfrac{1}{4}(\tfrac{1}{4})^2 + \tfrac{1}{4}(\tfrac{1}{2})^2 + \tfrac{1}{4}(\tfrac{3}{4})^2 + \tfrac{1}{4}(1)^2 \\ &\approx 0.469 \end{aligned}$$

Now suppose there are n subintervals between $x = 0$ and $x = 1$, each of width $\dfrac{1}{n}$.

AREA FINDER

We can use the **area finder** software to help calculate A_L and A_U for large values of n.

The table alongside summarises the results you should obtain for $n = 4$, 10, 25, and 50.

The exact value of A is in fact $\frac{1}{3}$, as we will find later in the chapter. Notice how both A_L and A_U are converging to this value as n increases.

n	A_L	A_U	*Average*
4	0.218 75	0.468 75	0.343 75
10	0.285 00	0.385 00	0.335 00
25	0.313 60	0.353 60	0.333 60
50	0.323 40	0.343 40	0.333 40

EXERCISE 15A.1

1 Consider the area between $y = x$ and the x-axis from $x = 0$ to $x = 1$.

- **a** Divide the interval into 5 strips of equal width, then estimate the area using:
  - **i** upper rectangles
  - **ii** lower rectangles.
- **b** Calculate the actual area and compare it with your answers in **a**.

2 Consider the area between $y = \frac{1}{x}$ and the x-axis from $x = 2$ to $x = 4$. Divide the interval into 6 strips of equal width, then estimate the area using:

- **a** upper rectangles
- **b** lower rectangles.

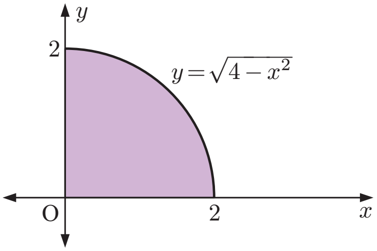
3 Use rectangles to find lower and upper sums for the area between the graph of $y = x^2$ and the x-axis for $1 \leqslant x \leqslant 2$. Use $n = 10$, 25, 50, 100, and 500. Give your answers to 4 decimal places.
As n gets larger, both A_L and A_U converge to the same number which is a simple fraction. What is it?

AREA FINDER

4
- **a** Use lower and upper sums to estimate the area between each of the following functions and the x-axis for $0 \leqslant x \leqslant 1$. Use values of $n = 5$, 10, 50, 100, 500, 1000, and 10 000. Give your answer to 5 decimal places in each case.
  - **i** $y = x^3$
  - **ii** $y = x$
  - **iii** $y = x^{\frac{1}{2}}$
  - **iv** $y = x^{\frac{1}{3}}$
- **b** For each case in **a**, A_L and A_U converge to the same number which is a simple fraction. What fractions are they?
- **c** Using your answer to **b**, predict the area between the graph of $y = x^a$ and the x-axis for $0 \leqslant x \leqslant 1$ and any number $a > 0$.

5 Consider the quarter circle of centre $(0, 0)$ and radius 2 units illustrated.

Its area is $\frac{1}{4}$(full circle of radius 2)
$$= \frac{1}{4} \times \pi \times 2^2$$
$$= \pi \text{ units}^2$$

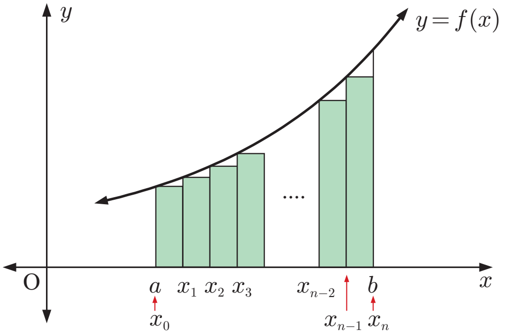


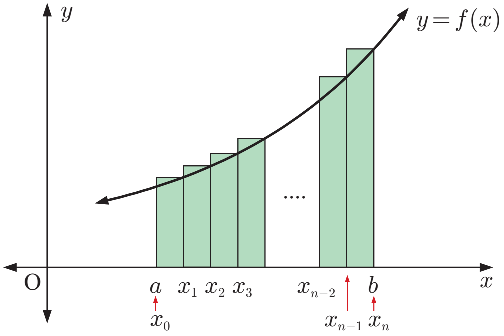
- **a** Estimate the area using lower and upper rectangles for $n = 10$, 50, 100, 200, 1000, and 10 000. Hence, find rational bounds for π.
- **b** Archimedes found the famous approximation $3\frac{10}{71} < \pi < 3\frac{1}{7}$.
  For what value of n is your estimate for π better than that of Archimedes?

THE DEFINITE INTEGRAL

Consider the lower and upper rectangle sums for a function which is positive and increasing on the interval $a \leqslant x \leqslant b$.

We divide the interval into n subintervals, each of width $w = \dfrac{b-a}{n}$.

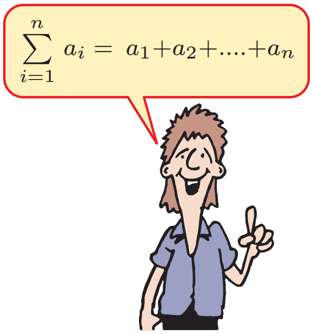




Since the function is increasing:

$$A_L = w\,f(x_0) + w\,f(x_1) + ... + w\,f(x_{n-2}) + w\,f(x_{n-1}) = w\sum_{i=0}^{n-1} f(x_i)$$

$$A_U = w\,f(x_1) + w\,f(x_2) + ... + w\,f(x_{n-1}) + w\,f(x_n) = w\sum_{i=1}^{n} f(x_i)$$

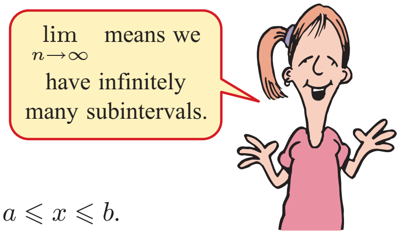


Notice that
$$\begin{aligned} A_U - A_L &= w\,(f(x_n) - f(x_0)) \\ &= \frac{1}{n}(b-a)\,(f(b) - f(a)) \end{aligned}$$

$$\therefore \quad \lim_{n\to\infty}(A_U - A_L) = 0 \qquad \{\text{since } \lim_{n\to\infty}\frac{1}{n} = 0\}$$

$$\therefore \quad \lim_{n\to\infty} A_L = \lim_{n\to\infty} A_U \quad \{\text{when both limits exist}\}$$

$\therefore$ since $A_L < A < A_U$ for all values of n, it follows that
$\lim\limits_{n\to\infty} A_L = A = \lim\limits_{n\to\infty} A_U$

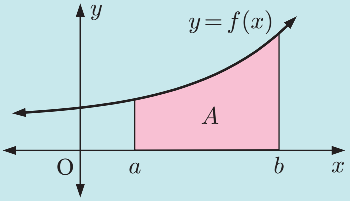


This fact is true for all positive continuous functions on an interval $a \leqslant x \leqslant b$.

The value A is known as the "**definite integral** of $f(x)$ from a to b", written $A = \displaystyle\int_a^b f(x)\,dx$.

If $f(x) \geqslant 0$ for all $a \leqslant x \leqslant b$, then

$\displaystyle\int_a^b f(x)\,dx$ is equal to the shaded area.



Historical note

The word **integration** means "*to put together into a whole*". An **integral** is the "whole" produced from integration, since the areas $f(x_i) \times w$ of the thin rectangular strips are put together into one whole area.

The symbol $\int$ is called an **integral sign**. In the time of **Newton** and **Leibniz** it was the stretched out letter s, but it is no longer part of the alphabet.

Example 1

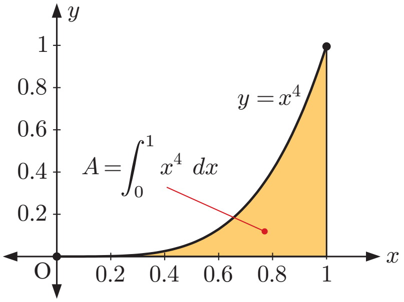
Self Tutor

a Sketch the graph of $y = x^4$ for $0 \leqslant x \leqslant 1$. Shade the area described by $\int_0^1 x^4 \, dx$.

b Use technology to calculate the lower and upper rectangle sums for n equal subintervals where $n = 5$, 10, 50, 100, and 500.

c Hence find $\int_0^1 x^4 \, dx$ to 2 significant figures.

a



b

n	A_L	A_U
5	0.1133	0.3133
10	0.1533	0.2533
50	0.1901	0.2101
100	0.1950	0.2050
500	0.1990	0.2010

c When $n = 500$, $A_L \approx A_U \approx 0.20$, to 2 significant figures.

$\therefore$ since $A_L < \int_0^1 x^4 \, dx < A_U$, $\int_0^1 x^4 \, dx \approx 0.20$

EXERCISE 15A.2

1 **a** Sketch the graph of $y = \sqrt{x}$ for $0 \leqslant x \leqslant 1$.

Shade the area described by $\int_0^1 \sqrt{x} \, dx$.

b Find the lower and upper rectangle sums for $n = 5$, 10, 50, 100, and 500.

c Hence find $\int_0^1 \sqrt{x} \, dx$ to 2 significant figures.

AREA FINDER

2 Consider the region enclosed by $y = \sqrt{1 + x^3}$ and the x-axis for $0 \leqslant x \leqslant 2$.

a Write expressions for the lower and upper rectangle sums using n subintervals, $n \in \mathbb{N}$.

b Find the lower and upper rectangle sums for $n = 50$, 100, and 500.

c Hence estimate $\int_0^2 \sqrt{1 + x^3} \, dx$.

GRAPHING PACKAGE

3 The integral $\displaystyle\int_{-3}^{3} e^{-\frac{x^2}{2}}\, dx$ is of considerable interest to statisticians.

- **a** Use the graphing package to help sketch $y = e^{-\frac{x^2}{2}}$ for $-3 \leqslant x \leqslant 3$.
- **b** Calculate the upper and lower rectangular sums for the interval $0 \leqslant x \leqslant 3$ using $n = 2250$.
- **c** Use the symmetry of $y = e^{-\frac{x^2}{2}}$ to find upper and lower rectangular sums for $-3 \leqslant x \leqslant 0$ for $n = 2250$.
- **d** Hence estimate $\displaystyle\int_{-3}^{3} e^{-\frac{x^2}{2}}\, dx$.

  How accurate is your estimate compared with $\sqrt{2\pi}$?

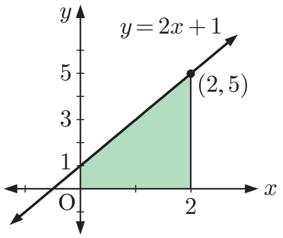
AREA FINDER

Example 2

Self Tutor

Use graphical evidence and known area facts to find: **a** $\displaystyle\int_0^2 (2x+1)\, dx$ **b** $\displaystyle\int_0^1 \sqrt{1-x^2}\, dx$

a



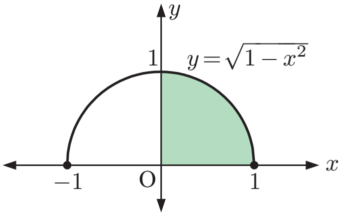
$$\int_0^2 (2x+1)\, dx$$

$= \text{shaded area}$

$= \left(\frac{1+5}{2}\right) \times 2$

$= 6$

b If $y = \sqrt{1-x^2}$ then $y^2 = 1 - x^2$ and so $x^2 + y^2 = 1$ which is the equation of the unit circle. $y = \sqrt{1-x^2}$ is the upper half.



$$\int_0^1 \sqrt{1-x^2}\, dx$$

$= \text{shaded area}$

$= \frac{1}{4}(\pi r^2)$ where $r = 1$

$= \frac{\pi}{4}$

4 Use graphical evidence and known area facts to find:

- **a** $\displaystyle\int_1^3 (1+4x)\, dx$
- **b** $\displaystyle\int_{-1}^2 (2-x)\, dx$
- **c** $\displaystyle\int_{-2}^2 \sqrt{4-x^2}\, dx$

B ANTIDIFFERENTIATION

In many problems in calculus we know the rate of change of one variable with respect to another, but we do not have a formula which relates the variables. In other words, we know $\dfrac{dy}{dx}$, but we need to know y in terms of x.

The process of finding y from $\frac{dy}{dx}$, or $f(x)$ from $f'(x)$, is the reverse process of differentiation. We call it **antidifferentiation**.

$$\begin{tikzcd}
y \text{ or } f(x) \arrow[r, bend left, "\text{differentiation}"] & \frac{dy}{dx} \text{ or } f'(x) \arrow[l, bend left, "\text{antidifferentiation}"]
\end{tikzcd}$$

Consider $\frac{dy}{dx} = x^2$.

From our work on differentiation, we know that when we differentiate power functions the index reduces by 1. We hence know that y must involve x^3.

Now if $y = x^3$ then $\frac{dy}{dx} = 3x^2$, so if we start with $y = \frac{1}{3}x^3$ then $\frac{dy}{dx} = x^2$.

However, for all of the cases $y = \frac{1}{3}x^3 + 2$, $y = \frac{1}{3}x^3 + 100$, and $y = \frac{1}{3}x^3 - 7$, we find that $\frac{dy}{dx} = x^2$.

In fact, there are infinitely many functions of the form $y = \frac{1}{3}x^3 + c$ where c is an arbitrary constant, which will give $\frac{dy}{dx} = x^2$. Ignoring the arbitrary constant, we say that $\frac{1}{3}x^3$ is the **antiderivative** of x^2. It is the simplest function which, when differentiated, gives x^2.

If $F(x)$ is a function where $F'(x) = f(x)$ we say that:

- the **derivative** of $F(x)$ is $f(x)$ and
- the **antiderivative** of $f(x)$ is $F(x)$.

Example 3 Self Tutor

Find the antiderivative of: **a** x^3 **b** e^{2x} **c** $\frac{1}{\sqrt{x}}$

a $\frac{d}{dx}\left(x^4\right) = 4x^3$

$\therefore \frac{d}{dx}\left(\frac{1}{4}x^4\right) = x^3$

$\therefore$ the antiderivative of x^3 is $\frac{1}{4}x^4$.

b $\frac{d}{dx}\left(e^{2x}\right) = e^{2x} \times 2$

$\therefore \frac{d}{dx}\left(\frac{1}{2}e^{2x}\right) = \frac{1}{2} \times e^{2x} \times 2 = e^{2x}$

$\therefore$ the antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$.

c $\frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

Now $\frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}}$

$\therefore \frac{d}{dx}(2x^{\frac{1}{2}}) = 2(\frac{1}{2})x^{-\frac{1}{2}} = x^{-\frac{1}{2}}$

$\therefore$ the antiderivative of $\frac{1}{\sqrt{x}}$ is $2\sqrt{x}$.

EXERCISE 15B

1 **a** Find the antiderivative of:

i x **ii** x^2 **iii** x^5 **iv** x^{-2} **v** x^{-4} **vi** $x^{\frac{1}{3}}$ **vii** $x^{-\frac{1}{2}}$

b Predict a general rule for the antiderivative of x^n, for $n \neq -1$.

2 **a** Find the antiderivative of:

i e^{2x} **ii** e^{5x} **iii** $e^{\frac{1}{2}x}$ **iv** $e^{0.01x}$ **v** $e^{\pi x}$ **vi** $e^{\frac{x}{3}}$

b Predict a general rule for the antiderivative of e^{kx} where k is a constant, $k \neq 0$.

3 Find the antiderivative of:

a $6x^2 + 4x$ by first differentiating $x^3 + x^2$

b e^{3x+1} by first differentiating e^{3x+1}

c $\sqrt{x}$ by first differentiating $x\sqrt{x}$

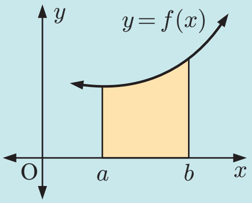
d $(2x+1)^3$ by first differentiating $(2x+1)^4$.

C THE FUNDAMENTAL THEOREM OF CALCULUS

Sir Isaac Newton and **Gottfried Wilhelm Leibniz** showed the link between differential calculus and the definite integral or limit of an area sum we saw in **Section A**. This link is called the **fundamental theorem of calculus**. The beauty of this theorem is that it enables us to evaluate complicated summations.

We have already observed that:

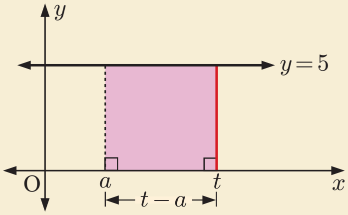
If $f(x)$ is a continuous positive function on an interval $a \leqslant x \leqslant b$ then the area under the curve between $x = a$ and $x = b$ is $\int_a^b \boldsymbol{f(x)\,dx}$.



Discovery — The area function

Consider the constant function $f(x) = 5$.

We wish to find an **area function** which will give the area under the function between $x = a$ and some other value of x which we will call t.



$$\begin{aligned} \text{The area function is}\quad A(t) &= \int_a^t 5\,dx \\ &= \text{shaded area in graph} \\ &= (t-a)5 \\ &= 5t - 5a \end{aligned}$$

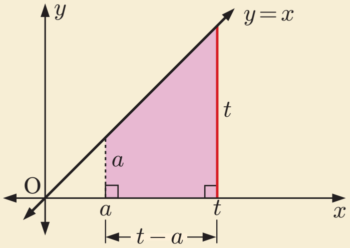
$\therefore$ we can write $A(t)$ in the form $F(t) - F(a)$ where $F(t) = 5t$ or equivalently $F(x) = 5x$

What to do:

1 What is the derivative $F'(x)$ of the function $F(x) = 5x$? How does this relate to the function $f(x)$?

2 Consider the simplest linear function $f(x) = x$.
The corresponding area function is

$$A(t) = \int_a^t x\,dx$$
$$= \text{shaded area in graph}$$
$$= \left(\frac{t+a}{2}\right)(t-a)$$

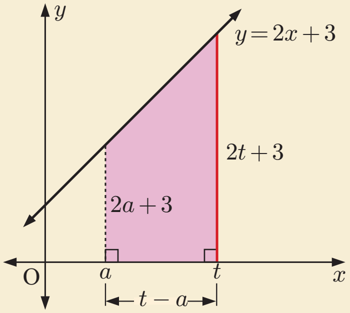


a Write $A(t)$ in the form $F(t) - F(a)$.

b What is the derivative $F'(x)$? How does it relate to the function $f(x)$?

3 Consider $f(x) = 2x + 3$. The corresponding area function is

$$A(t) = \int_a^t (2x+3)\,dx$$
$$= \text{shaded area in graph}$$
$$= \left(\frac{2t+3+2a+3}{2}\right)(t-a)$$



a Write $A(t)$ in the form $F(t) - F(a)$.

b What is the derivative $F'(x)$?
How does it relate to the function $f(x)$?

4 Repeat the procedure in **2** and **3** to find area functions for:

a $f(x) = \frac{1}{2}x + 3$ **b** $f(x) = 5 - 2x$

Do your results fit with your earlier observations?

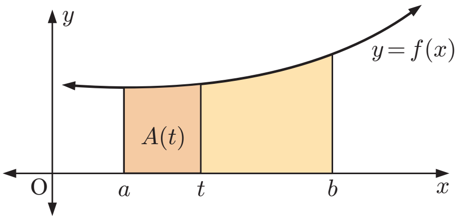
5 If $f(x) = 3x^2 + 4x + 5$, predict what $F(x)$ would be without performing the algebraic procedure.

From the **Discovery** you should have found that, for $f(x) \geqslant 0$,

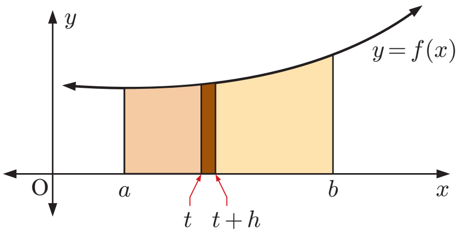
$$\int_a^t f(x)\,dx = F(t) - F(a) \quad \text{where} \quad F'(x) = f(x).$$ $F(x)$ is the **antiderivative** of $f(x)$.

The following argument shows why this is true for all functions $f(x) \geqslant 0$.

Consider a function $y = f(x)$ which has antiderivative $F(x)$ and an area function $A(t) = \int_a^t f(x)\,dx$ which is the area from $x = a$ to $x = t$.



$A(t)$ is clearly an increasing function and
$A(a) = 0$ .... (1)



Consider the narrow strip between $x = t$ and $x = t + h$. The area of this strip is $A(t+h) - A(t)$, but we also know it must lie between a lower and upper rectangle on the interval $t \leqslant x \leqslant t+h$ of width h.

$$\text{area of smaller rectangle} \leqslant A(t+h) - A(t) \leqslant \text{area of larger rectangle}$$

If $f(x)$ is increasing on this interval then

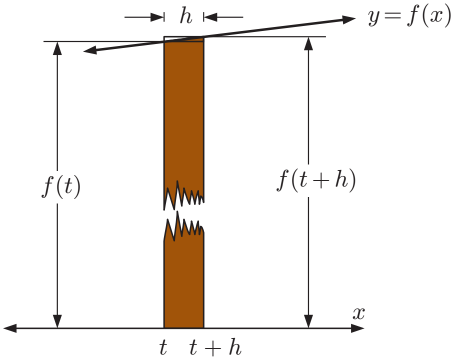
$$hf(t) \leqslant A(t+h) - A(t) \leqslant hf(t+h)$$

$$\therefore \quad f(t) \leqslant \frac{A(t+h) - A(t)}{h} \leqslant f(t+h)$$

Equivalently, if $f(x)$ is decreasing on this interval then $f(t+h) \leqslant \dfrac{A(t+h) - A(t)}{h} \leqslant f(t)$.

Taking the limit as $h \to 0$ gives $f(t) \leqslant A'(t) \leqslant f(t)$

$$\therefore \quad A'(t) = f(t)$$



enlarged strip

So, the area function $A(t)$ must only differ from the antiderivative of $f(t)$ by a constant.

$\therefore \quad A(t) = F(t) + c$

Letting $t = a$, $A(a) = F(a) + c$

But from (1), $A(a) = 0$

$\therefore \quad c = -F(a)$

$\therefore \quad A(t) = F(t) - F(a)$

Letting $t = b$, $\displaystyle\int_a^b f(x)\, dx = F(b) - F(a)$

This result is in fact true for all continuous functions $f(x)$.

THE FUNDAMENTAL THEOREM OF CALCULUS

From the geometric argument above, the Fundamental Theorem of Calculus can be stated in two forms:

For a continuous function $f(x)$, if we define the area function from $x = a$ as $A(t) = \displaystyle\int_a^t f(x)\, dx$, then $A'(x) = f(x)$.

or more commonly:

For a continuous function $f(x)$ with antiderivative $F(x)$, $\displaystyle\int_a^b \boldsymbol{f(x)\, dx = F(b) - F(a)}$.

PROPERTIES OF DEFINITE INTEGRALS

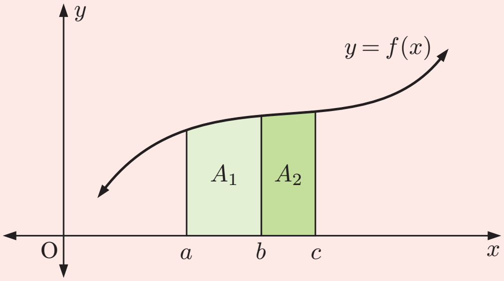
The following properties of definite integrals can all be deduced from the fundamental theorem of calculus:

- $\displaystyle\int_a^a f(x)\, dx = 0$
- $\displaystyle\int_a^b c\, dx = c(b-a)$ {c is a constant}
- $\displaystyle\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx$
- $\displaystyle\int_a^b c\, f(x)\, dx = c\int_a^b f(x)\, dx$

- $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Example proof:

$$\begin{aligned} &\int_a^b f(x)\,dx + \int_b^c f(x)\,dx \\ &= F(b) - F(a) + F(c) - F(b) \\ &= F(c) - F(a) \\ &= \int_a^c f(x)\,dx \end{aligned}$$



In particular, for the case where $a \leqslant b \leqslant c$ and $f(x) \geqslant 0$ for $a \leqslant x \leqslant c$, we observe that

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = A_1 + A_2 = \int_a^c f(x)\,dx$$

Example 4

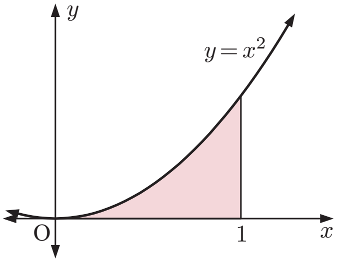
Self Tutor

Use the fundamental theorem of calculus to find the area between:

a the x-axis and $y = x^2$ from $x = 0$ to $x = 1$

b the x-axis and $y = \sqrt{x}$ from $x = 1$ to $x = 9$.

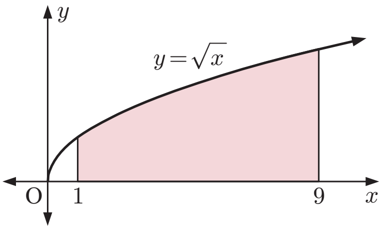
a



$f(x) = x^2$ has antiderivative $F(x) = \frac{x^3}{3}$

$$\begin{aligned} \therefore \text{ the area } &= \int_0^1 x^2\,dx \\ &= F(1) - F(0) \\ &= \tfrac{1}{3} - 0 \\ &= \tfrac{1}{3} \text{ units}^2 \end{aligned}$$

b



$f(x) = \sqrt{x} = x^{\frac{1}{2}}$ has antiderivative

$F(x) = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}x\sqrt{x}$

$$\begin{aligned} \therefore \text{ the area } &= \int_1^9 x^{\frac{1}{2}}\,dx \\ &= F(9) - F(1) \\ &= \tfrac{2}{3} \times 27 - \tfrac{2}{3} \times 1 \\ &= 17\tfrac{1}{3} \text{ units}^2 \end{aligned}$$

EXERCISE 15C

1 Use the fundamental theorem of calculus to find the area between:

- **a** the x-axis and $y = x^3$ from $x = 0$ to $x = 1$
- **b** the x-axis and $y = x^2$ from $x = 1$ to $x = 2$
- **c** the x-axis and $y = \sqrt{x}$ from $x = 0$ to $x = 1$.

2 Use the fundamental theorem of calculus to show that:

- **a** $\displaystyle\int_a^a f(x)\,dx = 0$ and explain the result graphically
- **b** $\displaystyle\int_a^b c\,dx = c(b-a)$ where c is a constant
- **c** $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- **d** $\displaystyle\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$ where c is a constant
- **e** $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

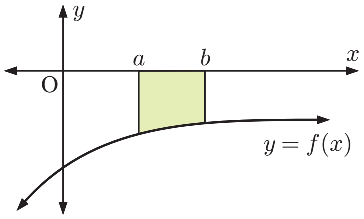
3 Use the fundamental theorem of calculus to find the area between the x-axis and:

- **a** $y = x^3$ from $x = 1$ to $x = 2$
- **b** $y = x^2 + 3x + 2$ from $x = 1$ to $x = 3$
- **c** $y = \sqrt{x}$ from $x = 1$ to $x = 2$
- **d** $y = e^x$ from $x = 0$ to $x = 1.5$
- **e** $y = \dfrac{1}{\sqrt{x}}$ from $x = 1$ to $x = 4$

4 **a** Use the fundamental theorem of calculus to show that

$$\int_a^b (-f(x))\,dx = -\int_a^b f(x)\,dx$$

b Hence show that if $f(x) \leqslant 0$ for all x on $a \leqslant x \leqslant b$ then the shaded area $= -\displaystyle\int_a^b f(x)\,dx$.



c Calculate the following integrals, and give graphical interpretations of your answers:

- **i** $\displaystyle\int_0^1 (-x^2)\,dx$
- **ii** $\displaystyle\int_0^1 (x^2 - x)\,dx$
- **iii** $\displaystyle\int_{-2}^0 3x\,dx$

d Use graphical evidence and known area facts to find $\displaystyle\int_0^2 \left(-\sqrt{4-x^2}\right)\,dx$.

D INTEGRATION

Earlier, we showed that the **antiderivative** of x^2 is $\frac{1}{3}x^3$, and that any function of the form $\frac{1}{3}x^3+c$ where c is a constant, has derivative x^2.

We say that the **indefinite integral** or **integral** of x^2 is $\frac{1}{3}x^3+c$, and write $\int x^2\,dx = \frac{1}{3}x^3+c$.

We read this as "the integral of x^2 with respect to x is $\frac{1}{3}x^3+c$, where c is a constant".

$$\text{If} \quad F'(x)=f(x) \quad \text{then} \quad \int f(x)\,dx = F(x)+c.$$

This process is known as **indefinite integration**. It is indefinite because it is not being applied to a particular interval.

DISCOVERING INTEGRALS

Since integration is the reverse process of differentiation we can sometimes discover integrals by differentiation. For example:

- if $F(x)=x^4$ then $F'(x)=4x^3$
  $$\therefore \int 4x^3\,dx = x^4+c$$
- if $F(x)=\sqrt{x}=x^{\frac{1}{2}}$ then $F'(x)=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$
  $$\therefore \int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x}+c$$

The following rules may prove useful:

- Any constant may be written in front of the integral sign.

$$\int k\,f(x)\,dx = k\int f(x)\,dx, \quad k \text{ is a constant}$$

Proof: Consider differentiating $kF(x)$ where $F'(x)=f(x)$.

$$\frac{d}{dx}(k\,F(x)) = k\,F'(x) = k\,f(x)$$

$$\therefore \int k\,f(x)\,dx = k\,F(x)$$

$$= k\int f(x)\,dx$$

- The integral of a sum is the sum of the separate integrals. This rule enables us to integrate term by term.

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Example 5

Self Tutor

If $y = x^4 + 2x^3$, find $\frac{dy}{dx}$. Hence find $\int (2x^3 + 3x^2)\,dx$.

If $y = x^4 + 2x^3$ then $\frac{dy}{dx} = 4x^3 + 6x^2$

$$\therefore \int (4x^3 + 6x^2)\,dx = x^4 + 2x^3 + c$$

$$\therefore \int 2(2x^3 + 3x^2)\,dx = x^4 + 2x^3 + c$$

$$\therefore 2\int (2x^3 + 3x^2)\,dx = x^4 + 2x^3 + c$$

$$\therefore \int (2x^3 + 3x^2)\,dx = \tfrac{1}{2}x^4 + x^3 + c$$

c represents a general constant, so is simply any value $c \in \mathbb{R}$.
Instead of writing $\frac{c}{2}$, we can therefore still write just c.

EXERCISE 15D

1 If $y = x^7$, find $\frac{dy}{dx}$. Hence find $\int x^6\,dx$.

2 If $y = x^3 + x^2$, find $\frac{dy}{dx}$. Hence find $\int (3x^2 + 2x)\,dx$.

3 If $y = e^{2x+1}$, find $\frac{dy}{dx}$. Hence find $\int e^{2x+1}\,dx$.

4 If $y = (2x+1)^4$ find $\frac{dy}{dx}$. Hence find $\int (2x+1)^3\,dx$.

Example 6

Self Tutor

Suppose $y = \sqrt{5x-1}$.

a Find $\frac{dy}{dx}$.

b Hence find $\int \frac{1}{\sqrt{5x-1}}\,dx$.

a

$$y = \sqrt{5x-1}$$

$$= (5x-1)^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx} = \tfrac{1}{2}(5x-1)^{-\frac{1}{2}}(5) \quad \{\text{chain rule}\}$$

$$= \frac{5}{2\sqrt{5x-1}}$$

b Using **a**, $\int \frac{5}{2\sqrt{5x-1}}\,dx = \sqrt{5x-1} + c$

$$\therefore \tfrac{5}{2}\int \frac{1}{\sqrt{5x-1}}\,dx = \sqrt{5x-1} + c$$

$$\therefore \int \frac{1}{\sqrt{5x-1}}\,dx = \tfrac{2}{5}\sqrt{5x-1} + c$$

5 If $y = x\sqrt{x}$, find $\frac{dy}{dx}$. Hence find $\int \sqrt{x}\,dx$.

6 If $y = \frac{1}{\sqrt{x}}$, find $\frac{dy}{dx}$. Hence find $\int \frac{1}{x\sqrt{x}}\,dx$.

7 If $y = \cos 2x$, find $\frac{dy}{dx}$. Hence find $\int \sin 2x \, dx$.

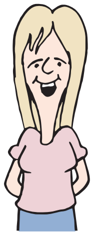
8 If $y = \sin(1 - 5x)$, find $\frac{dy}{dx}$. Hence find $\int \cos(1 - 5x) \, dx$.

9 By considering $\frac{d}{dx}(x^2 - x)^3$, find $\int (2x - 1)(x^2 - x)^2 \, dx$.

10 Prove the rule $\int [f(x) + g(x)] \, dx = \int f(x) \, dx + \int g(x) \, dx$.

11 Find $\frac{dy}{dx}$ if $y = \sqrt{1 - 4x}$. Hence find $\int \frac{1}{\sqrt{1 - 4x}} \, dx$.

We can check that an integral is correct by differentiating the answer. It should give us the **integrand**, the function we originally integrated.



E RULES FOR INTEGRATION

In **Chapter 13** we developed a set of rules to help us differentiate functions more efficiently:

Function	*Derivative*	*Name*
c, a constant	0	
$mx + c$, m and c are constants	m	
x^n	nx^{n-1}	**power rule**
$cu(x)$	$cu'(x)$	
$u(x) + v(x)$	$u'(x) + v'(x)$	**addition rule**
$u(x)v(x)$	$u'(x)v(x) + u(x)v'(x)$	**product rule**
$\frac{u(x)}{v(x)}$	$\frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$	**quotient rule**
$y = f(u)$ where $u = u(x)$	$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$	**chain rule**
e^x	e^x	
$e^{f(x)}$	$e^{f(x)} f'(x)$	
$\ln x$	$\frac{1}{x}$	
$\ln f(x)$	$\frac{f'(x)}{f(x)}$	
$[f(x)]^n$	$n[f(x)]^{n-1} f'(x)$	
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	

These rules or combinations of them can be used to differentiate all of the functions we consider in this course. Given an algebraic formula, we can repeatedly apply these rules until we get to basic functions such as x^n or $\sin x$, which we know how to differentiate.

However, the task of finding **antiderivatives** is not so easy. Given an algebraic formula there is no simple list of rules to find the antiderivative.

The problem was finally solved in 1968 by Robert Henry Risch. He devised a method for deciding if a function has an elementary antiderivative, and if it does, finding it. The original summary of his method took over 100 pages. Later developments from this are now used in all computer algebra systems.

Fortunately, our course is restricted to a few special cases.

RULES FOR INTEGRATION

For k a constant, $\frac{d}{dx}(kx+c)=k$ $\quad\therefore\quad$ $\boxed{\int k\,dx = kx + c}$

If $n \neq -1$, $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}+c\right)=\frac{(n+1)x^n}{n+1}=x^n$ $\quad\therefore\quad$ $\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1}+c, \quad n \neq -1}$

$$\frac{d}{dx}(e^x+c)=e^x \quad\therefore\quad \boxed{\int e^x\,dx = e^x + c}$$

$$\frac{d}{dx}(\sin x + c)=\cos x \quad\therefore\quad \boxed{\int \cos x\,dx = \sin x + c}$$

$$\frac{d}{dx}(-\cos x + c)=\sin x \quad\therefore\quad \boxed{\int \sin x\,dx = -\cos x + c}$$

Function	*Integral*
k, a constant	$kx+c$
x^n, $n \neq -1$	$\frac{x^{n+1}}{n+1}+c$
e^x	e^x+c
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$

c is an arbitrary constant called the **constant of integration** or **integrating constant**.

Remember that you can always check your integration by differentiating the resulting function.

Example 7

Self Tutor

Find:

a $\int (x^3 - 2x^2 + 5)\,dx$ **b** $\int \left(\frac{1}{x^3} - \sqrt{x}\right) dx$ **c** $\int (2\sin x - \cos x)\,dx$

a

$$\int (x^3 - 2x^2 + 5)\,dx$$
$$= \frac{x^4}{4} - \frac{2x^3}{3} + 5x + c$$

b

$$\int \left(\frac{1}{x^3} - \sqrt{x}\right) dx$$
$$= \int (x^{-3} - x^{\frac{1}{2}})\,dx$$
$$= \frac{x^{-2}}{-2} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= -\frac{1}{2x^2} - \tfrac{2}{3}x^{\frac{3}{2}} + c$$

c

$$\int (2\sin x - \cos x)\,dx$$
$$= 2(-\cos x) - \sin x + c$$
$$= -2\cos x - \sin x + c$$

There is no product or quotient rule for integration. Consequently we often have to carry out multiplication or division before we integrate.

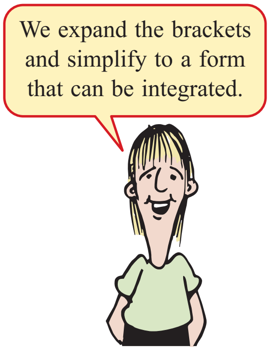
Example 8

Self Tutor

Find: **a** $\int \left(3x + \frac{2}{x}\right)^2 dx$ **b** $\int \left(\frac{x^2 - 2}{\sqrt{x}}\right) dx$

a

$$\int \left(3x + \frac{2}{x}\right)^2 dx$$
$$= \int \left(9x^2 + 12 + \frac{4}{x^2}\right) dx$$
$$= \int (9x^2 + 12 + 4x^{-2})\,dx$$
$$= \frac{9x^3}{3} + 12x + \frac{4x^{-1}}{-1} + c$$
$$= 3x^3 + 12x - \frac{4}{x} + c$$



b

$$\int \left(\frac{x^2 - 2}{\sqrt{x}}\right) dx$$
$$= \int \left(\frac{x^2}{\sqrt{x}} - \frac{2}{\sqrt{x}}\right) dx$$
$$= \int (x^{\frac{3}{2}} - 2x^{-\frac{1}{2}})\,dx$$
$$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= \tfrac{2}{5}x^2\sqrt{x} - 4\sqrt{x} + c$$

EXERCISE 15E.1

1 Find:

a $\int (x^4 - x^2 - x + 2)\,dx$ **b** $\int (5x^4 - 4x^3 - 6x^2 - 7)\,dx$ **c** $\int (\sqrt{x} + e^x)\,dx$

d $\int (3e^x + x^2)\,dx$ **e** $\int (x\sqrt{x} - 2)\,dx$ **f** $\int \left(\frac{1}{x\sqrt{x}} + 4x\right) dx$

g $\int \left(\tfrac{1}{2}x^3 - x^4 + x^{\frac{1}{3}}\right) dx$ **h** $\int \left(\frac{x}{2} + x^2 - e^x\right) dx$ **i** $\int (5e^x + \tfrac{1}{3}x^3 - \sqrt{x})\,dx$

2 Integrate with respect to x:

a $3\sin x - 2$ **b** $4x - 2\cos x$ **c** $\sin x - 2\cos x + e^x$

d $x^2\sqrt{x} - 10\sin x$ **e** $\dfrac{x(x-1)}{3} + \cos x$ **f** $-\sin x + 2\sqrt{x}$

3 Find:

a $\displaystyle\int (x^2 + 3x - 2)\, dx$ **b** $\displaystyle\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$ **c** $\displaystyle\int \left(2e^x - \frac{1}{x^2}\right) dx$

d $\displaystyle\int \frac{1-4x}{x\sqrt{x}}\, dx$ **e** $\displaystyle\int (2x+1)^2\, dx$ **f** $\displaystyle\int \left(x + \frac{1}{x}\right)^2 dx$

g $\displaystyle\int \frac{2x-1}{\sqrt{x}}\, dx$ **h** $\displaystyle\int \frac{x^2 - 4x + 10}{x^2\sqrt{x}}\, dx$ **i** $\displaystyle\int (x+1)^3\, dx$

4 Find:

a $\displaystyle\int \left(\sqrt{x} + \tfrac{1}{2}\cos x\right) dx$ **b** $\displaystyle\int (2e^t - 4\sin t)\, dt$ **c** $\displaystyle\int (3\cos t - \sin t)\, dt$

5 Find y if:

a $\dfrac{dy}{dx} = 6$ **b** $\dfrac{dy}{dx} = 4x^2$ **c** $\dfrac{dy}{dx} = 5\sqrt{x} - x^2$

d $\dfrac{dy}{dx} = \dfrac{1}{x^2}$ **e** $\dfrac{dy}{dx} = 2e^x - 5$ **f** $\dfrac{dy}{dx} = 4x^3 + 3x^2$

6 Find $f(x)$ if:

a $f'(x) = (1-2x)^2$ **b** $f'(x) = \sqrt{x} - \dfrac{2}{\sqrt{x}}$ **c** $f'(x) = \dfrac{x^2 - 5}{x^2}$

PARTICULAR VALUES

We can find the constant of integration c if we are given a particular value of the function.

Example 9

Self Tutor

Find $f(x)$ given that:

a $f'(x) = x^3 - 2x^2 + 3$ and $f(0) = 2$ **b** $f'(x) = 2\sin x - \sqrt{x}$ and $f(0) = 4$.

a Since $f'(x) = x^3 - 2x^2 + 3$,

$$f(x) = \int (x^3 - 2x^2 + 3)\, dx$$

$$\therefore\ f(x) = \frac{x^4}{4} - \frac{2x^3}{3} + 3x + c$$

But $f(0) = 2$, so $c = 2$

Thus $f(x) = \dfrac{x^4}{4} - \dfrac{2x^3}{3} + 3x + 2$

b $$f(x) = \int \left(2\sin x - x^{\frac{1}{2}}\right) dx$$

$$\therefore\ f(x) = 2 \times (-\cos x) - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$\therefore\ f(x) = -2\cos x - \tfrac{2}{3}x^{\frac{3}{2}} + c$$

But $f(0) = 4$,

so $-2\cos 0 - 0 + c = 4$

$\therefore\ c = 6$

Thus $f(x) = -2\cos x - \tfrac{2}{3}x^{\frac{3}{2}} + 6$.

If we are given the second derivative we need to integrate twice to find the function. This creates two integrating constants, so we need two other facts about the curve in order to determine these constants.

Example 10

Self Tutor

Find $f(x)$ given that $f''(x) = 12x^2 - 4$, $f'(0) = -1$, and $f(1) = 4$.

If $f''(x) = 12x^2 - 4$

then $f'(x) = \frac{12x^3}{3} - 4x + c$ {integrating with respect to x}

$\therefore$ $f'(x) = 4x^3 - 4x + c$

But $f'(0) = -1$, so $c = -1$

Thus $f'(x) = 4x^3 - 4x - 1$

$\therefore$ $f(x) = \frac{4x^4}{4} - \frac{4x^2}{2} - x + d$ {integrating again}

$\therefore$ $f(x) = x^4 - 2x^2 - x + d$

But $f(1) = 4$, so $1 - 2 - 1 + d = 4$ and hence $d = 6$

Thus $f(x) = x^4 - 2x^2 - x + 6$

EXERCISE 15E.2

1 Find $f(x)$ given that:

- **a** $f'(x) = 2x - 1$ and $f(0) = 3$
- **b** $f'(x) = 3x^2 + 2x$ and $f(2) = 5$
- **c** $f'(x) = e^x + \frac{1}{\sqrt{x}}$ and $f(1) = 1$
- **d** $f'(x) = x - \frac{2}{\sqrt{x}}$ and $f(1) = 2$

2 Find $f(x)$ given that:

- **a** $f'(x) = x^2 - 4\cos x$ and $f(0) = 3$
- **b** $f'(x) = 2\cos x - 3\sin x$ and $f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

3 Find $f(x)$ given that:

- **a** $f''(x) = 2x + 1$, $f'(1) = 3$, and $f(2) = 7$
- **b** $f''(x) = 15\sqrt{x} + \frac{3}{\sqrt{x}}$, $f'(1) = 12$, and $f(0) = 5$
- **c** $f''(x) = \cos x$, $f'(\frac{\pi}{2}) = 0$, and $f(0) = 3$
- **d** $f''(x) = 2x$ and the points $(1, 0)$ and $(0, 5)$ lie on the curve.

F INTEGRATING $f(ax+b)$

In this section we deal with integrals of functions which are composite with the linear function $ax + b$.

Notice that $\frac{d}{dx}\left(\frac{1}{a}e^{ax+b}\right) = \frac{1}{a}e^{ax+b} \times a = e^{ax+b}$

$$\therefore \quad \boxed{\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c \quad \text{for } a \neq 0}$$

Likewise if $n \neq -1$,

$$\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+1}\right) = \frac{1}{a(n+1)}(n+1)(ax+b)^n \times a,$$
$$= (ax+b)^n$$

$$\therefore \quad \boxed{\int (ax+b)^n \, dx = \frac{1}{a}\frac{(ax+b)^{n+1}}{(n+1)} + c \quad \text{for } n \neq -1}$$

We can perform the same process for the circular functions:

$$\frac{d}{dx}(\sin(ax+b)) = a\cos(ax+b)$$
$$\therefore \quad \int a\cos(ax+b)\,dx = \sin(ax+b) + c$$
$$\therefore \quad a\int \cos(ax+b)\,dx = \sin(ax+b) + c$$

So,
$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c \quad \text{for } a \neq 0.$$

Likewise we can show
$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + c \quad \text{for } a \neq 0.$$

For a, b constants with $a \neq 0$, we have:

Function	*Integral*
e^{ax+b}	$\frac{1}{a}e^{ax+b} + c$
$(ax+b)^n, \ n \neq -1$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1} + c$
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + c$
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + c$

Example 11

Self Tutor

Find: **a** $\int (2x+3)^4\,dx$ **b** $\int \frac{1}{\sqrt{1-2x}}\,dx$

a
$$\int (2x+3)^4\,dx$$
$$= \tfrac{1}{2} \times \frac{(2x+3)^5}{5} + c$$
$$= \tfrac{1}{10}(2x+3)^5 + c$$

b
$$\int \frac{1}{\sqrt{1-2x}}\,dx$$
$$= \int (1-2x)^{-\frac{1}{2}}\,dx$$
$$= \tfrac{1}{-2} \times \frac{(1-2x)^{\frac{1}{2}}}{\frac{1}{2}} + c$$
$$= -\sqrt{1-2x} + c$$

Example 12

Self Tutor

Integrate with respect to x:

a $2e^{2x} - e^{-3x}$

b $2\sin(3x) + \cos(4x + \pi)$

a $\int (2e^{2x} - e^{-3x})\, dx$

$= 2(\frac{1}{2})e^{2x} - (\frac{1}{-3})e^{-3x} + c$

$= e^{2x} + \frac{1}{3}e^{-3x} + c$

b $\int (2\sin(3x) + \cos(4x + \pi))\, dx$

$= 2 \times -\frac{1}{3}\cos(3x) + \frac{1}{4}\sin(4x + \pi) + c$

$= -\frac{2}{3}\cos(3x) + \frac{1}{4}\sin(4x + \pi) + c$

EXERCISE 15F

1 Find:

a $\int (2x+5)^3\, dx$ **b** $\int \frac{1}{(3-2x)^2}\, dx$ **c** $\int \frac{4}{(2x-1)^4}\, dx$

d $\int (4x-3)^7\, dx$ **e** $\int \sqrt{3x-4}\, dx$ **f** $\int \frac{10}{\sqrt{1-5x}}\, dx$

g $\int 3(1-x)^4\, dx$ **h** $\int \frac{4}{\sqrt{3-4x}}\, dx$

2 Integrate with respect to x:

a $\sin(3x)$ **b** $2\cos(-4x) + 1$ **c** $3\cos\left(\frac{x}{2}\right)$

d $3\sin(2x) - e^{-x}$ **e** $2\sin\left(2x + \frac{\pi}{6}\right)$ **f** $-3\cos\left(\frac{\pi}{4} - x\right)$

g $\cos(2x) + \sin(2x)$ **h** $2\sin(3x) + 5\cos(4x)$ **i** $\frac{1}{2}\cos(8x) - 3\sin x$

3 Find $y = f(x)$ given $\frac{dy}{dx} = \sqrt{2x-7}$ and that $y = 11$ when $x = 8$.

4 The function $f(x)$ has gradient function $f'(x) = \frac{4}{\sqrt{1-x}}$, and the curve $y = f(x)$ passes through the point $(-3, -11)$.
Find the point on the graph of $y = f(x)$ with x-coordinate -8.

5 Find:

a $\int 3(2x-1)^2\, dx$ **b** $\int (x^2-x)^2\, dx$ **c** $\int (1-3x)^3\, dx$

d $\int (1-x^2)^2\, dx$ **e** $\int 4\sqrt{5-x}\, dx$ **f** $\int (x^2+1)^3\, dx$

6 Find:

a $\int \left(2e^x + 5e^{2x}\right)\, dx$ **b** $\int \left(3e^{5x-2}\right)\, dx$ **c** $\int \left(e^{7-3x}\right)\, dx$

d $\int (e^x + e^{-x})^2\, dx$ **e** $\int (e^{-x} + 2)^2\, dx$ **f** $\int \left(x - \frac{5}{(1-x)^2}\right)\, dx$

7 Find an expression for y given that $\frac{dy}{dx} = (1 - e^x)^2$, and that the graph has y-intercept 4.

8 Suppose $f'(x) = p\sin\left(\frac{1}{2}x\right)$ where p is a constant. $f(0) = 1$ and $f(2\pi) = 0$. Find p and hence $f(x)$.

9 Consider a function g such that $g''(x) = -\sin 2x$.
Show that the gradients of the tangents to $y = g(x)$ when $x = \pi$ and $x = -\pi$ are equal.

10 Find $f(x)$ given $f'(x) = 2e^{-2x}$ and $f(0) = 3$.

11 A curve has gradient function $\sqrt{x} + \frac{1}{2}e^{-4x}$ and passes through $(1, 0)$. Find the equation of the function.

G DEFINITE INTEGRALS

Earlier we saw the **fundamental theorem of calculus**:

If $F(x)$ is the antiderivative of $f(x)$ where $f(x)$ is continuous on the interval $a \leqslant x \leqslant b$, then the **definite integral** of $f(x)$ on this interval is $\displaystyle\boldsymbol{\int_a^b f(x)\,dx = F(b) - F(a)}$.

$\displaystyle\int_a^b f(x)\,dx$ reads "the integral from $x = a$ to $x = b$ of $f(x)$ with respect to x"
or "the integral from a to b of $f(x)$ with respect to x".

It is called a **definite** integral because there are lower and upper limits for the integration, and it therefore results in a numerical answer.

When calculating definite integrals we can omit the constant of integration c as this will always cancel out in the subtraction process.

It is common to write $F(b) - F(a)$ as $[F(x)]_a^b$, and so

$$\boldsymbol{\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)}$$

Earlier in the chapter we proved the following properties of definite integrals using the fundamental theorem of calculus:

- $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, c is any constant
- $\displaystyle\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
- $\displaystyle\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

Example 13

Self Tutor

Find:

a $\displaystyle\int_1^3 (x^2+2)\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{3}} \sin x\,dx$

a
$$\begin{aligned}&\int_1^3 (x^2+2)\,dx\\ &= \left[\frac{x^3}{3}+2x\right]_1^3\\ &= \left(\tfrac{3^3}{3}+2(3)\right)-\left(\tfrac{1^3}{3}+2(1)\right)\\ &= (9+6)-(\tfrac{1}{3}+2)\\ &= 12\tfrac{2}{3}\end{aligned}$$

b
$$\begin{aligned}&\int_0^{\frac{\pi}{3}} \sin x\,dx\\ &= [-\cos x]_0^{\frac{\pi}{3}}\\ &= (-\cos\tfrac{\pi}{3})-(-\cos 0)\\ &= -\tfrac{1}{2}+1\\ &= \tfrac{1}{2}\end{aligned}$$

EXERCISE 15G

Use questions **1** to **4** to check the properties of definite integrals.

1 Find: **a** $\displaystyle\int_1^4 \sqrt{x}\,dx$ and $\displaystyle\int_1^4 (-\sqrt{x})\,dx$ **b** $\displaystyle\int_0^1 x^7\,dx$ and $\displaystyle\int_0^1 (-x^7)\,dx$

2 Find: **a** $\displaystyle\int_0^1 x^2\,dx$ **b** $\displaystyle\int_1^2 x^2\,dx$ **c** $\displaystyle\int_0^2 x^2\,dx$ **d** $\displaystyle\int_0^1 3x^2\,dx$

3 Find: **a** $\displaystyle\int_0^2 (x^3-4x)\,dx$ **b** $\displaystyle\int_2^3 (x^3-4x)\,dx$ **c** $\displaystyle\int_0^3 (x^3-4x)\,dx$

4 Find: **a** $\displaystyle\int_0^1 x^2\,dx$ **b** $\displaystyle\int_0^1 \sqrt{x}\,dx$ **c** $\displaystyle\int_0^1 (x^2+\sqrt{x})\,dx$

5 Evaluate:

a $\displaystyle\int_0^1 x^3\,dx$ **b** $\displaystyle\int_0^2 (x^2-x)\,dx$ **c** $\displaystyle\int_0^1 e^x\,dx$

d $\displaystyle\int_0^{\frac{\pi}{6}} \cos x\,dx$ **e** $\displaystyle\int_1^4 \left(x-\frac{3}{\sqrt{x}}\right)dx$ **f** $\displaystyle\int_4^9 \frac{x-3}{\sqrt{x}}\,dx$

g $\displaystyle\int_1^3 \frac{1}{x}\,dx$ **h** $\displaystyle\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x\,dx$ **i** $\displaystyle\int_1^2 (e^{-x}+1)^2\,dx$

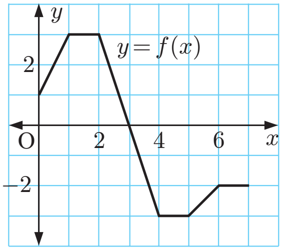
j $\displaystyle\int_2^6 \frac{1}{\sqrt{2x-3}}\,dx$ **k** $\displaystyle\int_0^1 e^{1-x}\,dx$ **l** $\displaystyle\int_0^{\frac{\pi}{6}} \sin(3x)\,dx$

6 Find m such that $\displaystyle\int_m^{2m} (2x-1)\,dx = 4$.

7 **a** Use the identity $\cos^2 x = \frac{1}{2} + \frac{1}{2}\cos(2x)$ to help evaluate $\int_0^{\frac{\pi}{4}} \cos^2 x \, dx$.

b Use the identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$ to help evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$.

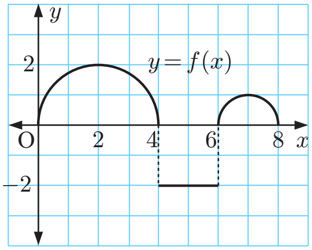
8 Evaluate the following integrals using area interpretation:



a $\int_0^3 f(x)\,dx$ **b** $\int_3^7 f(x)\,dx$

c $\int_2^4 f(x)\,dx$ **d** $\int_0^7 f(x)\,dx$

9 Evaluate the following integrals using area interpretation:



a $\int_0^4 f(x)\,dx$ **b** $\int_4^6 f(x)\,dx$

c $\int_6^8 f(x)\,dx$ **d** $\int_0^8 f(x)\,dx$

10 Write as a single integral:

a $\int_2^4 f(x)\,dx + \int_4^7 f(x)\,dx$ **b** $\int_1^3 g(x)\,dx + \int_3^8 g(x)\,dx + \int_8^9 g(x)\,dx$

11 **a** If $\int_1^3 f(x)\,dx = 2$ and $\int_1^6 f(x)\,dx = -3$, find $\int_3^6 f(x)\,dx$.

b If $\int_0^2 f(x)\,dx = 5$, $\int_4^6 f(x)\,dx = -2$, and $\int_0^6 f(x)\,dx = 7$, find $\int_2^4 f(x)\,dx$.

12 Given that $\int_{-1}^{1} f(x)\,dx = -4$, determine the value of:

a $\int_1^{-1} f(x)\,dx$ **b** $\int_{-1}^{1} (2 + f(x))\,dx$ **c** $\int_{-1}^{1} 2f(x)\,dx$

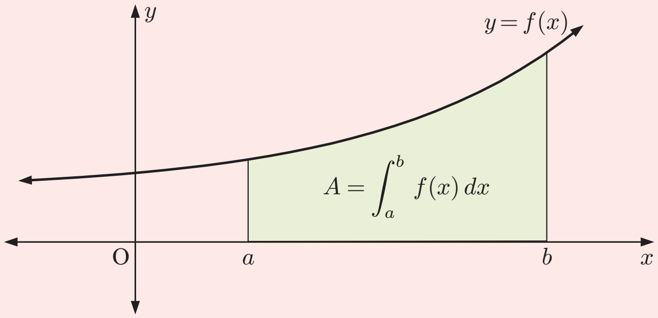
d k such that $\int_{-1}^{1} kf(x)\,dx = 7$

13 If $g(2) = 4$ and $g(3) = 5$, calculate $\int_2^3 (g'(x) - 1)\,dx$.

Historical note

Following the work of Newton and Leibniz, integration was rigorously formalised using limits by the German mathematician **Bernhard Riemann** (1826 - 1866).

If $f(x) \geqslant 0$ on the interval $a \leqslant x \leqslant b$, we have seen that the area under the curve is $A = \int_a^b f(x)\,dx$. This is known as the **Riemann integral**.



Bernhard Riemann

Review set 15A

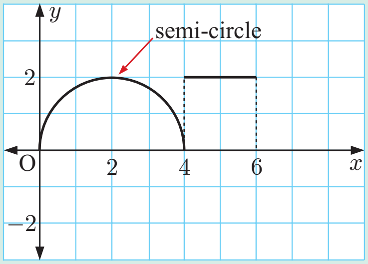
1 **a** Sketch the region between the curve $y = \dfrac{4}{1+x^2}$ and the x-axis for $0 \leqslant x \leqslant 1$.
Divide the interval into 5 equal parts and display the 5 upper and lower rectangles.

b Use the **area finder** software to find the lower and upper rectangle sums for $n = 5$, 50, 100, and 500.

AREA FINDER

c Give your best estimate for $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx$ and compare this answer with π.

2 The graph of $y = f(x)$ is illustrated:
Evaluate the following using area interpretation:



a $\displaystyle\int_0^4 f(x)\,dx$ **b** $\displaystyle\int_4^6 f(x)\,dx$

3 Integrate with respect to x:

a $\dfrac{4}{\sqrt{x}}$ **b** $\sin(4x-5)$ **c** e^{4-3x}

4 Find the exact value of:

a $\displaystyle\int_{-5}^{-1} \sqrt{1-3x}\,dx$ **b** $\displaystyle\int_0^{\frac{\pi}{2}} \cos\left(\frac{x}{2}\right)\,dx$

5 By differentiating $y = \sqrt{x^2-4}$, find $\displaystyle\int \frac{x}{\sqrt{x^2-4}}\,dx$.

6 Find the values of b such that $\displaystyle\int_0^b \cos x\,dx = \frac{1}{\sqrt{2}}$, $0 < b < \pi$.

7 Find y if:

a $\dfrac{dy}{dx} = (x^2-1)^2$

b $\dfrac{dy}{dx} = 400 - 20e^{-\frac{x}{2}}$

8 A curve $y = f(x)$ has $f''(x) = 18x + 10$. Find $f(x)$ if $f(0) = -1$ and $f(1) = 13$.

9 If $\displaystyle\int_0^a e^{1-2x}\,dx = \frac{e}{4}$, find a in the form $\ln k$.

10 Suppose $f''(x) = 3x^2 + 2x$ and $f(0) = f(2) = 3$. Find:

a $f(x)$

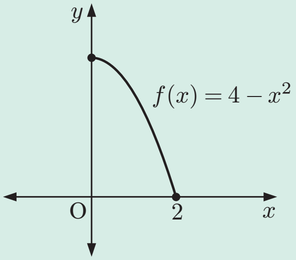
b the equation of the normal to $y = f(x)$ at $x = 2$.

11 **a** Find $(e^x+2)^3$ using the binomial expansion.

b Hence find the exact value of $\displaystyle\int_0^1 (e^x+2)^3\,dx$.

Review set 15B

1



a Use *four* upper and lower rectangles to find rational numbers A and B such that:

$$A < \int_0^2 (4-x^2)\,dx < B.$$

b Hence, find a good estimate for

$$\int_0^2 (4-x^2)\,dx.$$

2 Find:

a $\displaystyle\int (2e^{-x}+3)\,dx$

b $\displaystyle\int \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right) dx$

c $\displaystyle\int \left(3 + e^{2x-1}\right)^2 dx$

3 Given that $f'(x) = x^2 - 3x + 2$ and $f(1) = 3$, find $f(x)$.

4 Find the exact value of $\displaystyle\int_2^3 \frac{1}{\sqrt{3x-4}}\,dx$.

5 By differentiating $(3x^2+x)^3$, find $\displaystyle\int (3x^2+x)^2(6x+1)\,dx$.

6 If $\displaystyle\int_1^4 f(x)\,dx = 3$, determine:

a $\displaystyle\int_1^4 (f(x)+1)\,dx$

b $\displaystyle\int_1^2 f(x)\,dx - \int_4^2 f(x)\,dx$

7 Given that $f''(x) = 2\sin(2x)$, $f'(\frac{\pi}{2}) = 0$, and $f(0) = 3$, find the exact value of $f(\frac{\pi}{2})$.

8 Find $\dfrac{d}{dx}(e^{-2x}\sin x)$ and hence find $\displaystyle\int_0^{\frac{\pi}{2}} \left[e^{-2x}(\cos x - 2\sin x)\right] dx$

9 Find $\int (2x+3)^n\,dx$ for all integers $n \neq -1$.

10 A function has gradient function $2\sqrt{x} + \dfrac{a}{\sqrt{x}}$ and passes through the points $(0, 2)$ and $(1, 4)$.
Find a and hence explain why the function $y = f(x)$ has no stationary points.

11 $\displaystyle\int_a^{2a} (x^2 + ax + 2)\,dx = \frac{73a}{2}$. Find a.

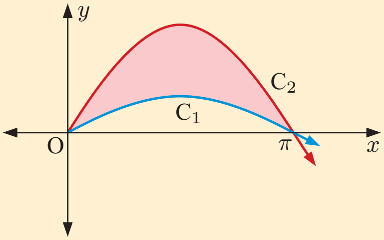
16

Applications of integration

Contents:

Opening problem

The illustrated curves are those of $y = \sin x$ and $y = 3\sin x$.



Things to think about:

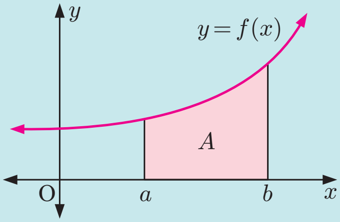
- **a** Can you identify each curve?
- **b** Can you find the shaded area enclosed by C_1 and C_2 for $0 \leqslant x \leqslant \pi$?

We have already seen how definite integrals can be related to the areas between functions and the x-axis. In this chapter we explore this relationship further, and consider other applications of integral calculus including kinematics.

A THE AREA UNDER A CURVE

We have already established in **Chapter 15** that:

If $f(x)$ is positive and continuous on the interval $a \leqslant x \leqslant b$, then the area bounded by $y = f(x)$, the x-axis, and the vertical lines $x = a$ and $x = b$ is given by $A = \int_a^b f(x)\,dx$ or $\int_a^b y\,dx$.



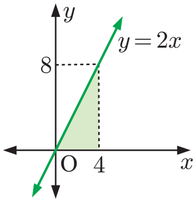
Example 1

Self Tutor

Find the area of the region enclosed by $y = 2x$, the x-axis, $x = 0$, and $x = 4$ by using:

- **a** a geometric argument
- **b** integration.

a



$$\text{Area} = \tfrac{1}{2} \times 4 \times 8$$
$$= 16 \text{ units}^2$$

b
$$\text{Area} = \int_0^4 2x\,dx$$
$$= \left[x^2\right]_0^4$$
$$= 4^2 - 0^2$$
$$= 16 \text{ units}^2$$

EXERCISE 16A

1 Find the area of each of the regions described below by using:

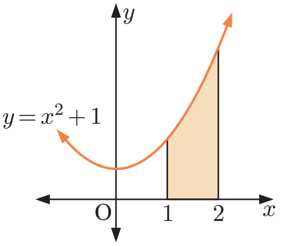
i a geometric argument **ii** integration

- **a** $y = 5$, the x-axis, $x = -6$, and $x = 0$
- **b** $y = x$, the x-axis, $x = 4$, and $x = 5$
- **c** $y = -3x$, the x-axis, $x = -3$, and $x = 0$
- **d** $y = -x$, the x-axis, $x = 0$, and $x = 2$

Example 2

Self Tutor

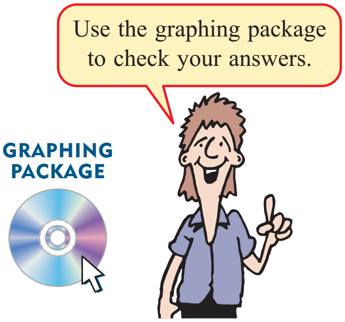
Find the area of the region enclosed by $y = x^2+1$, the x-axis, $x = 1$, and $x = 2$.



$$\text{Area} = \int_1^2 (x^2+1)\,dx$$
$$= \left[\frac{x^3}{3} + x\right]_1^2$$
$$= \left(\tfrac{8}{3} + 2\right) - \left(\tfrac{1}{3} + 1\right)$$
$$= 3\tfrac{1}{3} \text{ units}^2$$

2 Find the area of the region bounded by:

- **a** $y = x^2$, the x-axis, and $x = 1$
- **b** $y = \sin x$, the x-axis, $x = 0$, and $x = \pi$
- **c** $y = x^3$, the x-axis, $x = 1$, and $x = 4$
- **d** $y = e^x$, the x-axis, the y-axis, and $x = 1$
- **e** the x-axis and the part of $y = 6 + x - x^2$ above the x-axis
- **f** the axes and $y = \sqrt{9-x}$
- **g** $y = \dfrac{1}{x^2}$, the x-axis, $x = 1$, and $x = 2$
- **h** $y = 2 - \dfrac{1}{\sqrt{x}}$, the x-axis, and $x = 4$
- **i** $y = e^x + e^{-x}$, the x-axis, $x = -1$, and $x = 1$

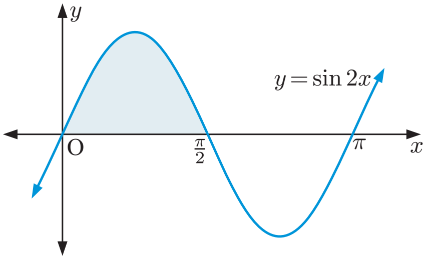


Example 3

Self Tutor

Find the area enclosed by one arch of the curve $y = \sin 2x$ and the x-axis.

The period of $y = \sin 2x$ is $\frac{2\pi}{2} = \pi$, so the first positive x-intercept is $\frac{\pi}{2}$.



$$\text{The required area} = \int_0^{\frac{\pi}{2}} \sin 2x\,dx$$
$$= \left[\tfrac{1}{2}(-\cos 2x)\right]_0^{\frac{\pi}{2}}$$
$$= -\tfrac{1}{2}\left[\cos 2x\right]_0^{\frac{\pi}{2}}$$
$$= -\tfrac{1}{2}(\cos \pi - \cos 0)$$
$$= 1 \text{ unit}^2$$

3 Find the area enclosed by one arch of the curve $y = \cos 3x$ and the x-axis.

Discovery $\int_a^b f(x)\,dx$ and areas

Does $\int_a^b f(x)\,dx$ always give us an area?

What to do:

1. Find $\int_0^1 x^3\,dx$ and $\int_{-1}^1 x^3\,dx$.
2. Using a graph, explain why the first integral in **1** gives an area, whereas the second integral does not.
3. Find $\int_{-1}^0 x^3\,dx$ and explain why the answer is negative.
4. Show that $\int_{-1}^0 x^3\,dx + \int_0^1 x^3\,dx = \int_{-1}^1 x^3\,dx$.
5. Find $\int_0^{-1} x^3\,dx$ and interpret its meaning.
6. Suppose $f(x)$ is a function such that $f(x) \leqslant 0$ for all $a \leqslant x \leqslant b$. Suggest an expression for the area between the curve and the function for $a \leqslant x \leqslant b$.

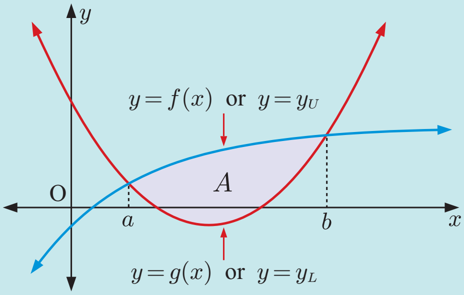
B THE AREA BETWEEN TWO FUNCTIONS

If two functions $f(x)$ and $g(x)$ intersect at $x = a$ and $x = b$, and $f(x) \geqslant g(x)$ for all $a \leqslant x \leqslant b$, then the area of the shaded region between their points of intersection is given by

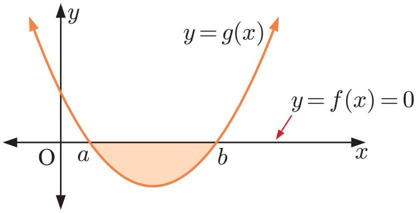
$$A = \int_a^b [f(x) - g(x)]\,dx.$$

Alternatively, if the upper and lower functions are $y = y_U$ and $y = y_L$ respectively, then the area is

$$A = \int_a^b [y_U - y_L]\,dx.$$



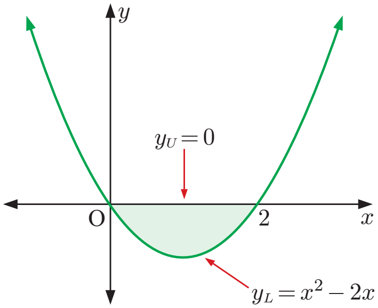
We can see immediately that if $f(x)$ is the x-axis $f(x) = 0$, then the enclosed area is $\int_a^b [-g(x)]\,dx$ or $-\int_a^b g(x)\,dx$.



Example 4

Self Tutor

Use $\int_a^b [y_U - y_L]\,dx$ to find the area bounded by the x-axis and $y = x^2 - 2x$.



The curve cuts the x-axis when $y = 0$

$\therefore \quad x^2 - 2x = 0$

$\therefore \quad x(x-2) = 0$

$\therefore \quad x = 0$ or 2

$\therefore$ the x-intercepts are 0 and 2.

$$\begin{aligned}\text{Area} &= \int_0^2 [y_U - y_L]\,dx \\ &= \int_0^2 [0 - (x^2 - 2x)]\,dx \\ &= \int_0^2 (2x - x^2)\,dx \\ &= \left[x^2 - \frac{x^3}{3}\right]_0^2 \\ &= \left(4 - \tfrac{8}{3}\right) - (0)\end{aligned}$$

$\therefore$ the area is $\frac{4}{3}$ units^2.

Example 5

Self Tutor

Find the area of the region enclosed by $y = x + 2$ and $y = x^2 + x - 2$.

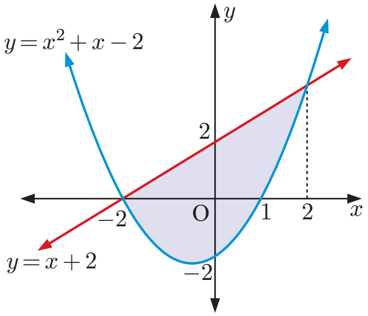
$y = x + 2$ meets $y = x^2 + x - 2$

where $x^2 + x - 2 = x + 2$

$\therefore \quad x^2 - 4 = 0$

$\therefore \quad (x+2)(x-2) = 0$

$\therefore \quad x = \pm 2$



$$\begin{aligned}\text{Area} &= \int_{-2}^2 [y_U - y_L]\,dx \\ &= \int_{-2}^2 [(x+2) - (x^2 + x - 2)]\,dx \\ &= \int_{-2}^2 (4 - x^2)\,dx \\ &= \left[4x - \frac{x^3}{3}\right]_{-2}^2 \\ &= \left(8 - \tfrac{8}{3}\right) - \left(-8 + \tfrac{8}{3}\right) \\ &= 10\tfrac{2}{3} \text{ units}^2\end{aligned}$$

$\therefore$ the area is $10\frac{2}{3}$ units^2.

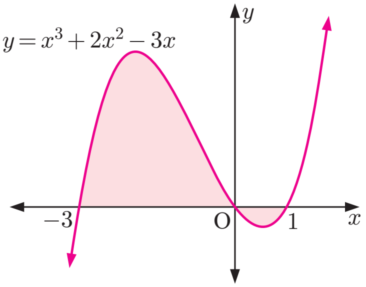
Example 6

Self Tutor

Find the total area of the regions contained by $y = f(x)$ and the x-axis for $f(x) = x^3 + 2x^2 - 3x$.

$$\begin{aligned} f(x) &= x^3 + 2x^2 - 3x \\ &= x(x^2 + 2x - 3) \\ &= x(x-1)(x+3) \end{aligned}$$

$\therefore$ $y = f(x)$ cuts the x-axis at 0, 1, and -3.



Total area

$$\begin{aligned} &= \int_{-3}^{0} (x^3 + 2x^2 - 3x)\, dx - \int_{0}^{1} (x^3 + 2x^2 - 3x)\, dx \\ &= \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{-3}^{0} - \left[\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2}\right]_{0}^{1} \\ &= \left(0 - -11\tfrac{1}{4}\right) - \left(-\tfrac{7}{12} - 0\right) \\ &= 11\tfrac{5}{6} \text{ units}^2 \end{aligned}$$

EXERCISE 16B

1 Find the exact value of the area bounded by:

- **a** the x-axis and $y = x^2 + x - 2$
- **b** the x-axis, $y = e^{-x} - 1$, and $x = 2$
- **c** the x-axis and the part of $y = 3x^2 - 8x + 4$ below the x-axis
- **d** $y = \cos x$, the x-axis, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$
- **e** $y = x^3 - 4x$, the x-axis, $x = 1$, and $x = 2$
- **f** $y = \sin x - 1$, the x-axis, $x = 0$, and $x = \frac{\pi}{2}$

2 Find the area of the region enclosed by $y = x^2 - 2x$ and $y = 3$.

3 Consider the graphs of $y = x - 3$ and $y = x^2 - 3x$.

- **a** Sketch the graphs on the same set of axes.
- **b** Find the coordinates of the points where the graphs meet.
- **c** Find the area of the region enclosed by the two graphs.

4 Determine the area of the region enclosed by $y = \sqrt{x}$ and $y = x^2$.

5
- **a** On the same set of axes, graph $y = e^x - 1$ and $y = 2 - 2e^{-x}$, showing axes intercepts and asymptotes.
- **b** Find algebraically the points of intersection of $y = e^x - 1$ and $y = 2 - 2e^{-x}$.
- **c** Find the area of the region enclosed by the two curves.

6 Find the area of the region bounded by $y = 2e^x$, $y = e^{2x}$, and $x = 0$.

7 On the same set of axes, sketch $y = 2x$ and $y = 4x^2$.
Find the area of the region enclosed by these functions.

8 Sketch the circle with equation $x^2 + y^2 = 9$.

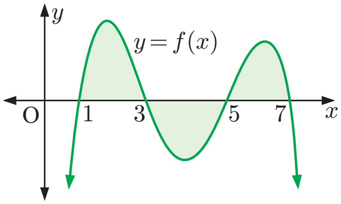
- **a** Explain why the upper half of the circle has equation $y = \sqrt{9 - x^2}$.
- **b** Hence, determine $\int_0^3 \sqrt{9 - x^2}\, dx$ without actually integrating the function.

9 Find the area enclosed by the function $y = f(x)$ and the x-axis for:

- **a** $f(x) = x^3 - 9x$
- **b** $f(x) = -x(x-2)(x-4)$
- **c** $f(x) = x^4 - 5x^2 + 4$.

10 Answer the **Opening Problem** on page **438**.

11
- **a** Explain why the total area shaded is *not* equal to $\int_1^7 f(x)\, dx$.
- **b** Write an expression for the total shaded area in terms of integrals.



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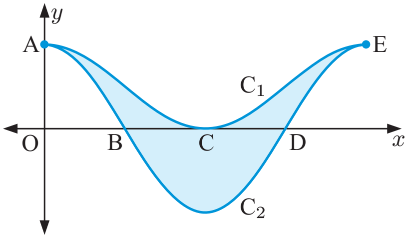
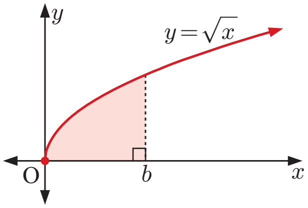


The illustrated curves are $y = \cos(2x)$ and $y = \frac{1}{2} + \frac{1}{2}\cos(2x)$.

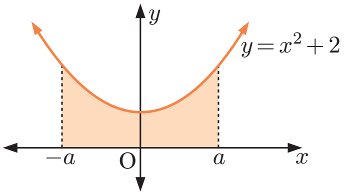
- **a** Identify each curve as C_1 or C_2.
- **b** Determine the coordinates of A, B, C, D, and E.
- **c** Show that the area of the shaded region is $\frac{\pi}{2}$ units^2.

13 Explain why the area between two functions $f(x)$ and $g(x)$ on the interval $a \leqslant x \leqslant b$ is given by $A = \int_a^b |f(x) - g(x)|\, dx$.

14 The shaded area is 1 unit^2.
Find b, correct to 4 decimal places.



15 The shaded area is $6a$ units^2.
Find the exact value of a.



C KINEMATICS

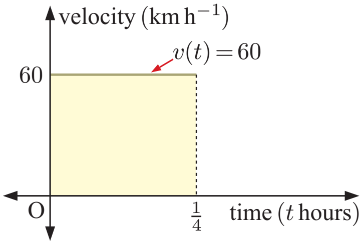
DISTANCES FROM VELOCITY GRAPHS

Suppose a car travels at a constant positive velocity of $60\ \text{km h}^{-1}$ for 15 minutes.

We know the distance travelled $=$ speed $\times$ time

$$= 60\ \text{km h}^{-1} \times \tfrac{1}{4}\ \text{h}$$

$$= 15\ \text{km}.$$



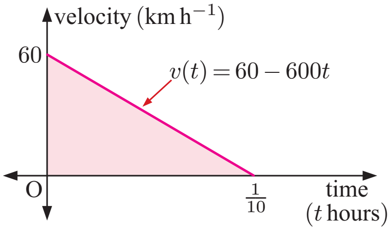
When we graph *velocity* against *time*, the graph is a horizontal line, and we can see that the distance travelled is the area shaded.

So, the distance travelled can also be found by the definite integral $\displaystyle\int_0^{\frac{1}{4}} 60\, dt = 15$ km.

Now suppose the velocity decreases at a constant rate, so that the car, initially travelling at $60\ \text{km h}^{-1}$, stops in 6 minutes or $\frac{1}{10}$ hour.

In this case the *average* speed is $30\ \text{km h}^{-1}$, so the distance travelled $= 30\ \text{km h}^{-1} \times \frac{1}{10}$ h

$= 3$ km



But the triangle has area $= \frac{1}{2} \times$ base $\times$ altitude

$$= \tfrac{1}{2} \times \tfrac{1}{10} \times 60 = 3$$

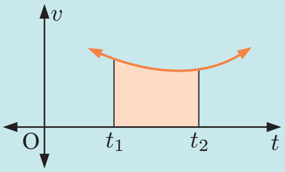
So, once again the shaded area gives us the distance travelled, and we can find it using the definite integral

$$\int_0^{\frac{1}{10}} (60 - 600t)\, dt = 3.$$

These results suggest that:

For a velocity-time function $v(t)$ where $v(t) \geqslant 0$ on the interval $t_1 \leqslant t \leqslant t_2$,

$$\textbf{distance travelled} = \int_{t_1}^{t_2} v(t)\, dt.$$

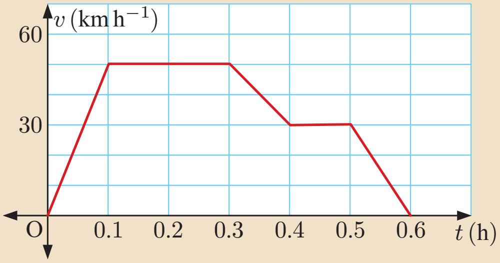


If we have a change of direction within the time interval then the velocity will change sign. We therefore need to add the components of area above and below the t-axis to find the total distance travelled.

Example 7

Self Tutor

The velocity-time graph for a train journey is illustrated in the graph alongside. Find the total distance travelled by the train.



Total distance travelled

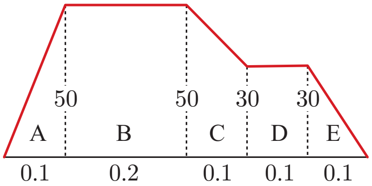
$= $ total area under the graph

$= $ area A + area B + area C + area D + area E

$= \frac{1}{2}(0.1)50 + (0.2)50 + \left(\frac{50+30}{2}\right)(0.1) + (0.1)30 + \frac{1}{2}(0.1)30$

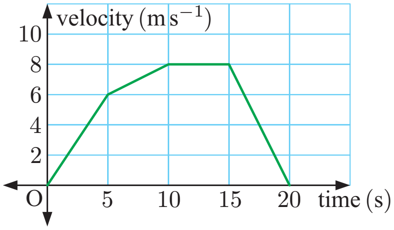
$= 2.5 + 10 + 4 + 3 + 1.5$

$= 21$ km



EXERCISE 16C.1

1 A runner has the velocity-time graph shown. Find the total distance travelled by the runner.



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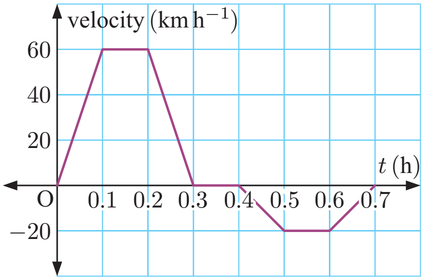


A car travels along a straight road with the velocity-time function illustrated.

- **a** What is the significance of the graph:
  - **i** above the t-axis
  - **ii** below the t-axis?
- **b** Find the total *distance* travelled by the car.
- **c** Find the final *displacement* of the car from its starting point.

3 A cyclist rides off from rest, accelerating at a constant rate for 3 minutes until she reaches 40 km h^{-1}. She then maintains a constant speed for 4 minutes until reaching a hill. She slows down at a constant rate over one minute to 30 km h^{-1}, then continues at this rate for 10 minutes. At the top of the hill she reduces her speed uniformly and is stationary 2 minutes later.

- **a** Draw a graph to show the cyclist's motion.
- **b** How far has the cyclist travelled?

DISPLACEMENT AND VELOCITY FUNCTIONS

In this section we are concerned with **motion in a straight line**.

For some displacement function $s(t)$, the velocity function is $v(t) = s'(t)$.

So, given a velocity function we can determine the displacement function by the integral

$$\boxed{s(t) = \int v(t)\,dt}$$

The constant of integration determines the **initial position** on the line where the object begins.

Using the displacement function we can determine the change in displacement in a time interval $t_1 \leqslant t \leqslant t_2$ using the integral:

$$\boxed{\textbf{Displacement} = s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)\,dt}$$

TOTAL DISTANCE TRAVELLED

To determine the total distance travelled in a time interval $t_1 \leqslant t \leqslant t_2$, we need to account for any changes of direction in the motion.

To find the total distance travelled given a velocity function $v(t) = s'(t)$ on $t_1 \leqslant t \leqslant t_2$:

- Draw a sign diagram for $v(t)$ so we can determine any changes of direction.
- Determine $s(t)$ by integration, including a constant of integration.
- Find $s(t_1)$ and $s(t_2)$. Also find $s(t)$ at each time the direction changes.
- Draw a motion diagram.
- Determine the total distance travelled from the motion diagram.

VELOCITY AND ACCELERATION FUNCTIONS

We know that the acceleration function is the derivative of the velocity function, so $a(t) = v'(t)$.

So, given an acceleration function, we can determine the velocity function by integration:

$$\boxed{v(t) = \int a(t)\,dt}$$

Summary

$$\begin{tikzcd}
\boxed{\begin{array}{c} s(t) \\ \textbf{displacement} \end{array}} \arrow[r, bend left, "\text{differentiate}"] & \boxed{\begin{array}{c} v(t) = \dfrac{ds}{dt} \\ \textbf{velocity} \end{array}} \arrow[l, bend left, "\text{integrate}"] \arrow[r, bend left, "\text{differentiate}"] & \boxed{\begin{array}{c} a(t) = \dfrac{dv}{dt} = \dfrac{d^2s}{dt^2} \\ \textbf{acceleration} \end{array}} \arrow[l, bend left, "\text{integrate}"]
\end{tikzcd}$$

Example 8

Self Tutor

A particle P moves in a straight line with velocity function $v(t) = t^2 - 3t + 2$ m s^{-1}.

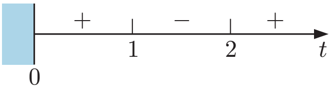
a How far does P travel in the first 4 seconds of motion?

b Find the displacement of P after 4 seconds.

a $v(t) = s'(t) = t^2 - 3t + 2$

$= (t-1)(t-2)$

$\therefore$ the sign diagram of v is:



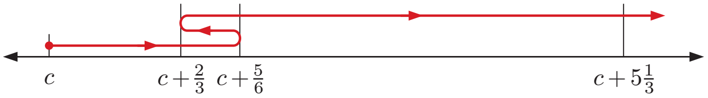
Since the signs change, P reverses direction at $t = 1$ and $t = 2$ seconds.

Now $s(t) = \int (t^2 - 3t + 2)\,dt = \frac{t^3}{3} - \frac{3t^2}{2} + 2t + c$

Hence $s(0) = c$ $\qquad s(1) = \frac{1}{3} - \frac{3}{2} + 2 + c = c + \frac{5}{6}$

$s(2) = \frac{8}{3} - 6 + 4 + c = c + \frac{2}{3}$ $\qquad s(4) = \frac{64}{3} - 24 + 8 + c = c + 5\frac{1}{3}$

Motion diagram:



$\therefore$ total distance travelled $= (c + \frac{5}{6} - c) + (c + \frac{5}{6} - [c + \frac{2}{3}]) + (c + 5\frac{1}{3} - [c + \frac{2}{3}])$

$= \frac{5}{6} + \frac{5}{6} - \frac{2}{3} + 5\frac{1}{3} - \frac{2}{3}$

$= 5\frac{2}{3}$ m

b Displacement = final position − original position

$= s(4) - s(0)$

$= c + 5\frac{1}{3} - c$

$= 5\frac{1}{3}$ m

So, the displacement is $5\frac{1}{3}$ m to the right.

EXERCISE 16C.2

1 A particle has velocity function $v(t) = 1 - 2t$ cm s^{-1} as it moves in a straight line. The particle is initially 2 cm to the right of O.

- **a** Write a formula for the displacement function $s(t)$.
- **b** Find the total distance travelled in the first second of motion.
- **c** Find the displacement of the particle at the end of one second.

2 Particle P is initially at the origin O. It moves with the velocity function $v(t) = t^2 - t - 2$ cm s^{-1}.

- **a** Write a formula for the displacement function $s(t)$.
- **b** Find the total distance travelled in the first 3 seconds of motion.
- **c** Find the displacement of the particle at the end of three seconds.

3 An object has velocity function $v(t) = \cos(2t)$ m s^{-1}. If $s(\frac{\pi}{4}) = 1$ m, determine $s(\frac{\pi}{3})$ exactly.

4 The velocity of a moving object is given by $v(t) = 32 + 4t$ m s^{-1}.

a If $s = 16$ m when $t = 0$ seconds, find the displacement function.

b Explain why the displacement of the object and its total distance travelled in the interval $0 \leqslant t \leqslant t_1$, can both be represented by the definite integral $\int_0^{t_1} (32 + 4t)\, dt$.

c Show that the object is travelling with constant acceleration.

5 A particle moves along the x-axis with velocity function $s'(t) = 16t - 4t^3$ units per second. Find the total distance travelled in the time interval:

a $0 \leqslant t \leqslant 3$ seconds **b** $1 \leqslant t \leqslant 3$ seconds.

6 A particle moves in a straight line with velocity function $v(t) = \cos t$ m s^{-1}.

a Show that the particle oscillates between two points.

b Find the distance between the two points in **a**.

7 The velocity of a particle travelling in a straight line is given by $v(t) = 50 - 10e^{-0.5t}$ m s^{-1}, where $t \geqslant 0$, t in seconds.

a State the initial velocity of the particle.

b Find the velocity of the particle after 3 seconds.

c How long will it take for the particle's velocity to increase to 45 m s^{-1}?

d Discuss $v(t)$ as $t \to \infty$.

e Show that the particle's acceleration is always positive.

f Draw the graph of $v(t)$ against t.

g Find the total distance travelled by the particle in the first 3 seconds of motion.

Example 9

Self Tutor

A particle is initially at the origin and moving to the right at 5 cm s^{-1}. It accelerates with time according to $a(t) = 4 - 2t$ cm s^{-2}.

a Find the velocity function of the particle, and sketch its graph for $0 \leqslant t \leqslant 6$ s.

b For the first 6 seconds of motion, determine the:

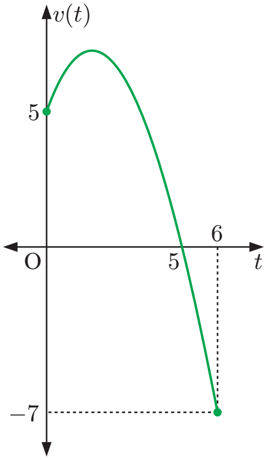
i displacement of the particle **ii** total distance travelled.

a $v(t) = \int a(t)\, dt = \int (4 - 2t)\, dt$

$= 4t - t^2 + c$

But $v(0) = 5$, so $c = 5$

$\therefore$ $v(t) = -t^2 + 4t + 5$ cm s^{-1}



b $s(t) = \int v(t)\, dt = \int (-t^2 + 4t + 5)\, dt$

$= -\frac{1}{3}t^3 + 2t^2 + 5t + c$ cm

But $s(0) = 0$, so $c = 0$

$\therefore$ $s(t) = -\frac{1}{3}t^3 + 2t^2 + 5t$ cm

i Displacement $= s(6) - s(0)$

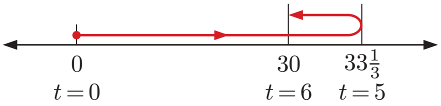
$= -\frac{1}{3}(6)^3 + 2(6)^2 + 5(6)$

$= 30$ cm

ii The particle changes direction when $t = 5$ s.

Now $s(5) = -\frac{1}{3}(5)^3 + 2(5)^2 + 5(5) = 33\frac{1}{3}$ cm

Motion diagram:



$\therefore$ the total distance travelled $= 33\frac{1}{3} + 3\frac{1}{3}$

$= 36\frac{2}{3}$ cm

8 A particle is initially stationary at the origin. It accelerates according to the function $a(t) = \dfrac{2}{(t+1)^3}$ m s^{-2}.

- **a** Find the velocity function $v(t)$ for the particle.
- **b** Find the displacement function $s(t)$ for the particle.
- **c** Describe the motion of the particle at the time $t = 2$ seconds.

9 A train moves along a straight track with acceleration $\dfrac{t}{10} - 3$ m s^{-2}. The initial velocity of the train is 45 m s^{-1}.

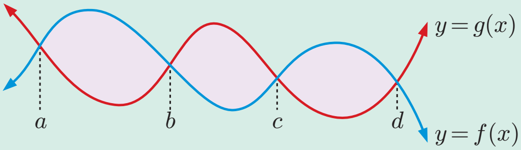
- **a** Determine the velocity function $v(t)$.
- **b** Evaluate $\displaystyle\int_0^{60} v(t)\,dt$ and explain what this value represents.

10 An object has initial velocity 20 m s^{-1} as it moves in a straight line with acceleration function $4e^{-\frac{t}{20}}$ m s^{-2}.

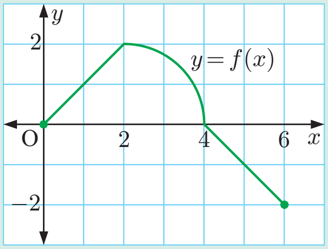
- **a** Show that as t increases the object approaches a limiting velocity.
- **b** Find the total distance travelled in the first 10 seconds of motion.

Review set 16A

1 Write an expression for the total shaded area.



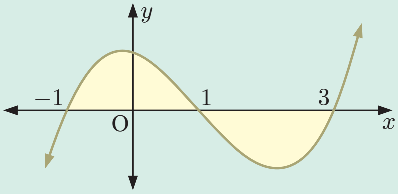
2



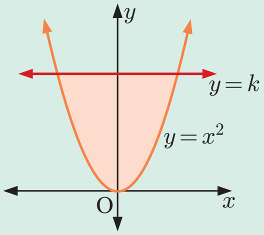
Find:

- **a** $\displaystyle\int_0^4 f(x)\,dx$
- **b** $\displaystyle\int_4^6 f(x)\,dx$
- **c** $\displaystyle\int_0^6 f(x)\,dx$

3 Does $\int_{-1}^{3} f(x)\,dx$ represent the area of the shaded region?
Explain your answer briefly.



4 Determine k if the enclosed region has area $5\frac{1}{3}$ units^2.



5 Find the area of the region enclosed by $y = x^2 + 4x + 1$ and $y = 3x + 3$.

6 A particle moves in a straight line with velocity $v(t) = t^2 - 6t + 8$ m s^{-1}, for $t \geqslant 0$ seconds.

- **a** Draw a sign diagram for $v(t)$.
- **b** Describe what happens to the particle in the first 5 seconds of motion.
- **c** After 5 seconds, how far is the particle from its original position?
- **d** Find the total distance travelled in the first 5 seconds of motion.

7 Determine the area enclosed by the axes and $y = 4e^x - 1$.

8 A particle moves in a straight line with velocity given by $v(t) = \sin t$ m s^{-1}, where $t \geqslant 0$ seconds. Find the total distance travelled by the particle in the first 4 seconds of motion.

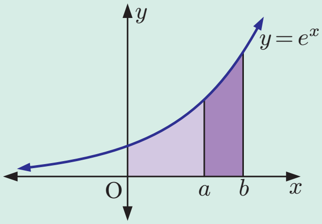
Review set 16B

1 At time $t = 0$ a particle passes through the origin with velocity 27 cm s^{-1}. Its acceleration t seconds later is $6t - 30$ cm s^{-2}.

- **a** Write an expression for the particle's velocity.
- **b** Calculate the displacement from the origin after 6 seconds.

2
- **a** Sketch the graphs of $y = \frac{1}{2} - \frac{1}{2}\cos 2x$ and $y = \sin x$ on the same set of axes for $0 \leqslant x \leqslant \pi$.
- **b** Verify that both graphs pass through the points $(0, 0)$ and $(\frac{\pi}{2}, 1)$.
- **c** Find the area enclosed by these curves for $0 \leqslant x \leqslant \frac{\pi}{2}$.

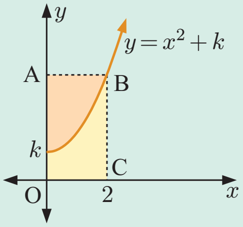
3 Find a given that the area of the region between $y = e^x$ and the x-axis from $x = 0$ to $x = a$ is 2 units^2.
Hence determine b such that the area of the region from $x = a$ to $x = b$ is also 2 units^2.



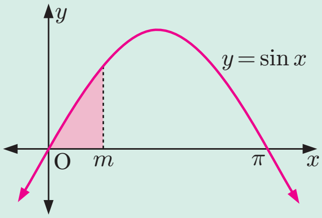
4 A particle moves in a straight line with velocity $v(t) = 2t - 3t^2$ m s^{-1}, for $t \geqslant 0$ seconds.

- **a** Find a formula for the acceleration function $a(t)$.
- **b** Find a formula for the displacement function $s(t)$.
- **c** Find the change in displacement after two seconds.

5 OABC is a rectangle and the two shaded regions are equal in area. Find k.



6 The shaded region has area $\frac{1}{2}$ unit^2. Find the value of m.



7 Find the area of the region enclosed by $y = x$ and $y = \sin\left(\frac{\pi x}{2}\right)$.

8 A boat travelling in a straight line has its engine turned off at time $t = 0$. Its velocity at time t seconds thereafter is given by $v(t) = \dfrac{100}{(t+2)^2}$ m s^{-1}.

- **a** Find the initial velocity of the boat, and its velocity after 3 seconds.
- **b** Discuss $v(t)$ as $t \to \infty$.
- **c** Sketch the graph of $v(t)$ against t.
- **d** Find how long it takes for the boat to travel 30 metres from when the engine is turned off.
- **e** Find the acceleration of the boat at any time t.
- **f** Show that $\dfrac{dv}{dt} = -kv^{\frac{3}{2}}$, and find the value of the constant k.

ANSWERS

EXERCISE 1A

1 **a** $5 \in D$ **b** $6 \notin G$ **c** $d \notin \{a, e, i, o, u\}$
d $\{2, 5\} \subseteq \{1, 2, 3, 4, 5, 6\}$
e $\{3, 8, 6\} \nsubseteq \{1, 2, 3, 4, 5, 6\}$

2 **a** **i** $\{9\}$ **ii** $\{5, 6, 7, 8, 9, 10, 11, 12, 13\}$
b **i** $\varnothing$ **ii** $\{1, 2, 3, 4, 5, 6, 7, 8\}$
c **i** $\{1, 3, 5, 7\} = A$ **ii** $\{1, 2, 3, 4, 5, 6, 7, 8, 9\} = B$

3 **a** 5 **b** 6 **c** 2 **d** 9

4 **a** true **b** true **c** true **d** true
e false **f** true **g** true **h** false

5 **a** finite **b** infinite **c** infinite **d** infinite

6 **a** true **b** true **c** false **d** true

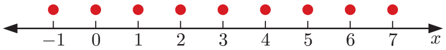
7 **a** disjoint **b** not disjoint **8** true

9 **a** 15 subsets **b** $2^n - 1, \ n \in \mathbb{Z}^+$

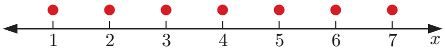
EXERCISE 1B

1 **a** finite **b** infinite **c** infinite **d** infinite
e infinite **f** infinite **g** infinite

2 **a** **i** The set of all integers x such that x is between -1 and 7, including -1 and 7.
ii $\{-1, 0, 1, 2, 3, 4, 5, 6, 7\}$ **iii** 9
iv



b **i** The set of all natural numbers x such that x is between -2 and 8.
ii $\{1, 2, 3, 4, 5, 6, 7\}$ **iii** 7
iv

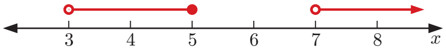


c **i** The set of all real numbers x such that x is between 0 and 1, including 0 and 1.
ii not possible **iii** infinite
iv

d **i** The set of all rational numbers x such that x is between 5 and 6, including 5 and 6.
ii not possible **iii** infinite
iv cannot be illustrated

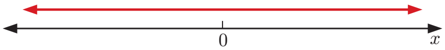
e **i** The set of all real numbers x such that x is between -1 and 5, including -1.
ii not possible **iii** infinite
iv

f **i** The set of all real numbers x such that x is between 3 and 5 (including 5), or greater than 7.
ii not possible **iii** infinite
iv



g **i** The set of all real numbers x such that x is less than or equal to 1, or greater than 2.
ii not possible **iii** infinite
iv

h **i** The set of all real numbers x such that x is less than 2, or greater than or equal to 1. (So, A is the set of all real numbers.)
ii not possible **iii** infinite
iv



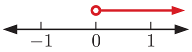
3 **a** $A = \{x \in \mathbb{Z} : -100 < x < 100\}$
b $A = \{x \in \mathbb{R} : x > 1000\}$
c $A = \{x \in \mathbb{Q} : 2 \leqslant x \leqslant 3\}$

4 **a** $A = \{x \in \mathbb{Z} : -2 \leqslant x \leqslant 3\}$
b $A = \{x \in \mathbb{Z} : x \leqslant -3\}$
c $A = \{x \in \mathbb{R} : -3 \leqslant x < 2\}$
d $A = \{x \in \mathbb{R} : 1 \leqslant x \leqslant 3 \cup x > 5\}$

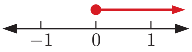
5 **a** $A \subseteq B$ **b** $A \nsubseteq B$ **c** $A \subseteq B$ **d** $A \subseteq B$
e $A \nsubseteq B$ **f** $A \nsubseteq B$

6 **a** neither **b** open **c** neither **d** open
e closed **f** neither

7 **a** There are infinitely many rational numbers within any given interval, so we cannot represent $\mathbb{Q}$ as a series of dots like we can with $\mathbb{Z}$. We cannot represent $\mathbb{Q}$ with a continuous line either (like we do with $\mathbb{R}$), as this would imply that irrational numbers are part of $\mathbb{Q}$.
b **i** the set of positive real numbers, $\{x \in \mathbb{R} : x > 0\}$

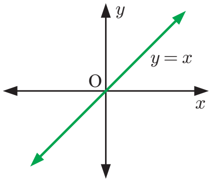


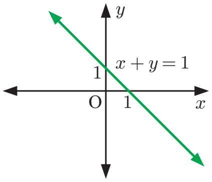
ii the set of positive real numbers and zero, $\{x \in \mathbb{R} : x \geqslant 0\}$



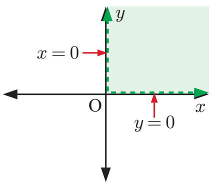
EXERCISE 1C

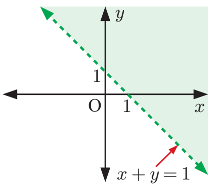
1 **a** infinite **b** infinite





c infinite **d** infinite

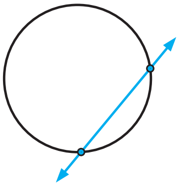




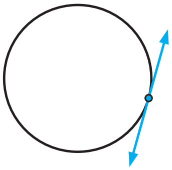
2 **a** infinite **b** finite **c** infinite

3 **a** **i** The set of all points of intersection between the line and the circle.
ii The set of all points that lie on either the straight line or the circle.

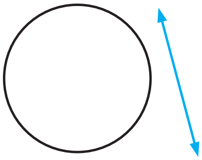
b **i** There are two points of intersection between the straight line and the circle.



ii There is one point of intersection between the straight line and the circle (that is, the straight line is a tangent to the circle).



iii The straight line and the circle do not intersect.



EXERCISE 1D

1 **a** $C' = \{\text{consonants}\}$ **b** $C' = \{x \in \mathbb{Z} : x \geqslant 0\}$
c $C' = \{x \in \mathbb{Z} : x \geqslant -4\}$ **d** $C' = \{x \in \mathbb{Q} : 2 < x < 8\}$

2 **a** $\{2, 3, 4, 5, 6, 7\}$ **b** $\{0, 1, 8\}$ **c** $\{5, 6, 7, 8\}$
d $\{0, 1, 2, 3, 4\}$ **e** $\{5, 6, 7\}$ **f** $\{2, 3, 4, 5, 6, 7, 8\}$
g $\{2, 3, 4\}$ **h** $\{0, 1, 2, 3, 4, 8\}$

3 **a** 9 **b** 11 **4** **a** false **b** true

5 **a** $\{1, 2, 10, 11, 12\}$ **b** $\{1, 2, 3, 4, 12\}$
c $\{1, 8, 9, 10, 11, 12\}$ **d** $\{3, 4, 5, 6, 7\}$
e $\{1, 2, 8, 9, 10, 11, 12\}$ **f** $\{8, 9, 10, 11\}$
g $\{1, 2, 5, 6, 7, 8, 9, 10, 11, 12\}$ **h** $\{2, 10, 11\}$

6 **a** $[0, \infty)$ **b** $(-\infty, 1)$ **c** $(-\infty, 3) \cup [2, \infty)$
d $(-\infty, -5] \cup (7, \infty)$ **e** $[1, 3)$
f $(-\infty, -5) \cup [0, 1]$

EXERCISE 1E

1 **a** $P = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$ **b** $\{2, 5, 11\}$
c $\{2, 3, 4, 5, 7, 11, 12, 13, 15, 17, 19, 23\}$
d $12 = 9 + 6 - 3$ ✓

2 **a** $P = \{1, 2, 4, 7, 14, 28\}$, $Q = \{1, 2, 4, 5, 8, 10, 20, 40\}$
b $\{1, 2, 4\}$ **c** $\{1, 2, 4, 5, 7, 8, 10, 14, 20, 28, 40\}$
d $11 = 6 + 8 - 3$ ✓

3 **a** $M = \{32, 36, 40, 44, 48, 52, 56\}$, $N = \{36, 42, 48, 54\}$
b $\{36, 48\}$ **c** $\{32, 36, 40, 42, 44, 48, 52, 54, 56\}$
d $9 = 7 + 4 - 2$ ✓

4 **a** $R = \{-2, -1, 0, 1, 2, 3, 4\}$, $S = \{0, 1, 2, 3, 4, 5, 6\}$
b $\{0, 1, 2, 3, 4\}$ **c** $\{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$
d $9 = 7 + 7 - 5$ ✓

5 **a** $C = \{-4, -3, -2, -1\}$
$D = \{-7, -6, -5, -4, -3, -2, -1\}$
b $\{-4, -3, -2, -1\}$ **c** $\{-7, -6, -5, -4, -3, -2, -1\}$
d $7 = 4 + 7 - 4$ ✓

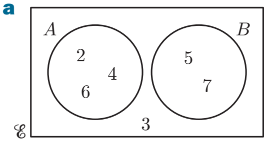
6 **a** $P = \{1, 2, 3, 4, 6, 12\}$, $Q = \{1, 2, 3, 6, 9, 18\}$,
$R = \{1, 3, 9, 27\}$
b **i** $\{1, 2, 3, 6\}$ **ii** $\{1, 3\}$ **iii** $\{1, 3, 9\}$
iv $\{1, 2, 3, 4, 6, 9, 12, 18\}$ **v** $\{1, 2, 3, 4, 6, 9, 12, 27\}$
vi $\{1, 2, 3, 6, 9, 18, 27\}$
c **i** $\{1, 3\}$ **ii** $\{1, 2, 3, 4, 6, 9, 12, 18, 27\}$

7 **a** $A = \{4, 8, 12, 16, 20, 24, 28, 32, 36\}$
$B = \{6, 12, 18, 24, 30, 36\}$, $C = \{12, 24, 36\}$
b **i** $\{12, 24, 36\}$ **ii** $\{12, 24, 36\}$
iii $\{12, 24, 36\}$ **iv** $\{12, 24, 36\}$
v $\{4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36\}$
c $12 = 9 + 6 + 3 - 3 - 3 - 3 + 3$ ✓

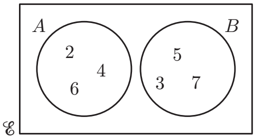
8 **a** $A = \{6, 12, 18, 24, 30\}$, $B = \{1, 2, 3, 5, 6, 10, 15, 30\}$
$C = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$
b **i** $\{6, 30\}$ **ii** $\{2, 3, 5\}$ **iii** $\varnothing$ **iv** $\varnothing$
v $\{1, 2, 3, 5, 6, 7, 10, 11, 12, 13, 15, 17, 18, 19, 23, 24, 29, 30\}$
c $18 = 5 + 8 + 10 - 2 - 3 - 0 + 0$ ✓

EXERCISE 1F.1

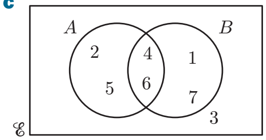
1 **a**



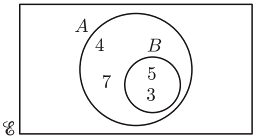
b



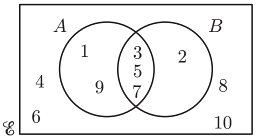
c



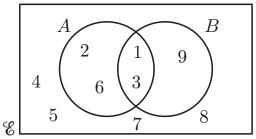
d



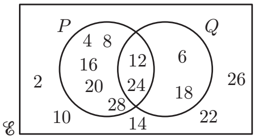
2 **a** $A = \{1, 3, 5, 7, 9\}$
$B = \{2, 3, 5, 7\}$
b $A \cap B = \{3, 5, 7\}$
$A \cup B = \{1, 2, 3, 5, 7, 9\}$
c



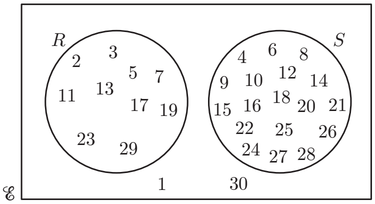
3 **a** $A = \{1, 2, 3, 6\}$
$B = \{1, 3, 9\}$
b $A \cap B = \{1, 3\}$
$A \cup B = \{1, 2, 3, 6, 9\}$
c



4 **a** $P = \{4, 8, 12, 16, 20, 24, 28\}$
$Q = \{6, 12, 18, 24\}$
b $P \cap Q = \{12, 24\}$
$P \cup Q = \{4, 6, 8, 12, 16, 18, 20, 24, 28\}$
c



5 **a** $R = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$
$S = \{4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28\}$
b $R \cap S = \varnothing$
$R \cup S = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29\}$
c



6 **a** $\{b, d, e, h\}$ **b** $\{e, f, h, i, j\}$ **c** $\{a, c, f, g, i, j, k\}$
d $\{a, b, c, d, g, k\}$ **e** $\{e, h\}$ **f** $\{b, d, e, f, h, i, j\}$
g $\{a, c, g, k\}$ **h** $\{a, b, c, d, f, g, i, j, k\}$

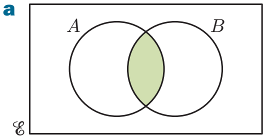
7 **a** **i** $\{a, b, c, d, h, j\}$ **ii** $\{a, c, d, e, f, g, k\}$
iii $\{a, b, e, f, i, l\}$ **iv** $\{a, c, d\}$
v $\{a, b, c, d, e, f, g, h, j, k\}$ **vi** $\{a, e, f\}$
vii $\{a\}$ **viii** $\{a, b, c, d, e, f, g, h, i, j, k, l\}$

b **i** 12 **ii** 12

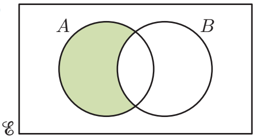
c
$$\begin{aligned} & n(A \cup B \cup C) \\ &= n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) \\ &\quad - n(B \cap C) + n(A \cap B \cap C) \end{aligned}$$

EXERCISE 1F.2

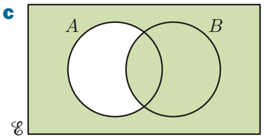
1 **a**



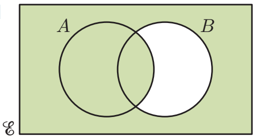
b



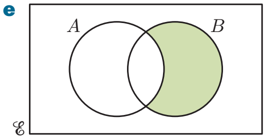
c



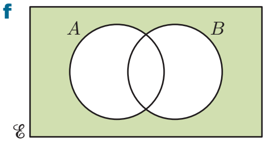
d



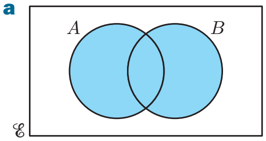
e



f



2 **a**



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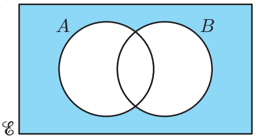

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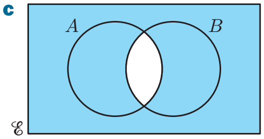
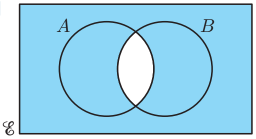


d



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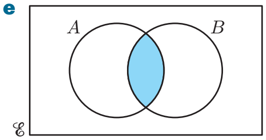

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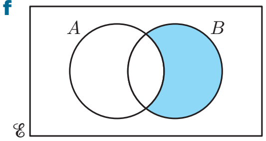
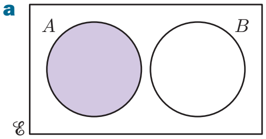
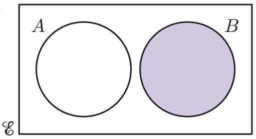


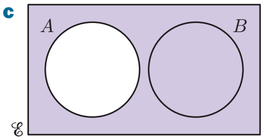
3 **a**



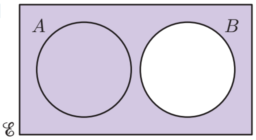
b



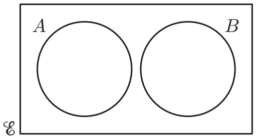
c



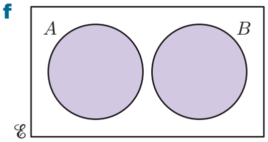
d



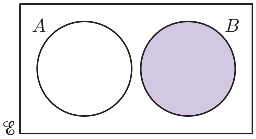
e



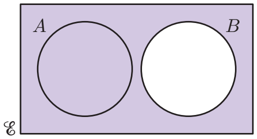
f



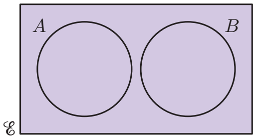
g



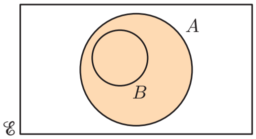
h



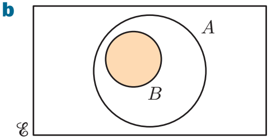
i



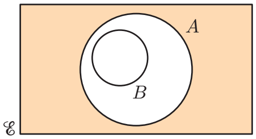
4 **a**



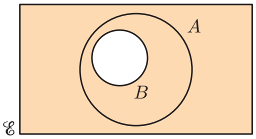
b



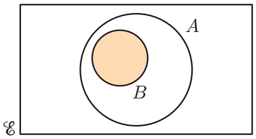
c



d



e



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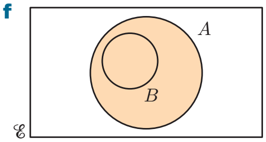
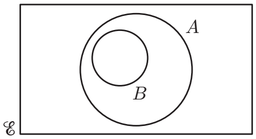


g



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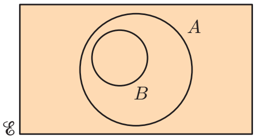
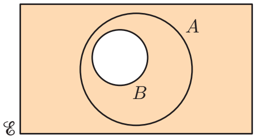
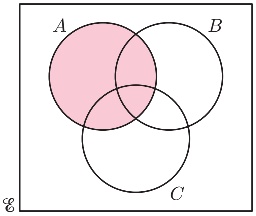


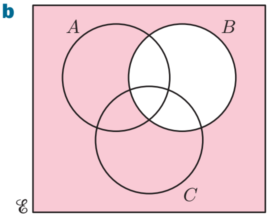
i



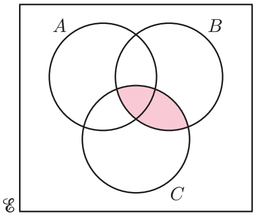
5 **a**



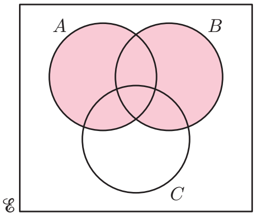
b



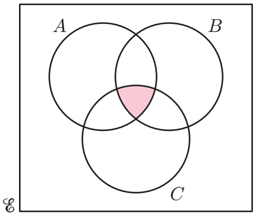
c



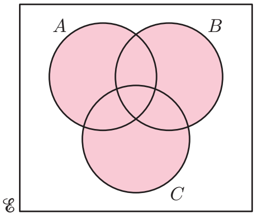
d



e



f



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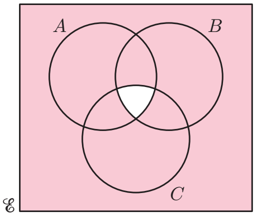

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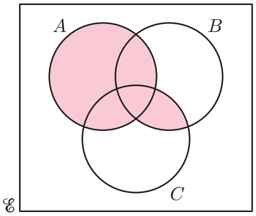
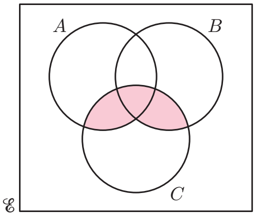
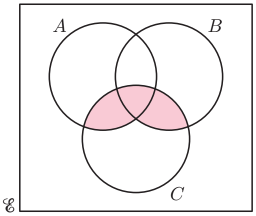


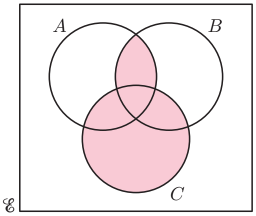
i



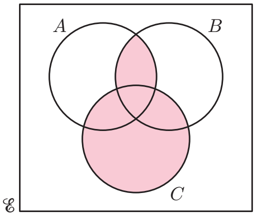
j



k



l



EXERCISE 1G

1 **a** 7 **b** 14 **c** 14 **d** 7 **e** 5 **f** 9

2 **a** $b+c$ **b** $c+d$ **c** b
d $a+b+c$ **e** $a+c+d$ **f** d

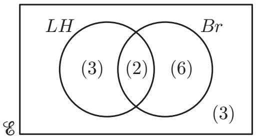
3 **a** **i** $2a+4$ **ii** $4a+4$ **iii** $3a-5$ **iv** $5a-1$
b **i** $a=6$ **ii** $a=\frac{32}{5}$

Since $a \in \mathbb{N}$, there cannot be 31 elements in ℰ, but it is possible to have 29 elements.

5 **a** 15 **b** 4 **6** **a** 18 **b** 6 **7** **a** 7 **b** 23

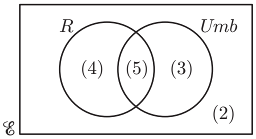
EXERCISE 1H

1 **a**



b **i** 9 **ii** 3 **iii** 3

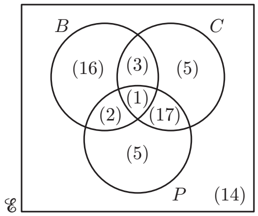
2 **a**



b **i** 4 **ii** 2

3 13 players **4** 20 people

5 **a**



b **i** 16
ii 33
iii 14
iv 7

6 **a** 29 **b** 6 **c** 1 **d** 11

7 **a** 3 **b** 5 **c** 5 **d** 21

8 **a** 3 **b** 4 **c** 9

REVIEW SET 1A

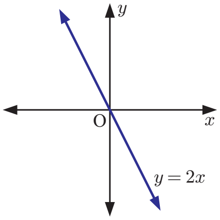
1 **a** $S=\{3, 4, 5, 6, 7\}$ **b** 5 **c** 31

2 **a** yes **b** yes **c** no **d** yes

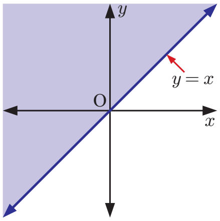
3 **a** $X'=\{$orange, yellow, green, blue$\}$
b $X'=\{-5, -3, -2, 0, 1, 2, 5\}$
c $X'=\{x \in \mathbb{Q} : x \geqslant -8\}$
d $X'=\{x \in (-\infty, -3) \cup [1, 4]\}$

4 **a** $\{x \in \mathbb{R} : -2 \leqslant x < 3\}$, neither
b $\{x \in \mathbb{R} : x < 3\}$, open

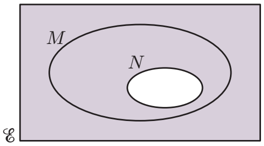
5 **a**



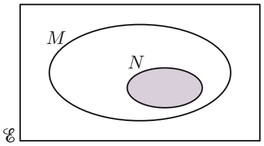
b



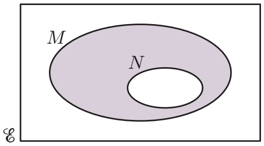
6 **a**



b



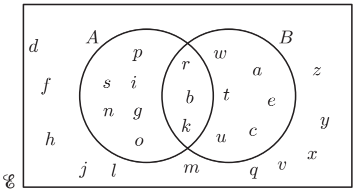
c



7 **a** **i** $\{s, p, r, i, n, g, b, o, k, w, a, t, e, u, c\}$
ii $\{r, b, k\}$ **iii** $\{g, i, n, o, p, s\}$

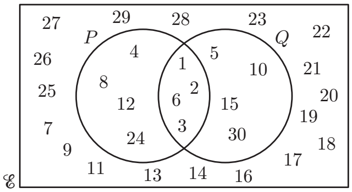
b **i** {the letters in ‘springbok’ or ‘waterbuck’}
ii {the letters common to both ‘springbok’ and ‘waterbuck’}
iii {the letters in ‘springbok’ but not ‘waterbuck’}

c

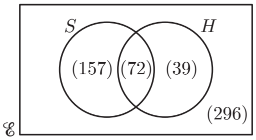


8 **a** **i** $\{1, 2, 3, 4, 6, 8, 12, 24\}$
ii $\{1, 2, 3, 5, 6, 10, 15, 30\}$ **iii** $\{1, 2, 3, 6\}$
iv $\{1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 24, 30\}$

b



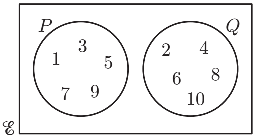
9 **a**



b **i** 72
ii 39
iii 268

10 8 **11** **a** 9 **b** 7 **c** 17

12 **a** $P = \{1, 3, 5, 7, 9\}$
$Q = \{2, 4, 6, 8, 10\}$
b They are disjoint.
c



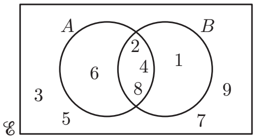
REVIEW SET 1B

1 **a** true **b** false **c** true **d** false **e** false

2 **a** **i** $\{x \in \mathbb{R} : 5 < x < 12\}$ **ii** $\{x \in \mathbb{Z} : -4 \leqslant x < 7\}$
iii $\{x \in \mathbb{N} : x > 45\}$
b **i** infinite **ii** finite **iii** infinite

3 $\varnothing$, $\{1\}$, $\{3\}$, $\{5\}$, $\{1, 3\}$, $\{3, 5\}$, $\{1, 5\}$, $\{1, 3, 5\}$

4 **a** **i** $\{2, 4, 6, 8\}$
ii $\{2, 4, 8\}$
iii $\{3, 5, 7, 9\}$
b



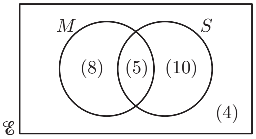
5 **a** $\varnothing$ **b** $s + t$

6 **a** neither **b** closed **c** neither

7 **a** **i** The set of points which lie on both A and B (that is, the point(s) of intersection of line A and line B).
ii The set of points which lie on line A or line B.
b No. If the lines are coincident (so, A and B describe the same line), then $A \cap B$ will be infinite.
c $n(A \cap B) = 0$ or 1

8 **a** C' **b** $(A \cap B) \cup (A \cap C)$ or $A \cap (B \cup C)$

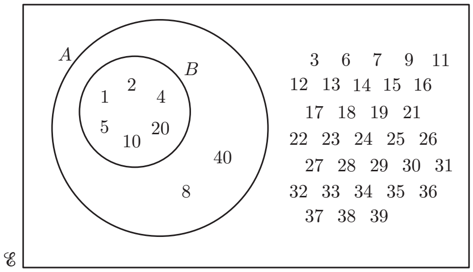
9 **a**



b **i** 27
ii 8
iii 14

10 4

11 **a** $A = \{1, 2, 4, 5, 8, 10, 20, 40\}$, $B = \{1, 2, 4, 5, 10, 20\}$
b $B \subset A$
c



12 **a** 1 **b** 7 **c** 15

EXERCISE 2A.1

1 **a, d, e** **2** **a, b, c, e, g, i**

3 No, for example $(0, 4)$ and $(0, -4)$ satisfy the relation.

EXERCISE 2A.2

1 **a, c, f**

2 **a** not a function **b** function, one-one
c function, not one-one

3 **a** **i** \$13 **ii** yes **iii** yes
b **i** no **ii** no

EXERCISE 2B

1 **a** 2 **b** 2 **c** -16 **d** -68 **e** $\frac{17}{4}$

2 **a** -3 **b** 3 **c** 3 **d** -3 **e** $\frac{15}{2}$

3 **a** **i** 1 **ii** -1 **b** $x = -4$

4 **a** $7 - 3a$ **b** $7 + 3a$ **c** $-3a - 2$ **d** $10 - 3b$
e $1 - 3x$ **f** $7 - 3x - 3h$

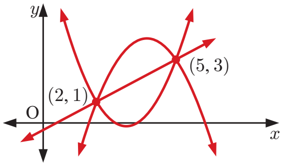
5 **a** $2x^2 + 19x + 43$ **b** $2x^2 - 11x + 13$
c $2x^2 - 3x - 1$ **d** $2x^4 + 3x^2 - 1$
e $2x^4 - x^2 - 2$ **f** $2x^2 + (4h + 3)x + 2h^2 + 3h - 1$

6 **a** **i** $-\frac{7}{2}$ **ii** $-\frac{3}{4}$ **iii** $-\frac{4}{9}$
b $x = 4$ **c** $\dfrac{2x + 7}{x - 2}$ **d** $x = \frac{9}{5}$

7 f is the function which converts x into $f(x)$ whereas $f(x)$ is the value of the function at any value of x.

8 **a** $V(4) = 6210$, the value in dollars after 4 years
b $t = 4.5$, the time in years for the photocopier to reach a value of 5780 dollars.
c 9650 dollars

9



10 $f(x) = -2x + 5$

11 $a = 3$, $b = -2$

12 $a = 3$, $b = -1$,
$c = -4$

EXERCISE 2C

1 **a** Domain $= \{x : x \geqslant -1\}$, Range $= \{y : y \leqslant 3\}$
b Domain $= \{x : -1 < x \leqslant 5\}$, Range $= \{y : 1 < y \leqslant 3\}$
c Domain $= \{x : x \neq 2\}$, Range $= \{y : y \neq -1\}$
d Domain $= \{x : x \in \mathbb{R}\}$, Range $= \{y : 0 < y \leqslant 2\}$
e Domain $= \{x : x \in \mathbb{R}\}$, Range $= \{y : y \geqslant -1\}$
f Domain $= \{x : x \in \mathbb{R}\}$, Range $= \{y : y \leqslant \frac{25}{4}\}$
g Domain $= \{x : x \geqslant -4\}$, Range $= \{y : y \geqslant -3\}$

h Domain $= \{x : x \in \mathbb{R}\}$, Range $= \{y : y > -2\}$

i Domain $= \{x : x \neq \pm 2\}$,
Range $= \{y : y \leqslant -1 \text{ or } y > 0\}$

2 **a** $f(x)$ defined for $x \geqslant -6$, Domain $= \{x : x \geqslant -6\}$

b $f(x)$ defined for $x \neq 0$, Domain $= \{x : x \neq 0\}$

c $f(x)$ defined for $x < \frac{3}{2}$, Domain $= \{x : x < \frac{3}{2}\}$

3 **a** Domain $= \{x : x \in \mathbb{R}\}$, Range $= \{y : y \in \mathbb{R}\}$

b Domain $= \{x : x \in \mathbb{R}\}$, Range $= \{3\}$

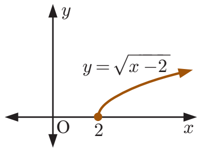
c Domain $= \{x : x \geqslant 0\}$, Range $= \{y : y \geqslant 0\}$

d Domain $= \{x : x \neq -1\}$, Range $= \{y : y \neq 0\}$

e Domain $= \{x : x > 0\}$, Range $= \{y : y < 0\}$

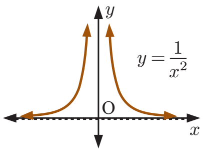
f Domain $= \{x : x \neq 3\}$, Range $= \{y : y \neq 0\}$

4 **a**



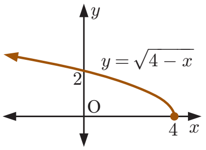
Domain $= \{x : x \geqslant 2\}$
Range $= \{y : y \geqslant 0\}$

b



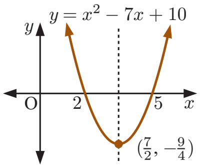
Domain $= \{x : x \neq 0\}$
Range $= \{y : y > 0\}$

c



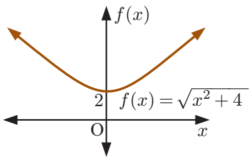
Domain $= \{x : x \leqslant 4\}$
Range $= \{y : y \geqslant 0\}$

d



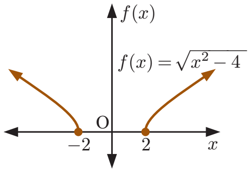
Domain $= \{x : x \in \mathbb{R}\}$
Range $= \{y : y \geqslant -2\frac{1}{4}\}$

e



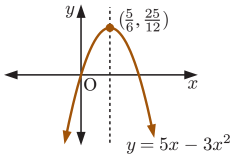
Domain $= \{x : x \in \mathbb{R}\}$
Range $= \{y : y \geqslant 2\}$

f



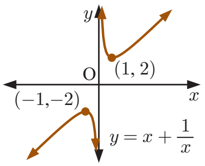
Domain $= \{x : x \leqslant -2$
or $x \geqslant 2\}$
Range $= \{y : y \geqslant 0\}$

g



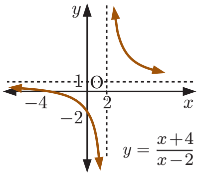
Domain $= \{x : x \in \mathbb{R}\}$
Range $= \{y : y \leqslant \frac{25}{12}\}$

h



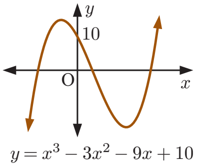
Domain $= \{x : x \neq 0\}$
Range $= \{y : y \leqslant -2$
or $y \geqslant 2\}$

i



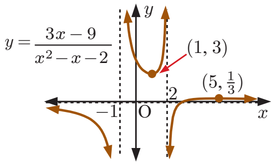
Domain $= \{x : x \neq 2\}$
Range $= \{y : y \neq 1\}$

j



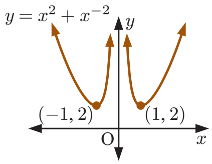
Domain $= \{x : x \in \mathbb{R}\}$
Range $= \{y : y \in \mathbb{R}\}$

k



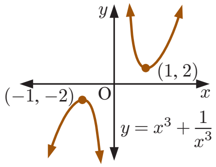
Domain $= \{x : x \neq -1$
and $x \neq 2\}$
Range $= \{y : y \leqslant \frac{1}{3}$
or $y \geqslant 3\}$

l



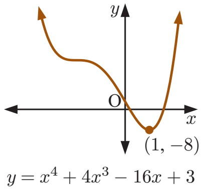
Domain $= \{x : x \neq 0\}$
Range $= \{y : y \geqslant 2\}$

m



Domain $= \{x : x \neq 0\}$
Range $= \{y : y \leqslant -2$
or $y \geqslant 2\}$

n



Domain $= \{x : x \in \mathbb{R}\}$
Range $= \{y : y \geqslant -8\}$

EXERCISE 2D.1

1 **a** 5 **b** 5 **c** 4 **d** 4 **e** 6 **f** 0 **g** $\frac{2}{7}$ **h** $\frac{8}{27}$

2 **a** 1 **b** 6 **c** 4 **d** 3

3 **a** 2 **b** -4 **c** -6 **d** -5

EXERCISE 2D.2

1 **a** $x = \pm 3$ **b** no solution **c** $x = 0$

d $x = 4$ or -2 **e** $x = -1$ or 7 **f** no solution

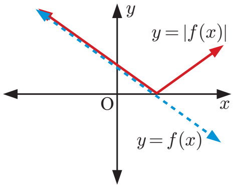
g $x = 1$ or $\frac{1}{3}$ **h** $x = 0$ or 3 **i** $x = -2$ or $\frac{14}{5}$

2 **a** $x = -\frac{1}{4}$ or $\frac{3}{2}$ **b** $x = -6$ or $-\frac{4}{3}$ **c** $x = \frac{1}{2}$

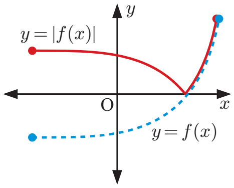
d $x = \frac{5}{2}$ **e** $x = 0$ or $\frac{2}{5}$ **f** $x = -2$ or 0

EXERCISE 2D.3

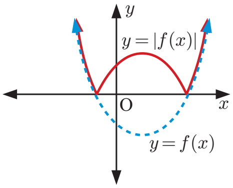
1 **a**



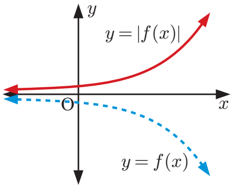
b



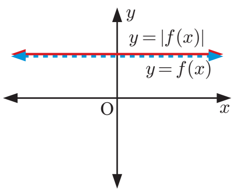
c



d



e



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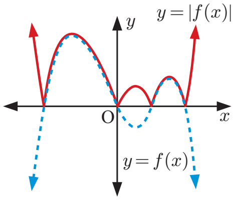
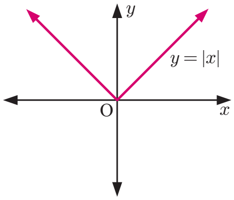


2 function **d** **3** $\{y : 0 \leqslant y \leqslant 6\}$

4 **a** false **b** true **c** true **d** false

5 **a**



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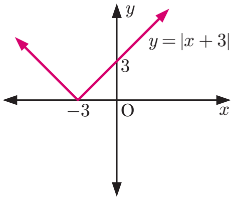

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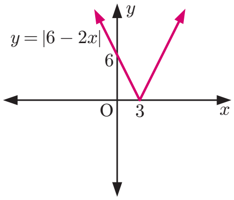

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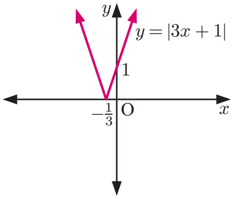

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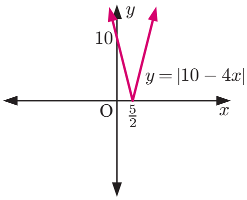

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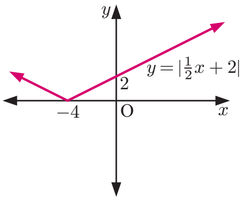


EXERCISE 2E

1 **a** $5 - 2x$ **b** $-2x - 2$ **c** 11

2 **a** $5 - x$ **b** $1 - x$ **c** $4 + x$

3 **a** $25x - 42$ **b** $\sqrt{8}$ **c** -7

4 $f(g(x)) = (2 - x)^2$,
Domain $= \{x : x \in \mathbb{R}\}$,
Range $= \{y : y \geqslant 0\}$

$g(f(x)) = 2 - x^2$,
Domain $= \{x : x \in \mathbb{R}\}$,
Range $= \{y : y \leqslant 2\}$

5 **a** $(f \circ g)(x) = 6x - 4$ **b** $x = -\frac{3}{4}$

6 **a** **i** $x^2 - 6x + 10$ **ii** $2 - x^2$ **b** $x = \pm\frac{1}{\sqrt{2}}$

7 **a** Let $x = 0$, $\therefore\ b = d$ and so
$ax + b = cx + b$
$\therefore\ ax = cx$ for all x
Let $x = 1$, $\therefore\ a = c$

b $(f \circ g)(x) = [2a]x + [2b + 3] = 1x + 0$ for all x
$\therefore\ 2a = 1$ and $2b + 3 = 0$

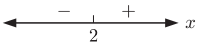
c Yes, $\{(g \circ f)(x) = [2a]x + [3a + b]\}$

8 **a** $(f \circ g)(x) = \sqrt{1 - x^2}$

b Domain $= \{x : -1 \leqslant x \leqslant 1\}$, Range $= \{y : 0 \leqslant y \leqslant 1\}$

EXERCISE 2F

1 **a**



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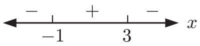

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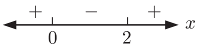

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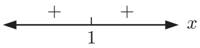

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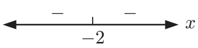

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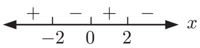

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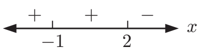

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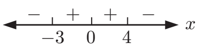

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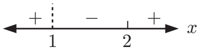

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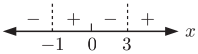

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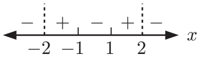
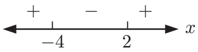


2 **a**



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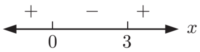

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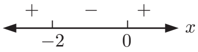

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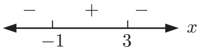

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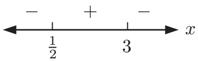

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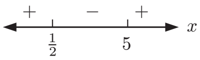

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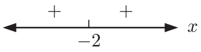

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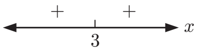

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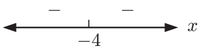
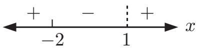


3 **a**



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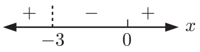

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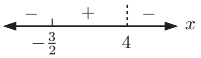

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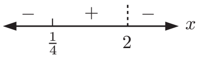
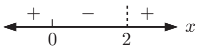


e



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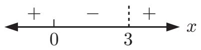

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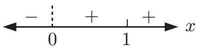

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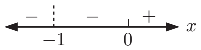

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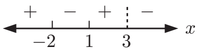

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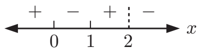

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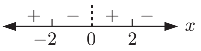

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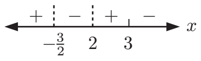
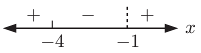


4 **a**



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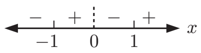

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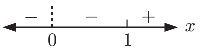
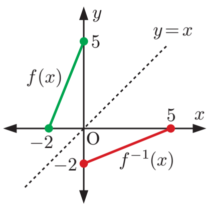


EXERCISE 2G

1 **a**



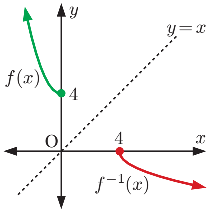
Domain of $f(x)$ is $\{x : -2 \leqslant x \leqslant 0\}$

Range of $f(x)$ is $\{y : 0 \leqslant y \leqslant 5\}$

Domain of $f^{-1}(x)$ is $\{x : 0 \leqslant x \leqslant 5\}$

Range of $f^{-1}(x)$ is $\{y : -2 \leqslant y \leqslant 0\}$

b



Domain of $f(x)$ is $\{x : x \leqslant 0\}$

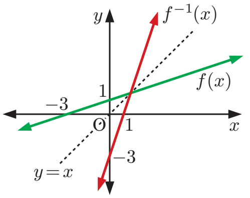
Range of $f(x)$ is $\{y : y \geqslant 4\}$

Domain of $f^{-1}(x)$ is $\{x : x \geqslant 4\}$

Range of $f^{-1}(x)$ is $\{y : y \leqslant 0\}$

c The function does not have an inverse, as it is not one-one.

d



Domain of $f(x)$ is $\{x : x \in \mathbb{R}\}$

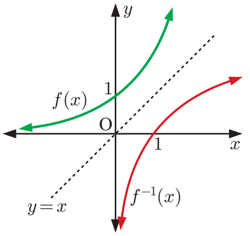
Range of $f(x)$ is $\{y : y \in \mathbb{R}\}$

Domain of $f^{-1}(x)$ is $\{x : x \in \mathbb{R}\}$

Range of $f^{-1}(x)$ is $\{y : y \in \mathbb{R}\}$

e The function does not have an inverse, as it is not one-one.

f



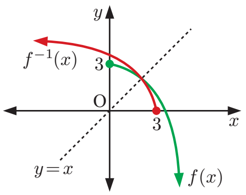
Domain of $f(x)$ is $\{x : x \in \mathbb{R}\}$

Range of $f(x)$ is $\{y : y > 0\}$

Domain of $f^{-1}(x)$ is $\{x : x > 0\}$

Range of $f^{-1}(x)$ is $\{y : y \in \mathbb{R}\}$

g



Domain of $f(x)$ is $\{x : x \geqslant 0\}$

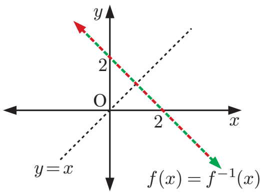
Range of $f(x)$ is $\{y : y \leqslant 3\}$

Domain of $f^{-1}(x)$ is $\{x : x \leqslant 3\}$

Range of $f^{-1}(x)$ is $\{y : y \geqslant 0\}$

h The function does not have an inverse, as it is not one-one.

i



Domain of $f(x)$ is $\{x : x \in \mathbb{R}\}$

Range of $f(x)$ is $\{y : y \in \mathbb{R}\}$

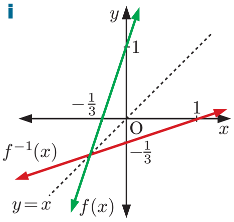
Domain of $f^{-1}(x)$ is $\{x : x \in \mathbb{R}\}$

Range of $f^{-1}(x)$ is $\{y : y \in \mathbb{R}\}$

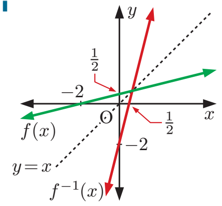
2 function **i**

3 Range $= \{y : -2 \leqslant y < 3\}$

4 **a** **i**



b **i**



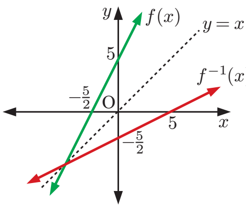
ii $f^{-1}(x) = \dfrac{x-1}{3}$

ii $f^{-1}(x) = 4x - 2$

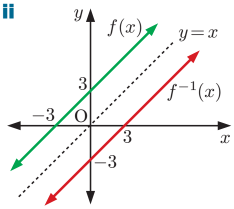
5 **a** **i** $f^{-1}(x) = \dfrac{x-5}{2}$

b **i** $f^{-1}(x) = x - 3$

ii

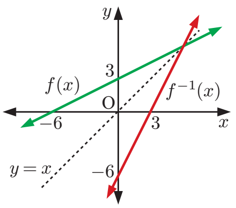


ii



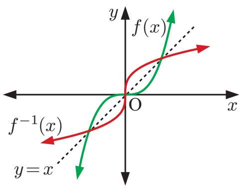
c **i** $f^{-1}(x) = 2x - 6$

ii



6 $f(x)$ is the same as $(f^{-1})^{-1}(x)$

7



8 $f^{-1}(x) = \dfrac{1}{x}$ and $f(x) = \dfrac{1}{x}$ $\therefore$ $f = f^{-1}$

$\therefore$ f is a self-inverse function

9 **a** $f^{-1}(x) = 2x + 2$

b **i** $(f \circ f^{-1})(x) = x$ **ii** $(f^{-1} \circ f)(x) = x$

10 **a** 10

b $f^{-1}(x) = \dfrac{x-5}{2}$ and $f^{-1}(-3) = -4$

$g^{-1}(x) = 8 - 2x$ and $g^{-1}(6) = -4$

$\therefore$ $f^{-1}(-3) - g^{-1}(6) = 0$

c $x = 3$

11 **a** **i** 25 **ii** 16 **b** $x = 1$

12 **a** Is not **b** Is **c** Is **d** Is **e** Is **f** Is not

13 $(f^{-1} \circ g^{-1})(x) = \dfrac{x+3}{8}$ and $(g \circ f)^{-1}(x) = \dfrac{x+3}{8}$

REVIEW SET 2A

1 **a** function **b** function **c** not a function **d** function

2 $a = -6$, $b = 13$

3 **a** $x = -2$ or 12 **b** $x = -5$ or 1

4 **a** 10 **b** $x^2 - x - 2$

5 **a** **i** Domain is $\{x : x \geqslant -3\}$, Range is $\{y : y \geqslant 2\}$

ii function is one-one

b **i** Domain is $\{x : x \in \mathbb{R}\}$, Range is $\{y : y \geqslant -5\}$

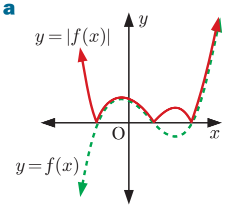
ii function is not one-one

c **i** Domain is $\{x : x \in \mathbb{R}\}$,

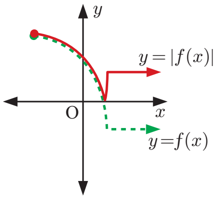
Range is $\{y : y = -3 \text{ or } y = 1\}$

ii function is not one-one

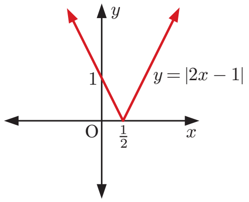
6 **a**



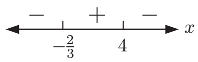
b



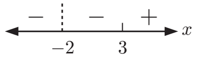
7



8 **a**

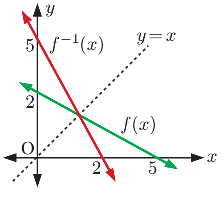


b

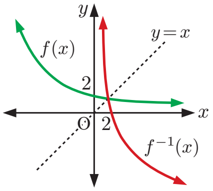


9 **a** $2x^2 + 1$ **b** $4x^2 - 12x + 11$

10 **a**



b



c The function does not have an inverse.

11 **a** $f^{-1}(x) = \dfrac{x-2}{4}$ **b** $f^{-1}(x) = \dfrac{3-4x}{5}$

12 **a** $f(-3) = (-3)^2 = 9$ **b** 169 **c** $x = -4$

$g(-\frac{4}{3}) = 1 - 6(-\frac{4}{3}) = 9$

13 $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x) = x - 2$

REVIEW SET 2B

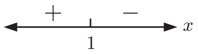
1 **a** not a function **b** function, one-one

c function, not one-one

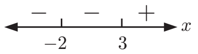
2 **a** 12 **b** $x = \pm 1$

3 **a** $x = -5$ or 6 **b** $x = 1$ or 3

4 **a**

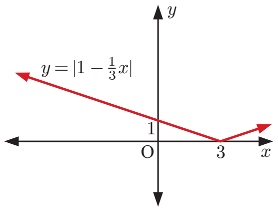


b



5 **a** $10 - 6x$ **b** $9x - 14$ **c** -23

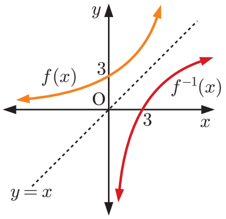
6 $\{y : 3 \leqslant y \leqslant 7\}$ **7**



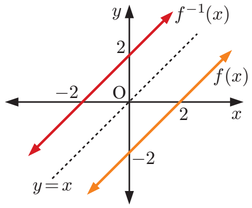
8 **a** **i** $1 - 10x$ **ii** $5 - 10x$ **b** $x = -\frac{3}{5}$

9 $a = 1$, $b = -6$, $c = 5$

10 **a**



c



b The function does not have an inverse.

11 **a** $f^{-1}(x) = \dfrac{7-x}{4}$ **b** $f^{-1}(x) = \dfrac{5x-3}{2}$

12 $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x) = \dfrac{4x+6}{15}$ **13** 16

EXERCISE 3A.1

1 **a** $x = 0, -\frac{7}{4}$ **b** $x = 0, -\frac{1}{3}$ **c** $x = 0, \frac{7}{3}$

d $x = 0, \frac{11}{2}$ **e** $x = 0, \frac{8}{3}$ **f** $x = 0, \frac{3}{2}$

g $x = 3, 2$ **h** $x = 4, -2$ **i** $x = 3, 7$

j $x = 3$ **k** $x = -4, 3$ **l** $x = -11, 3$

2 **a** $x = \frac{2}{3}$ **b** $x = -\frac{1}{2}, 7$ **c** $x = -\frac{2}{3}, 6$

d $x = \frac{1}{3}, -2$ **e** $x = \frac{3}{2}, 1$ **f** $x = -\frac{2}{3}, -2$

g $x = -\frac{2}{3}, 4$ **h** $x = \frac{1}{2}, -\frac{3}{2}$ **i** $x = -\frac{1}{4}, 3$

j $x = -\frac{3}{4}, \frac{5}{3}$ **k** $x = \frac{1}{7}, -1$ **l** $x = -2, \frac{28}{15}$

3 **a** $x = 2, 5$ **b** $x = -3, 2$ **c** $x = 0, -\frac{3}{2}$

d $x = 1, 2$ **e** $x = \frac{1}{2}, -1$ **f** $x = 3$

EXERCISE 3A.2

1 **a** $x = -5 \pm \sqrt{2}$ **b** no real solutions **c** $x = 4 \pm 2\sqrt{2}$
d $x = 8 \pm \sqrt{7}$ **e** $x = -3 \pm \sqrt{5}$ **f** $x = 2 \pm \sqrt{6}$
g $x = -1 \pm \sqrt{10}$ **h** $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{3}$ **i** $x = \frac{1}{3} \pm \frac{\sqrt{7}}{3}$

2 **a** $x = 2 \pm \sqrt{3}$ **b** $x = -3 \pm \sqrt{7}$ **c** $x = 7 \pm \sqrt{3}$
d $x = 2 \pm \sqrt{7}$ **e** $x = -3 \pm \sqrt{2}$ **f** $x = 1 \pm \sqrt{7}$
g $x = -3 \pm \sqrt{11}$ **h** $x = 4 \pm \sqrt{6}$ **i** no real solns.

3 **a** $x = -1 \pm \frac{1}{\sqrt{2}}$ **b** $x = \frac{5}{2} \pm \frac{\sqrt{19}}{2}$ **c** $x = -2 \pm \sqrt{\frac{7}{3}}$
d $x = 1 \pm \sqrt{\frac{7}{3}}$ **e** $x = \frac{3}{2} \pm \sqrt{\frac{37}{20}}$ **f** $x = -\frac{1}{2} \pm \frac{\sqrt{6}}{2}$

4 **a** $x = \frac{2}{3} \pm \frac{\sqrt{10}}{3}$ **b** $x = -\frac{1}{10} \pm \frac{\sqrt{21}}{10}$ **c** $x = -\frac{5}{6} \pm \frac{\sqrt{13}}{6}$

5 $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

EXERCISE 3A.3

1 **a** $x = 2 \pm \sqrt{7}$ **b** $x = -3 \pm \sqrt{2}$ **c** $x = 2 \pm \sqrt{3}$
d $x = -2 \pm \sqrt{5}$ **e** $x = 2 \pm \sqrt{2}$ **f** $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{7}$
g $x = \frac{5}{6} \pm \frac{\sqrt{37}}{6}$ **h** $x = 2 \pm \sqrt{10}$ **i** $x = \frac{7}{4} \pm \frac{\sqrt{33}}{4}$

2 **a** $x = -2 \pm 2\sqrt{2}$ **b** $x = -\frac{5}{8} \pm \frac{\sqrt{57}}{8}$ **c** $x = \frac{5}{2} \pm \frac{\sqrt{13}}{2}$
d $x = -\frac{4}{9} \pm \frac{\sqrt{7}}{9}$ **e** $x = -\frac{7}{4} \pm \frac{\sqrt{97}}{4}$ **f** $x = \frac{1}{8} \pm \frac{\sqrt{145}}{8}$
g $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{7}$ **h** $x = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$ **i** $x = \frac{3}{4} \pm \frac{\sqrt{17}}{4}$

EXERCISE 3B

1 **a** $x \leqslant -3$ or $x \geqslant 2$ **b** $-1 < x < 4$
c $x < -\frac{1}{2}$ or $x > 3$ **d** $x \leqslant 0$ or $x \geqslant 1$
e $x \leqslant 0$ or $x \geqslant 3$ **f** $-\frac{2}{3} < x < 0$
g $-2 < x < 2$ **h** $x \leqslant -3$ or $x \geqslant 3$
i $x \neq -2$ **j** $x < -5$ or $x > 3$
k $4 \leqslant x \leqslant 7$ **l** $-6 < x < -4$
m $x \leqslant -2$ or $x \geqslant 15$ **n** $x \leqslant -1$ or $x \geqslant \frac{3}{2}$
o no solutions **p** $-\frac{3}{2} < x < \frac{1}{3}$
q $x < -\frac{4}{3}$ or $x > 4$ **r** no solutions **s** $\frac{1}{3} \leqslant x \leqslant \frac{1}{2}$
t $x \in \mathbb{R}$ **u** $-\frac{11}{8} - \frac{\sqrt{73}}{8} < x < -\frac{11}{8} + \frac{\sqrt{73}}{8}$

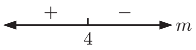
2 **a** $\square = <$ **b** $\square = \leqslant$ **c** $\square = \geqslant$ or $>$

EXERCISE 3C

1 **a** 2 distinct irrational roots **b** 2 distinct rational roots
c 2 distinct rational roots **d** 2 distinct irrational roots
e no real roots **f** a repeated root

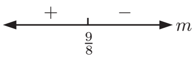
2 **a**, **c**, **d**, **f**

3 **a** $\Delta = 16 - 4m$



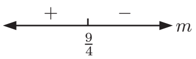
i $m = 4$ **ii** $m < 4$ **iii** $m > 4$

b $\Delta = 9 - 8m$



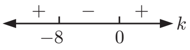
i $m = \frac{9}{8}$ **ii** $m < \frac{9}{8}$, $m \neq 0$ **iii** $m > \frac{9}{8}$

c $\Delta = 9 - 4m$



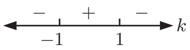
i $m = \frac{9}{4}$ **ii** $m < \frac{9}{4}$, $m \neq 0$ **iii** $m > \frac{9}{4}$

4 **a** $\Delta = k^2 + 8k$



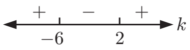
i $k < -8$ or $k > 0$ **ii** $k \leqslant -8$ or $k \geqslant 0$
iii $k = -8$ or 0 **iv** $-8 < k < 0$

b $\Delta = 4 - 4k^2$



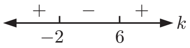
i $-1 < k < 1$, $k \neq 0$ **ii** $-1 \leqslant k \leqslant 1$, $k \neq 0$
iii $k = \pm 1$ **iv** $k < -1$ or $k > 1$

c $\Delta = k^2 + 4k - 12$



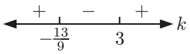
i $k < -6$ or $k > 2$ **ii** $k \leqslant -6$ or $k \geqslant 2$
iii $k = -6$ or 2 **iv** $-6 < k < 2$

d $\Delta = k^2 - 4k - 12$



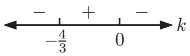
i $k < -2$ or $k > 6$ **ii** $k \leqslant -2$ or $k \geqslant 6$
iii $k = 6$ or -2 **iv** $-2 < k < 6$

e $\Delta = 9k^2 - 14k - 39$



i $k < -\frac{13}{9}$ or $k > 3$ **ii** $k \leqslant -\frac{13}{9}$ or $k \geqslant 3$
iii $k = -\frac{13}{9}$ or 3 **iv** $-\frac{13}{9} < k < 3$

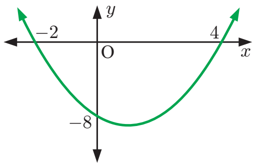
f $\Delta = -3k^2 - 4k$



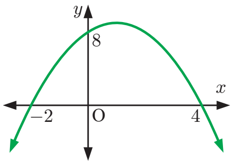
i $-\frac{4}{3} < k < 0$, $k \neq -1$ **ii** $-\frac{4}{3} \leqslant k \leqslant 0$, $k \neq -1$
iii $k = -\frac{4}{3}$ or 0 **iv** $k < -\frac{4}{3}$ or $k > 0$

EXERCISE 3D.1

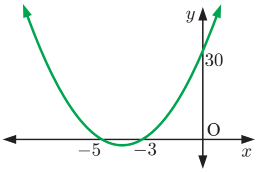
1 **a** $y = (x - 4)(x + 2)$



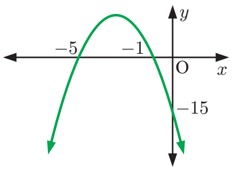
b $f(x) = -(x - 4)(x + 2)$



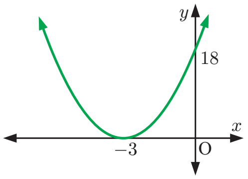
c $y = 2(x + 3)(x + 5)$



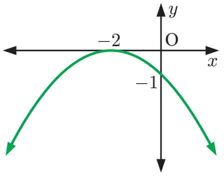
d $f(x) = -3(x + 1)(x + 5)$



e $f(x) = 2(x + 3)^2$



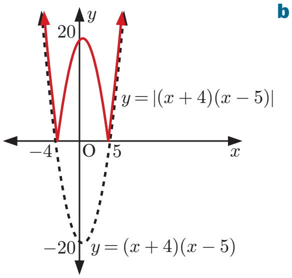
f $y = -\frac{1}{4}(x + 2)^2$



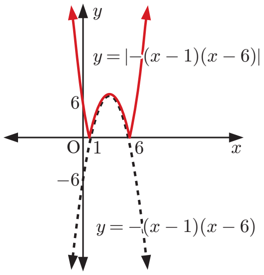
2 **a** $x = 1$ **b** $x = 1$ **c** $x = -4$
d $x = -3$ **e** $x = -3$ **f** $x = -2$

3 **a** C **b** E **c** B **d** F **e** G **f** H **g** A **h** D

4 **a**



b



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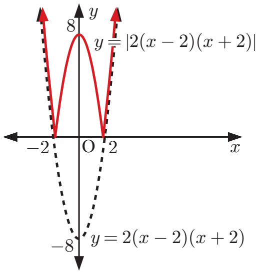

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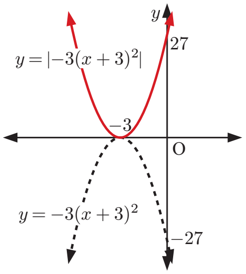
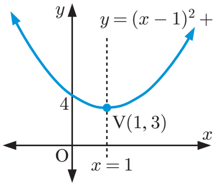


5 **a**



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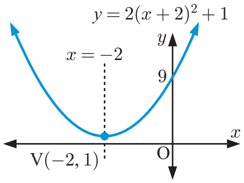
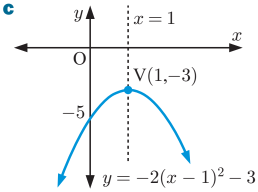
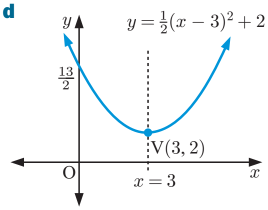


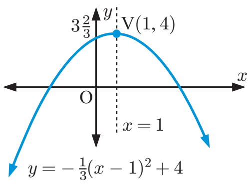
c



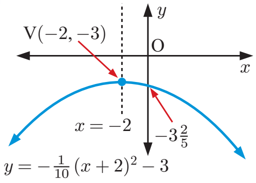
d



e



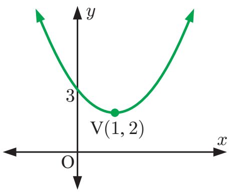
f



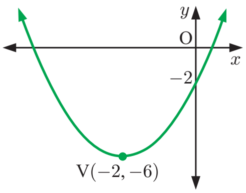
6 **a** G **b** A **c** E **d** B **e** I
f C **g** D **h** F **i** H

EXERCISE 3D.2

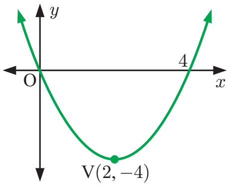
1 **a** $y = (x-1)^2 + 2$



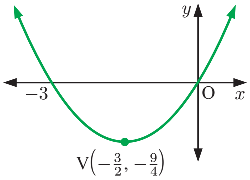
b $y = (x+2)^2 - 6$



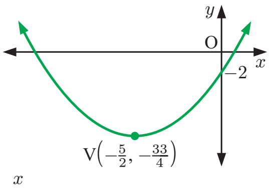
c $y = (x-2)^2 - 4$



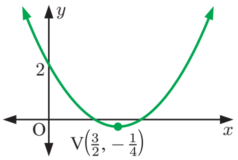
d $y = \left(x + \frac{3}{2}\right)^2 - \frac{9}{4}$



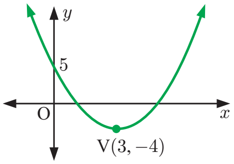
e $y = \left(x + \frac{5}{2}\right)^2 - \frac{33}{4}$



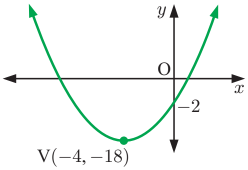
f $y = \left(x - \frac{3}{2}\right)^2 - \frac{1}{4}$



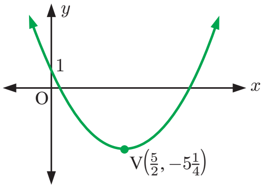
g $y = (x-3)^2 - 4$



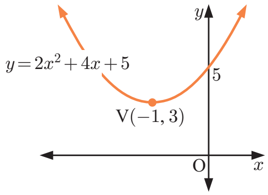
h $y = (x+4)^2 - 18$



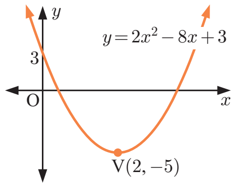
i $y = \left(x - \frac{5}{2}\right)^2 - 5\frac{1}{4}$



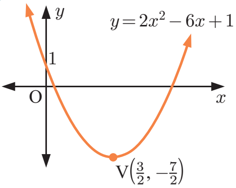
2 **a** **i** $y = 2(x+1)^2 + 3$
ii $(-1, 3)$ **iii** 5
iv



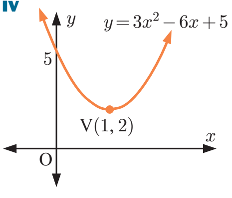
b **i** $y = 2(x-2)^2 - 5$
ii $(2, -5)$ **iii** 3
iv



c **i** $y = 2(x - \frac{3}{2})^2 - \frac{7}{2}$
ii $(\frac{3}{2}, -\frac{7}{2})$ **iii** 1
iv



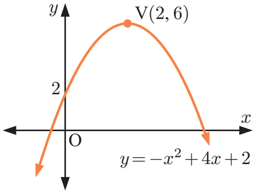
d **i** $y = 3(x-1)^2 + 2$
ii $(1, 2)$ **iii** 5
iv



e **i** $y = -(x-2)^2 + 6$

ii $(2, 6)$ **iii** 2

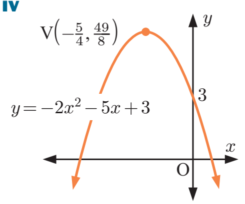
iv



f **i** $y = -2(x + \frac{5}{4})^2 + \frac{49}{8}$

ii $(-\frac{5}{4}, \frac{49}{8})$ **iii** 3

iv



EXERCISE 3D.3

1 **a** $(2, -2)$ **b** $(-1, -4)$ **c** $(0, 4)$

d $(0, 1)$ **e** $(-2, -15)$ **f** $(-2, -5)$

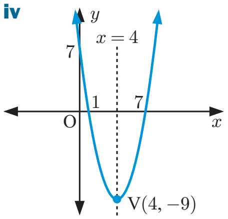
g $(-\frac{3}{2}, -\frac{11}{2})$ **h** $(\frac{5}{2}, -\frac{19}{2})$ **i** $(1, -\frac{9}{2})$

2 **a** **i** $x = 4$

ii $(4, -9)$

iii x-intercepts 1, 7,
y-intercept 7

iv



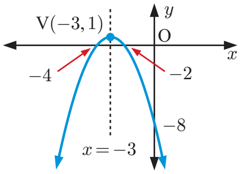
v $\{y : y \geqslant -9\}$

b **i** $x = -3$

ii $(-3, 1)$

iii x-int. $-2, -4$,
y-intercept -8

iv



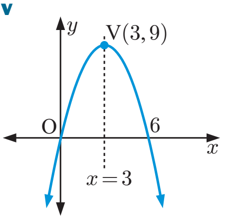
v $\{y : y \leqslant 1\}$

c **i** $x = 3$

ii $(3, 9)$

iii x-intercepts 0, 6,
y-intercept 0

iv



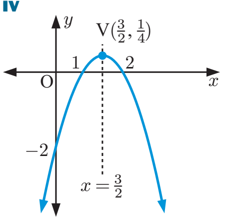
v $\{y : y \leqslant 9\}$

d **i** $x = \frac{3}{2}$

ii $(\frac{3}{2}, \frac{1}{4})$

iii x-intercepts 1, 2,
y-intercept -2

iv



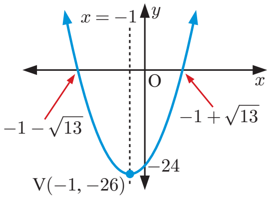
v $\{y : y \leqslant \frac{1}{4}\}$

e **i** $x = -1$

ii $(-1, -26)$

iii x-int. $-1 \pm \sqrt{13}$,
y-intercept -24

iv



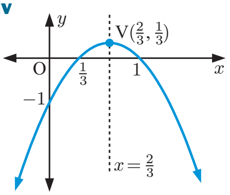
v $\{y : y \geqslant -26\}$

f **i** $x = \frac{2}{3}$

ii $(\frac{2}{3}, \frac{1}{3})$

iii x-intercepts $\frac{1}{3}$, 1,
y-intercept -1

iv



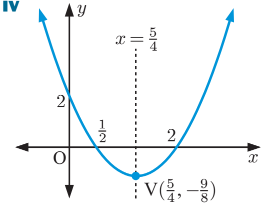
v $\{y : y \leqslant \frac{1}{3}\}$

g **i** $x = \frac{5}{4}$

ii $(\frac{5}{4}, -\frac{9}{8})$

iii x-intercepts $\frac{1}{2}$, 2,
y-intercept 2

iv



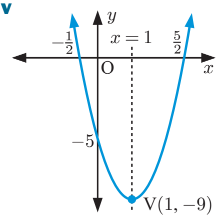
v $\{y : y \geqslant -\frac{9}{8}\}$

h **i** $x = 1$

ii $(1, -9)$

iii x-intercepts $-\frac{1}{2}$, $\frac{5}{2}$,
y-intercept -5

iv



v $\{y : y \geqslant -9\}$

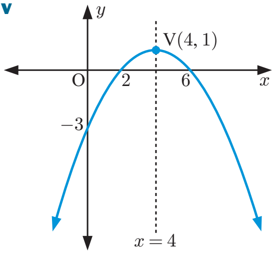
i **i** $x = 4$

ii $(4, 1)$

iii x-intercepts 2, 6,
y-intercept -3

v $\{y : y \leqslant 1\}$

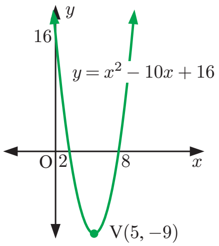
iv



3 **a** **i** $y = (x-2)(x-8)$,
roots are 2 and 8

ii $y = (x-5)^2 - 9$,
vertex is $(5, -9)$

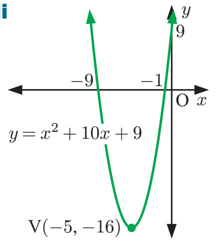
iii



b **i** $y = (x+1)(x+9)$,
roots are -1 and -9

ii $y = (x+5)^2 - 16$,
vertex is $(-5, -16)$

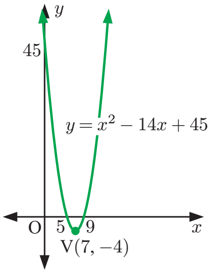
iii



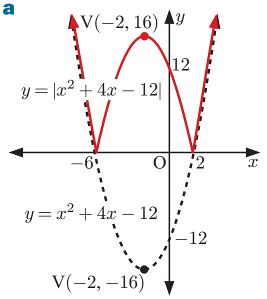
c **i** $y = (x-5)(x-9)$,
roots are 5 and 9

ii $y = (x-7)^2 - 4$,
vertex is $(7, -4)$

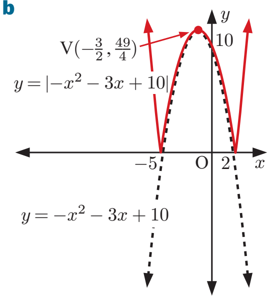
iii



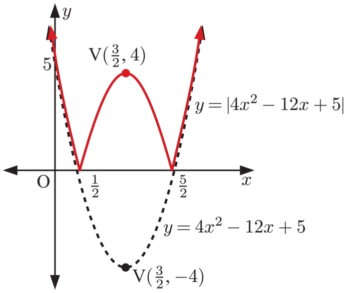
4 **a**



b



c

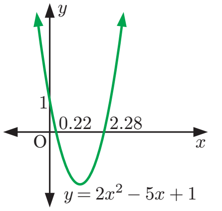


5 **a** $\{y : -10 \leqslant y \leqslant 15\}$ **b** $\{y : 3 \leqslant y \leqslant 19\}$
c $\{y : -13 \leqslant y \leqslant 37\}$ **d** $\{y : -8 \leqslant y \leqslant \frac{49}{4}\}$

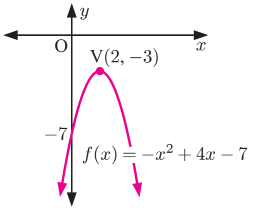
EXERCISE 3D.4

1 **a** cuts x-axis twice, concave up
b cuts x-axis twice, concave up
c lies entirely below the x-axis, concave down, negative definite
d cuts x-axis twice, concave up
e touches x-axis, concave up
f cuts x-axis twice, concave down
g cuts x-axis twice, concave up
h cuts x-axis twice, concave down
i touches x-axis, concave up

2 **a** concave up
b $\Delta = 17$ which is > 0
c x-intercepts ≈ 0.22 and 2.28
d y-intercept $= 1$
e



3 **a** $\Delta = -12$ which is < 0
b negative definite
c vertex is $(2, -3)$, y-intercept $= -7$
d



4 **a** $a = 1$ which is > 0 and $\Delta = -15$ which is < 0 so is entirely above the x-axis.
b $a = -1$ which is < 0 and $\Delta = -8$ which is < 0 so is entirely below the x-axis.
c $a = 2$ which is > 0 and $\Delta = -40$ which is < 0 so is entirely above the x-axis.
d $a = -2$ which is < 0 and $\Delta = -23$ which is < 0 so is entirely below the x-axis.

5 $a = 3$ which is > 0 and $\Delta = k^2 + 12$ which is always > 0 $\{$as $k^2 \geqslant 0$ for all $k\}$ $\therefore$ always cuts x-axis twice.

6 $-4 < k < 4$

EXERCISE 3E

1 **a** $y = 2(x-1)(x-2)$ **b** $y = 2(x-2)^2$
c $y = (x-1)(x-3)$ **d** $y = -(x-3)(x+1)$
e $y = -3(x-1)^2$ **f** $y = -2(x+2)(x-3)$

2 **a** $y = \frac{3}{2}(x-2)(x-4)$ **b** $y = -\frac{1}{2}(x+4)(x-2)$
c $y = -\frac{4}{3}(x+3)^2$

3 **a** $y = 3x^2 - 18x + 15$ **b** $y = -4x^2 + 6x + 4$
c $y = -x^2 + 6x - 9$ **d** $y = 4x^2 + 16x + 16$

4 **a** $y = \frac{3}{2}x^2 - 6x + \frac{9}{2}$ **b** $y = -\frac{1}{3}x^2 + \frac{2}{3}x + 5$

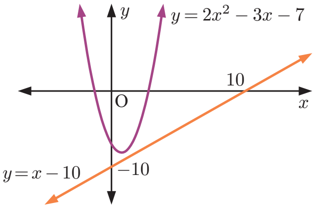
5 **a** $y = -(x-2)^2 + 4$ **b** $y = 2(x-2)^2 - 1$
c $y = -2(x-3)^2 + 8$ **d** $y = \frac{2}{3}(x-4)^2 - 6$
e $y = -2(x-2)^2 + 3$ **f** $y = 2(x - \frac{1}{2})^2 - \frac{3}{2}$

EXERCISE 3F

1 **a** $(1, 7)$ and $(2, 8)$ **b** $(4, 5)$ and $(-3, -9)$
c $(3, 0)$ (touching) **d** graphs do not meet

2 $c = -9$ **3** $m = 0$ or -8 **4** -1 or 11

5 **a** $c < -9$
b example: $c = -10$



6 **a** $c > -2$ **b** $c = -2$ **c** $c < -2$

7 **a** $m < -1$ or $m > 7$ **b** $m = -1$ or $m = 7$
c $-1 < m < 7$

8 **Hint:** A straight line through $(0, 3)$ will have an equation of the form $y = mx + 3$.

EXERCISE 3G

1 7 and -5 or -7 and 5 **2** 5 or $\frac{1}{5}$ **3** 14
4 18 and 20 or -18 and -20 **5** 15 sides
6 3.48 cm **7** **b** 6 cm by 6 cm by 7 cm
8 11.2 cm square **9** no

11 **a** $y = -\frac{8}{9}x^2 + 8$
b No, as the tunnel is only 4.44 m high when it is the same width as the truck.

12 **b** The graph is a parabola. **c** 21.25 m
d $f(x) = -0.05x^2 + 2x + 1.25$ **e** yes

EXERCISE 3H

1 **a** min. -1, when $x = 1$ **b** max. 8, when $x = -1$
c max. $8\frac{1}{3}$, when $x = \frac{1}{3}$ **d** min. $-1\frac{1}{8}$, when $x = -\frac{1}{4}$
e min. $4\frac{15}{16}$, when $x = \frac{1}{8}$ **f** max. $6\frac{1}{8}$, when $x = \frac{7}{4}$

2 **a** 40 refrigerators **b** \$4000

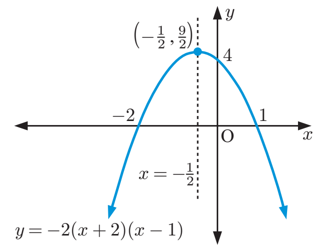
4 500 m by 250 m **5** **c** 100 m by 112.5 m

6 **a** $41\frac{2}{3}$ m by $41\frac{2}{3}$ m **b** 50 m by $31\frac{1}{4}$ m

7 **b** $3\frac{1}{8}$ units **8** **a** $y = 6 - \frac{3}{4}x$ **b** 3 cm by 4 cm

REVIEW SET 3A

1 **a** $-2, 1$
b $x = -\frac{1}{2}$
c 4
d $(-\frac{1}{2}, \frac{9}{2})$
e



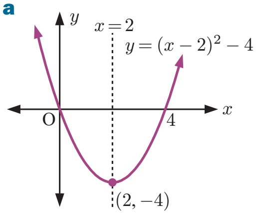
f $\{y : y \leqslant \frac{9}{2}\}$

2 **a** $x = 0$ or 4 **b** $x = -\frac{5}{3}$ or 2 **c** $x = 15$ or -4

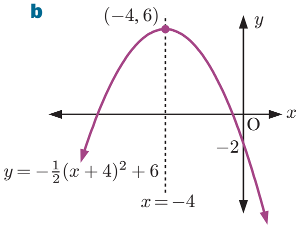
3 **a** $x = -\frac{5}{2} \pm \frac{\sqrt{13}}{2}$ **b** $x = -\frac{11}{6} \pm \frac{\sqrt{145}}{6}$

4 **a** $-3 < x < 7$ **b** $x \leqslant -\frac{1}{3}$ or $x \geqslant 2$

5 **a**



b



6 **a** $y = 3x^2 - 24x + 48$ **b** $y = \frac{2}{5}x^2 + \frac{16}{5}x + \frac{37}{5}$

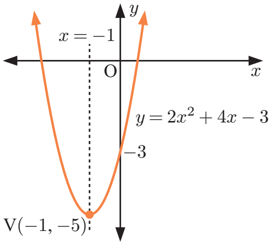
7 $a = -2$ which is < 0 $\therefore$  a max.
max. $= 5$ when $x = 1$

8 $(4, 4)$ and $(-3, 18)$ **9** $k < -3\frac{1}{8}$

10 **a** $m = \frac{9}{8}$ **b** $m < \frac{9}{8}$ **c** $m > \frac{9}{8}$ **11** $\frac{6}{5}$ or $\frac{5}{6}$

12 **Hint:** Let the line have equation $y = mx + 10$.

13 **a** $y = 2(x+1)^2 - 5$ **b**



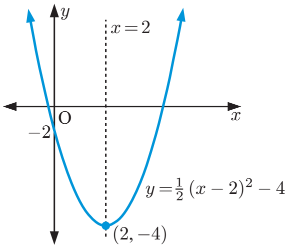
14 **a** $y = \frac{20}{9}(x-2)^2 - 20$ **b** $y = -\frac{2}{7}(x-1)(x-7)$
c $y = \frac{2}{9}(x+3)^2$

15 $\{y : -13 \leqslant y \leqslant 12\}$ **16** 21 m

REVIEW SET 3B

1 **a** $x = 2$
b $(2, -4)$
c -2
e $\{y : y \geqslant -4\}$

d

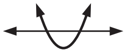


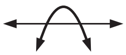
2 **a** $x = \frac{5}{2} \pm \frac{\sqrt{37}}{2}$ **b** $x = \frac{7}{4} \pm \frac{\sqrt{73}}{4}$

3 **a** $-7 \leqslant x \leqslant 2$ **b** $x < -4$ or $x > \frac{3}{2}$

4 $x = \frac{4}{3}$, $V(\frac{4}{3}, 12\frac{1}{3})$

5 **a** graph cuts x-axis twice **b** graph cuts x-axis twice



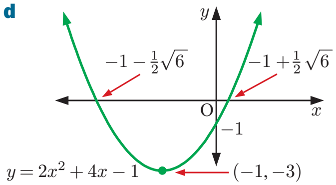


6 **a** $a < 0$, $\Delta > 0$, neither
b $a > 0$, $\Delta < 0$, positive definite

7 $y = -6(x-2)^2 + 25$ **8** $m < -5$ or $m > 19$

9 **a** $x = -1$
b $(-1, -3)$
c y-intercept -1,
x-ints. $-1 \pm \frac{1}{2}\sqrt{6}$

d



10 $\{y : -55 \leqslant y \leqslant \frac{11}{2}\}$

11 **a** $k = -8$ **b** $k < -8$ or $k > 0$ **c** $-8 < k < 0$

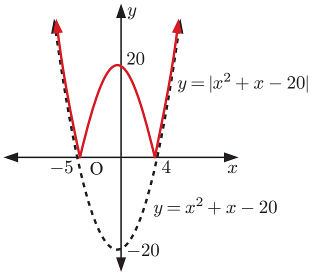
12 **a** $c > -6$
b example: $c = -2$, $(-1, -5)$ and $(3, 7)$

13 **a** $y = -\frac{2}{5}(x+5)(x-1)$ **b** $(-2, 3\frac{3}{5})$

14 **a** min. $= 5\frac{2}{3}$ when $x = -\frac{2}{3}$
b max. $= 5\frac{1}{8}$ when $x = -\frac{5}{4}$

15 **b** $37\frac{1}{2}$ m by $33\frac{1}{3}$ m **c** 1250 m^2

16



EXERCISE 4A.1

1 **a** 11 **b** $\sqrt{15}$ **c** 3 **d** $\sqrt{30}$
e 4 **f** 12 **g** 42 **h** 45
i $\sqrt{6}$ **j** $\sqrt{6}$ **k** 2 **l** $\sqrt{5}$

2 **a** $2\sqrt{2}$ **b** $2\sqrt{3}$ **c** $2\sqrt{5}$ **d** $4\sqrt{2}$
e $3\sqrt{3}$ **f** $3\sqrt{5}$ **g** $4\sqrt{3}$ **h** $3\sqrt{6}$
i $5\sqrt{2}$ **j** $4\sqrt{5}$ **k** $4\sqrt{6}$ **l** $6\sqrt{3}$

EXERCISE 4A.2

1 **a** $5\sqrt{2}$ **b** $-\sqrt{2}$ **c** $2\sqrt{5}$ **d** $8\sqrt{5}$
e $-2\sqrt{5}$ **f** $9\sqrt{3}$ **g** $-3\sqrt{6}$ **h** $3\sqrt{2}$

2 **a** $3\sqrt{2} - 2$ **b** $5 + \sqrt{5}$ **c** $3\sqrt{10} + 20$
d $21 - 4\sqrt{7}$ **e** $-5\sqrt{3} - 3$ **f** $12 - 14\sqrt{6}$
g $-8 + 5\sqrt{8}$ **h** $-12\sqrt{2} + 36$

3 **a** $22 + 9\sqrt{2}$ **b** $34 + 15\sqrt{3}$ **c** $22 + 14\sqrt{7}$
d $-7 - \sqrt{3}$ **e** $34 - 15\sqrt{8}$ **f** $-47 + 30\sqrt{5}$

4 **a** $11 + 6\sqrt{2}$ **b** $39 - 12\sqrt{3}$ **c** $6 + 2\sqrt{5}$
d $17 - 6\sqrt{8}$ **e** $28 + 16\sqrt{3}$ **f** $46 + 6\sqrt{5}$
g $89 - 28\sqrt{10}$ **h** $166 - 40\sqrt{6}$

5 **a** 2 **b** -23 **c** 13 **d** 7
e -56 **f** 218

EXERCISE 4A.3

1 **a** $\frac{\sqrt{3}}{3}$ **b** $\sqrt{3}$ **c** $3\sqrt{3}$ **d** $\frac{11\sqrt{3}}{3}$ **e** $\frac{\sqrt{6}}{9}$
f $\sqrt{2}$ **g** $3\sqrt{2}$ **h** $6\sqrt{2}$ **i** $\frac{\sqrt{6}}{2}$ **j** $\frac{\sqrt{2}}{8}$

2 **a** $\sqrt{5}$ **b** $3\sqrt{5}$ **c** $-\frac{3\sqrt{5}}{5}$ **d** $40\sqrt{5}$ **e** $\frac{\sqrt{5}}{15}$
f $\sqrt{7}$ **g** $3\sqrt{7}$ **h** $\frac{2\sqrt{11}}{11}$ **i** $2\sqrt{13}$ **j** $\frac{\sqrt{3}}{9}$

3 **a** $\frac{3-\sqrt{2}}{7}$ **b** $\frac{6+2\sqrt{2}}{7}$ **c** $-2+\sqrt{5}$
d $1+\sqrt{2}$ **e** $2+2\sqrt{6}$ **f** $\frac{\sqrt{21}-2\sqrt{3}}{3}$
g $-3-2\sqrt{2}$ **h** $\frac{3+4\sqrt{3}}{13}$ **i** $4+2\sqrt{2}$
j $-7-3\sqrt{5}$ **k** $\frac{5+3\sqrt{3}}{2}$ **l** $\frac{-38+11\sqrt{10}}{6}$

4 **a** $-\frac{9}{7}-\frac{3}{7}\sqrt{2}$ **b** $4-2\sqrt{2}$ **c** $-\frac{2}{23}-\frac{5}{23}\sqrt{2}$
d $-4+2\sqrt{2}$

5 **a** $-2-2\sqrt{3}$ **b** $12-6\sqrt{3}$ **c** $3+2\sqrt{3}$ **d** $-\frac{1}{2}+\frac{5}{6}\sqrt{3}$

6 **a** $(a+b\sqrt{c})(a-b\sqrt{c})=a^2-b^2c$
which is an integer as a, b, and c are integers.
b **i** $\frac{-1+2\sqrt{3}}{11}$ **ii** $\frac{-6-5\sqrt{2}}{7}$ **iii** $1+\sqrt{2}$

7 **a** $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$
which is an integer as a and b are integers.
b **i** $\sqrt{3}-\sqrt{2}$ **ii** $\frac{-3-\sqrt{15}}{2}$ **iii** $\frac{2\sqrt{154}-25}{3}$

8 $x=-7+5\sqrt{3}$ **9** $x=\frac{10}{19}+\frac{1}{19}\sqrt{5}$

EXERCISE 4B

1 **a** $2^1=2,\ 2^2=4,\ 2^3=8,\ 2^4=16,\ 2^5=32,\ 2^6=64$
b $3^1=3,\ 3^2=9,\ 3^3=27,\ 3^4=81,\ 3^5=243,$
$3^6=729$
c $4^1=4,\ 4^2=16,\ 4^3=64,\ 4^4=256,\ 4^5=1024,$
$4^6=4096$

2 **a** $5^1=5,\ 5^2=25,\ 5^3=125,\ 5^4=625$
b $6^1=6,\ 6^2=36,\ 6^3=216,\ 6^4=1296$
c $7^1=7,\ 7^2=49,\ 7^3=343,\ 7^4=2401$

3 **a** -1 **b** 1 **c** 1 **d** -1 **e** 1
f -1 **g** -1 **h** -32 **i** -32 **j** -64
k 625 **l** -625

4 **a** 16 384 **b** 2401 **c** -3125
d -3125 **e** 262 144 **f** 262 144
g $-262\,144$ **h** 902.436 039 6 **i** $-902.436\,039\,6$
j $-902.436\,039\,6$

5 **a** $0.\overline{1}$ **b** $0.\overline{1}$ **c** $0.02\overline{7}$ **d** $0.02\overline{7}$
e 0.012 345 679 **f** 0.012 345 679 **g** 1 **h** 1

Notice that $a^{-n}=\frac{1}{a^n}$

6 3 **7** 7

EXERCISE 4C

1 **a** 5^{11} **b** d^8 **c** k^5 **d** $\frac{1}{7}$ **e** x^{10} **f** 3^{16}
g p^{-4} **h** n^{12} **i** 5^{3t} **j** 7^{x+2} **k** 10^{3-q} **l** c^{4m}

2 **a** 2^2 **b** 2^{-2} **c** 2^3 **d** 2^{-3} **e** 2^5 **f** 2^{-5}
g 2^1 **h** 2^{-1} **i** 2^6 **j** 2^{-6} **k** 2^7 **l** 2^{-7}

3 **a** 3^2 **b** 3^{-2} **c** 3^3 **d** 3^{-3} **e** 3^1 **f** 3^{-1}
g 3^4 **h** 3^{-4} **i** 3^0 **j** 3^5 **k** 3^{-5}

4 **a** 2^{a+1} **b** 2^{b+2} **c** 2^{t+3} **d** 2^{2x+2} **e** 2^{n-1}
f 2^{c-2} **g** 2^{2m} **h** 2^{n+1} **i** 2^1 **j** 2^{3x-1}

5 **a** 3^{p+2} **b** 3^{3a} **c** 3^{2n+1} **d** 3^{d+3} **e** 3^{3t+2}
f 3^{y-1} **g** 3^{1-y} **h** 3^{2-3t} **i** 3^{3a-1} **j** 3^3

6 **a** $4a^2$ **b** $27b^3$ **c** a^4b^4 **d** p^3q^3 **e** $\frac{m^2}{n^2}$
f $\frac{a^3}{27}$ **g** $\frac{b^4}{c^4}$ **h** $1,\ a,b\neq 0$ **i** $\frac{m^4}{81n^4}$ **j** $\frac{x^3y^3}{8}$

7 **a** $4a^2$ **b** $36b^4$ **c** $-8a^3$ **d** $-27m^6n^6$
e $16a^4b^{16}$ **f** $\frac{-8a^6}{b^6}$ **g** $\frac{16a^6}{b^2}$ **h** $\frac{9p^4}{q^6}$
i $4x^3y^2$ **j** $32a^5b$ **k** $\frac{5a^{12}}{b^2}$ **l** $\frac{-2x^{18}}{y^3}$

8 **a** $\frac{a}{b^2}$ **b** $\frac{1}{a^2b^2}$ **c** $\frac{4a^2}{b^2}$ **d** $\frac{9b^2}{a^4}$ **e** $\frac{a^2}{bc^2}$
f $\frac{a^2c^2}{b}$ **g** a^3 **h** $\frac{b^3}{a^2}$ **i** $\frac{2}{ad^2}$ **j** $12am^3$

9 **a** a^{-n} **b** b^n **c** 3^{n-2} **d** a^nb^m **e** a^{-2n-2}

10 **a** 1 **b** $\frac{4}{7}$ **c** 6 **d** 27 **e** $\frac{9}{16}$ **f** $\frac{5}{2}$
g $\frac{27}{125}$ **h** $\frac{151}{5}$

11 **a** 3^{-2} **b** 2^{-4} **c** 5^{-3} **d** $3^1\times 5^{-1}$ **e** $2^2\times 3^{-3}$
f $2^{c-3}\times 3^{-2}$ **g** $3^{2k}\times 2^{-1}\times 5^{-1}$ **h** $2^p\times 3^{p-1}\times 5^{-2}$

12 **a** $5^3=21+23+25+27+29$
b $7^3=43+45+47+49+51+53+55$
c $12^3=133+135+137+139+141+143+145+147$
$+149+151+153+155$

EXERCISE 4D

1 **a** $2^{\frac{1}{5}}$ **b** $2^{-\frac{1}{5}}$ **c** $2^{\frac{3}{2}}$ **d** $2^{\frac{5}{2}}$ **e** $2^{-\frac{1}{3}}$
f $2^{\frac{4}{3}}$ **g** $2^{\frac{3}{2}}$ **h** $2^{\frac{3}{2}}$ **i** $2^{-\frac{4}{3}}$ **j** $2^{-\frac{3}{2}}$

2 **a** $3^{\frac{1}{3}}$ **b** $3^{-\frac{1}{3}}$ **c** $3^{\frac{1}{4}}$ **d** $3^{\frac{3}{2}}$ **e** $3^{-\frac{5}{2}}$

3 **a** $7^{\frac{1}{3}}$ **b** $3^{\frac{3}{4}}$ **c** $2^{\frac{4}{5}}$ **d** $2^{\frac{5}{3}}$ **e** $7^{\frac{2}{7}}$
f $7^{-\frac{1}{3}}$ **g** $3^{-\frac{3}{4}}$ **h** $2^{-\frac{4}{5}}$ **i** $2^{-\frac{5}{3}}$ **j** $7^{-\frac{2}{7}}$

4 **a** 2.28 **b** 1.83 **c** 0.794 **d** 0.435 **e** 1.68
f 1.93 **g** 0.523

5 **a** 8 **b** 32 **c** 8 **d** 125 **e** 4
f $\frac{1}{2}$ **g** $\frac{1}{27}$ **h** $\frac{1}{16}$ **i** $\frac{1}{81}$ **j** $\frac{1}{25}$

EXERCISE 4E.1

1 **a** $x^5+2x^4+x^2$ **b** 4^x+2^x **c** $x+1$
d $49^x+2(7^x)$ **e** $2(3^x)-1$ **f** x^2+2x+3
g $1+5(2^{-x})$ **h** 5^x+1 **i** $x^{\frac{3}{2}}+x^{\frac{1}{2}}+1$

2 **a** $4^x+2^{x+1}-3$ **b** $9^x+7(3^x)+10$
c $25^x-6(5^x)+8$ **d** $4^x+6(2^x)+9$
e $9^x-2(3^x)+1$ **f** $16^x+14(4^x)+49$

3 **a** $x-4$ **b** 4^x-9 **c** $x-x^{-1}$ **d** $x^2+4+\frac{4}{x^2}$
e $7^{2x}-2+7^{-2x}$ **f** $25-10(2^{-x})+4^{-x}$
g $x^{\frac{4}{3}}+2x+x^{\frac{2}{3}}$ **h** x^3-2x^2+x **i** $4x-4+x^{-1}$

EXERCISE 4E.2

1 **a** $5^x(5^x+1)$ **b** $10(3^n)$ **c** $7^n(1+7^{2n})$
d $5(5^n-1)$ **e** $6(6^{n+1}-1)$ **f** $16(4^n-1)$

2 **a** $(3^x+2)(3^x-2)$ **b** $(2^x+5)(2^x-5)$
c $(4+3^x)(4-3^x)$ **d** $(5+2^x)(5-2^x)$
e $(3^x+2^x)(3^x-2^x)$ **f** $(2^x+3)^2$
g $(3^x+5)^2$ **h** $(2^x-7)^2$ **i** $(5^x-2)^2$

3 **a** $(2^x+3)(2^x+6)$ **b** $(2^x+4)(2^x-5)$
c $(3^x+2)(3^x+7)$ **d** $(3^x+5)(3^x-1)$
e $(5^x+2)(5^x-1)$ **f** $(7^x-4)(7^x-3)$

4 **a** 2^n **b** 10^a **c** 3^b **d** $\frac{1}{5^n}$ **e** 5^x
f $(\frac{3}{4})^a$ **g** 5 **h** 5^n

5 **a** 3^m+1 **b** $1+6^n$ **c** 4^n+2^n **d** 4^x-1
e 6^n **f** 5^n **g** 4 **h** 2^n-1 **i** $\frac{1}{2}$

6 **a** $n\,2^{n+1}$ **b** -3^{n-1}

EXERCISE 4F

1 **a** $x=3$ **b** $x=2$ **c** $x=4$ **d** $x=0$
e $x=-1$ **f** $x=\frac{1}{2}$ **g** $x=-3$ **h** $x=2$
i $x=-3$ **j** $x=-4$ **k** $x=2$ **l** $x=1$

2 **a** $x=\frac{5}{3}$ **b** $x=-\frac{3}{2}$ **c** $x=-\frac{3}{2}$ **d** $x=-\frac{1}{2}$
e $x=-\frac{2}{3}$ **f** $x=-\frac{5}{4}$ **g** $x=\frac{3}{2}$ **h** $x=\frac{5}{2}$
i $x=\frac{1}{8}$ **j** $x=\frac{9}{2}$ **k** $x=-4$ **l** $x=-4$
m $x=0$ **n** $x=\frac{7}{2}$ **o** $x=-2$ **p** $x=-6$

3 **a** $x=\frac{1}{7}$ **b** has no solutions **c** $x=2\frac{1}{2}$

4 **a** $x=1$ **b** $x=2$ **c** $x=1$
d $x=\frac{5}{4}$ **e** $x=2$ **f** $x=-\frac{9}{7}$

5 **a** $x=3$ **b** $x=2$ **c** $x=2$
d $x=2$ **e** $x=-2$ **f** $x=-2$

6 **a** $x=1$ or 2 **b** $x=1$ **c** $x=1$ or 2
d $x=1$ **e** $x=2$ **f** $x=0$

EXERCISE 4G

1 **a** 1.4 **b** 1.7 **c** 2.8 **d** 0.4

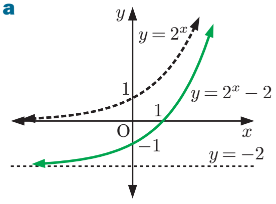
2 **a** $x\approx 1.6$ **b** $x\approx -0.7$ **c** $x\approx 2.1$ **d** $x\approx -1.7$

3 $y=2^x$ has a horizontal asymptote of $y=0$

4 **a** 2 **b** 54 **c** $\frac{2}{9}$

5 **a** $g(0)=3,\ \ g(-1)=\frac{11}{5}$ **b** $a=2$

6 **a**



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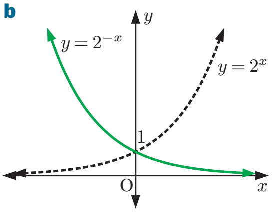

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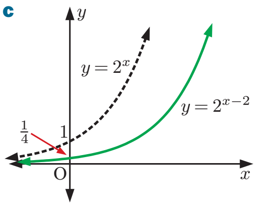
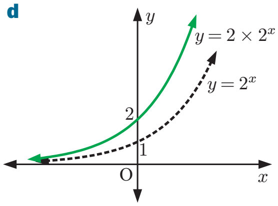
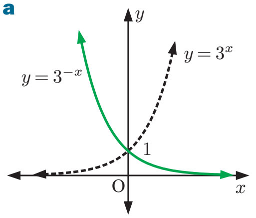


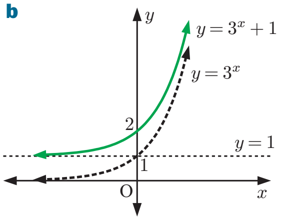
d



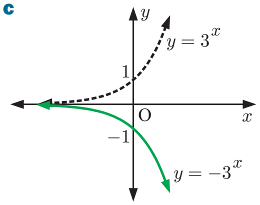
7 **a**



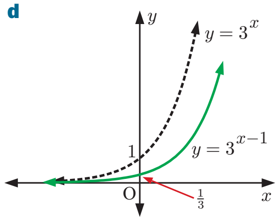
b



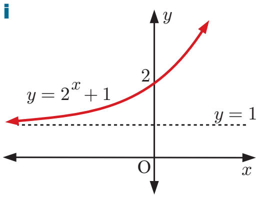
c



d



8 **a** **i**



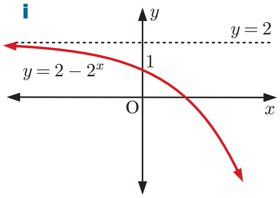
ii Domain: $\{x : x\in\mathbb{R}\}$
Range: $\{y : y>1\}$

iii $y\approx 3.67$

iv As $x\to\infty,\ y\to\infty$
As $x\to-\infty,\ y\to 1$ from above

v $y=1$

b **i**



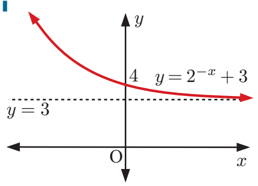
ii Domain: $\{x : x\in\mathbb{R}\}$
Range: $\{y : y<2\}$

iii $y\approx -0.665$

iv As $x\to\infty,\ y\to-\infty$
As $x\to-\infty,\ y\to 2$ from below

v $y=2$

c **i**



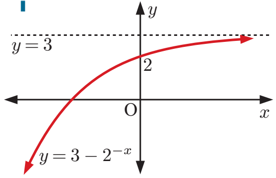
ii Domain: $\{x : x\in\mathbb{R}\}$
Range: $\{y : y>3\}$

iii $y\approx 3.38$

iv As $x\to\infty,\ y\to 3$ from above
As $x\to-\infty,\ y\to\infty$

v $y=3$

d **i**



ii Domain: $\{x : x\in\mathbb{R}\}$
Range: $\{y : y<3\}$

iii $y\approx 2.62$

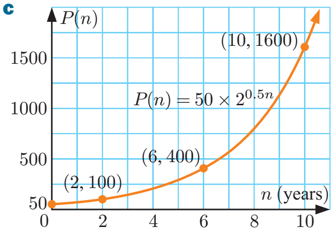
iv As $x\to\infty,\ y\to 3$ from below
As $x\to-\infty,\ y\to-\infty$

v $y=3$

9 **a** $P_0=50$

b **i** 100 possums
ii 400 possums
iii 1600 possums

c



d 8 years

10 **a** 100 g

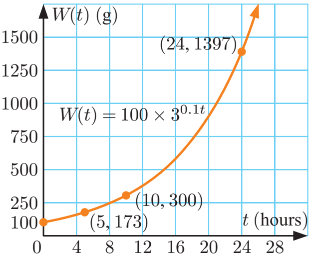
b **i** ≈ 173 g

ii 300 g

iii ≈ 1397 g

d 20 hours

c



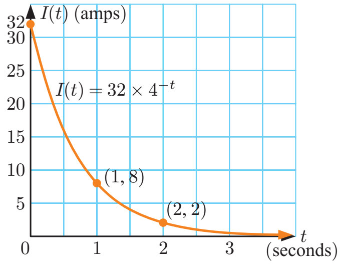
11 **a** 32 amps

b **i** 8 amps

ii 2 amps

d 3 seconds

c



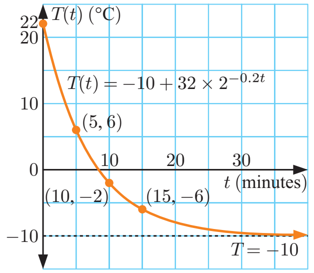
12 **a** **i** 22°C

ii 6°C

iii −2°C

iv −6°C

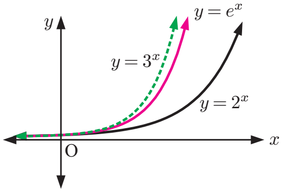
b



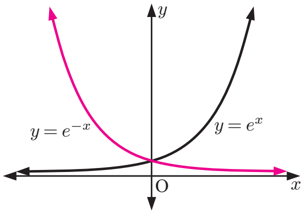
c The temperature will not reach −10°C according to this model, as the model has a horizontal asymptote at $T = -10$.

EXERCISE 4H

1 The graph of $y = e^x$ lies between $y = 2^x$ and $y = 3^x$.



2 One is the other reflected in the y-axis.



3 a

4 **a** $e^x > 0$ for all x

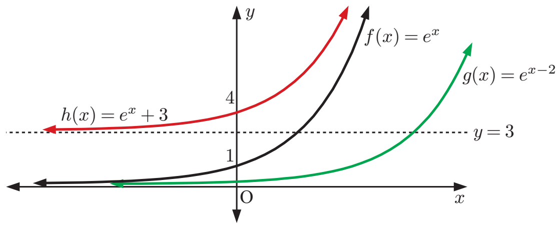
b **i** 0.000 000 004 12 **ii** 970 000 000

5 **a** ≈ 7.39 **b** ≈ 20.1 **c** ≈ 2.01 **d** ≈ 1.65

e ≈ 0.368

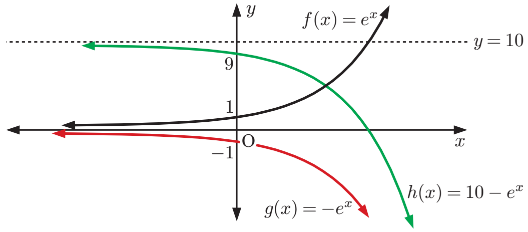
6 **a** $e^{\frac{1}{2}}$ **b** $e^{-\frac{1}{2}}$ **c** e^{-2} **d** $e^{\frac{3}{2}}$

7



Domain of f, g, and h is $\{x : x \in \mathbb{R}\}$
Range of f is $\{y : y > 0\}$, Range of g is $\{y : y > 0\}$
Range of h is $\{y : y > 3\}$

8



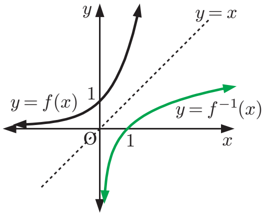
Domain of f, g, and h is $\{x : x \in \mathbb{R}\}$
Range of f is $\{y : y > 0\}$, Range of g is $\{y : y < 0\}$
Range of h is $\{y : y < 10\}$

9 **a** $e^{2x} + 2e^x + 1$ **b** $1 - e^{2x}$ **c** $1 - 3e^x$

10 **a** $x = \frac{1}{2}$ **b** $x = -4$

11 **a** $fg(x) = e^{3x+2}$, $gf(x) = 3e^x + 2$ **b** $x = -1$

12 **a**



b Domain of f^{-1} is $\{x : x > 0\}$,
Range of f^{-1} is $\{y : y \in \mathbb{R}\}$

REVIEW SET 4A

1 **a** $-15 + 20\sqrt{3}$ **b** $86 - 60\sqrt{2}$

2 **a** $\frac{2\sqrt{3}}{3}$ **b** $\frac{\sqrt{35}}{5}$ **c** $\frac{\sqrt{7}}{28}$

3 **a** a^6b^7 **b** $\frac{2}{3x}$ **c** $\frac{y^2}{5}$

4 **a** **i** 81 **ii** $\frac{1}{3}$ **b** $k = 9$

5 **a** $\frac{1}{x^5}$ **b** $\frac{2}{a^2b^2}$ **c** $\frac{2a}{b^2}$

6 **a** 3^{3-2a} **b** $3^{\frac{5}{2}-\frac{9}{2}x}$ **7** **a** 4 **b** $\frac{1}{9}$

8 **a** $\frac{m}{n^2}$ **b** $\frac{1}{m^3n^3}$ **c** $\frac{m^2p^2}{n}$ **d** $\frac{16n^2}{m^2}$

9 **a** $9 - 6e^x + e^{2x}$ **b** $x - 4$ **c** $2^x + 1$

10 $x = \frac{9}{34} + \frac{1}{34}\sqrt{13}$

11 **a** $x = -2$ **b** $x = \frac{3}{4}$ **c** $x = -\frac{1}{4}$

12 **a** **C** **b** **E** **c** **A** **d** **B** **e** **D**

13 **a** 3 **b** 24 **c** $\frac{3}{4}$

14 **a** Range of f is $\{y : y > -3\}$ **b** −2 **c** $x = \frac{1}{2}$

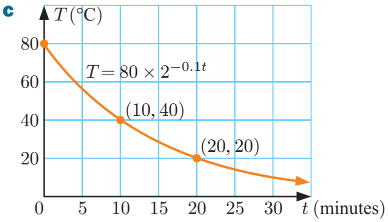
15 **a** 80°C

b **i** 40°C

ii 20°C

d 30 minutes

c



REVIEW SET 4B

1 **a** $17 - 11\sqrt{3}$ **b** 28

2 **a** $\dfrac{5+\sqrt{3}}{22}$ **b** $\dfrac{\sqrt{77}+2\sqrt{11}}{3}$ **c** $\dfrac{26+11\sqrt{2}}{7}$

d $\dfrac{-33-14\sqrt{5}}{3}$

3 **a** $x \approx 1.45$ **b** $x \approx -0.6$ **c** $x \approx 1.1$

4 **a** $3-2\sqrt{2}$ **b** $3-2\sqrt{2}$ **c** $3-2\sqrt{2}$ **d** $3-2\sqrt{2}$

5 **a** a^{21} **b** p^4q^6 **c** $\dfrac{4b}{a^3}$

6 **a** 2^{-3} **b** 2^7 **c** 2^{12}

7 **a** $4m^6$ **b** $\dfrac{-a^9}{b^3}$ **c** $3x^3y^2$ **d** $16ab^{\frac{4}{5}}$

8 2^{2x} **9** **a** 5^0 **b** $5^{\frac{3}{2}}$ **c** $5^{-\frac{1}{4}}$ **d** 5^{2a+6}

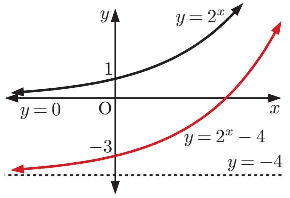
10 **a** $1+e^{2x}$ **b** $2^{2x}+10(2^x)+25$ **c** $x-49$

11 **a** $x=5$ **b** $x=-4$

12 **a** $x=-\frac{2}{5}$ **b** $x=1$ **c** $x=\frac{7}{11}$

13 **a** $\frac{1}{\sqrt{2}}+1 \approx 1.71$ **14**

b $a=-1$



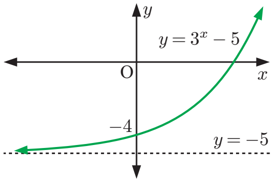
15 **a**

x	-2	-1	0	1	2
y	$-4\frac{8}{9}$	$-4\frac{2}{3}$	-4	-2	4

b as $x \to \infty$, $y \to \infty$;

as $x \to -\infty$, $y \to -5$ (above)

c



d $\{y : y > -5\}$

16 **a** Range of f is $\{y : y > 0\}$ **b** $g(\sqrt{2}) = e^2$

c $1+\frac{1}{2}\sqrt{2}$

EXERCISE 5A

1 **a** 4 **b** -3 **c** 1 **d** 0 **e** $\frac{1}{2}$ **f** $\frac{1}{3}$

g $-\frac{1}{4}$ **h** $1\frac{1}{2}$ **i** $\frac{2}{3}$ **j** $1\frac{1}{2}$ **k** $1\frac{1}{3}$ **l** $3\frac{1}{2}$

2 **a** n **b** $a+2$ **c** $1-m$ **d** $a-b$

3 **a** $\lg 41 \approx 1.6128$ **b** $41 \approx 10^{1.6128}$

4 **a** $10^{0.7782}$ **b** $10^{1.7782}$ **c** $10^{3.7782}$ **d** $10^{-0.2218}$

e $10^{-2.2218}$ **f** $10^{1.1761}$ **g** $10^{3.1761}$ **h** $10^{0.1761}$

i $10^{-0.8239}$ **j** $10^{-3.8239}$

5 A negative number cannot be written in the form 10^b where $b \in \mathbb{R}$, so its logarithm cannot be found.

6 **a** **i** 0.477 **ii** 2.477 **b** $\lg 300 = \lg(3 \times 10^2)$

7 **a** **i** 0.699 **ii** -1.301 **b** $\lg 0.05 = \lg(5 \times 10^{-2})$

8 **a** $x=100$ **b** $x=10$ **c** $x=1$

d $x=\frac{1}{10}$ **e** $x=10^{\frac{1}{2}}$ **f** $x=10^{-\frac{1}{2}}$

g $x=10\,000$ **h** $x=0.000\,01$ **i** $x \approx 6.84$

j $x \approx 140$ **k** $x \approx 0.0419$ **l** $x \approx 0.000\,631$

EXERCISE 5B

1 **a** $10^2 = 100$ **b** $10^4 = 10\,000$ **c** $10^{-1} = 0.1$

d $10^{\frac{1}{2}} = \sqrt{10}$ **e** $2^3 = 8$ **f** $3^2 = 9$

g $2^{-2} = \frac{1}{4}$ **h** $3^{1.5} = \sqrt{27}$ **i** $5^{-\frac{1}{2}} = \frac{1}{\sqrt{5}}$

2 **a** $\log_2 4 = 2$ **b** $\log_4 64 = 3$ **c** $\log_5 25 = 2$

d $\log_7 49 = 2$ **e** $\log_2 64 = 6$ **f** $\log_2(\frac{1}{8}) = -3$

g $\log_{10} 0.01 = -2$ **h** $\log_2(\frac{1}{2}) = -1$ **i** $\log_3(\frac{1}{27}) = -3$

3 **a** 5 **b** -2 **c** $\frac{1}{2}$ **d** 3 **e** 6 **f** 7 **g** 2

h 3 **i** -3 **j** $\frac{1}{2}$ **k** 2 **l** $\frac{1}{2}$ **m** 5 **n** $\frac{1}{3}$

o n, $a > 0$ **p** $\frac{1}{3}$ **q** -1, $t > 0$ **r** $\frac{3}{2}$ **s** 0

t 1

4 **a** ≈ 2.18 **b** ≈ 1.40 **c** ≈ 1.87 **d** ≈ -0.0969

5 **a** $x=8$ **b** $x=2$ **c** $x=3$ **d** $x=14$

6 **a** 2 **b** 2 **c** -1 **d** $\frac{3}{4}$ **e** $-\frac{1}{2}$ **f** $\frac{5}{2}$

g $-\frac{3}{2}$ **h** $-\frac{3}{4}$ **i** 2, $x > 0$ **j** $\frac{1}{2}$, $x > 0$

k 3, $m > 0$ **l** $\frac{3}{2}$, $x > 0$ **m** -1, $n > 0$

n -2, $a > 0$ **o** $-\frac{1}{2}$, $a > 0$ **p** $\frac{5}{2}$, $m > 0$

EXERCISE 5C

1 **a** $\lg 16$ **b** $\lg 20$ **c** $\lg 8$ **d** $\lg \dfrac{p}{m}$

e 1 **f** $\lg 2$ **g** $\lg 24$ **h** $\log_2 6$

i $\lg 0.4$ **j** 1 **k** $\lg 200$

l $\lg(10^t \times w)$ **m** $\log_m \left(\dfrac{40}{m^2}\right)$ **n** 0

o $\lg(0.005)$ **p** $\log_5(\frac{5}{2})$ **q** 2 **r** $\lg 28$

2 **a** $\lg 96$ **b** $\lg 72$ **c** $\lg 8$ **d** $\log_3(\frac{25}{8})$

e 1 **f** $\lg \frac{1}{2}$ **g** $\lg 20$ **h** $\lg 25$

i $\log_n \left(\dfrac{n^2}{10}\right)$

3 **a** 2 **b** $\frac{3}{2}$ **c** 3 **d** $\frac{1}{2}$ **e** -2 **f** $-\frac{3}{2}$

4 For example, for **a**, $\lg 9 = \lg 3^2 = 2\lg 3$

5 **a** 2 **b** -1 **c** 1

6 **a** $x+z$ **b** $z+2y$ **c** $x+z-y$ **d** $2x+\frac{1}{2}y$

e $3y-\frac{1}{2}z$ **f** $2z+\frac{1}{2}y-3x$

7 **a** $p+q$ **b** $2q+r$ **c** $2p+3q$ **d** $r+\frac{1}{2}q-p$

e $r-5p$ **f** $p-2q$

8 **a** 0.86 **b** 2.15 **c** 1.075 **9** $\log_b Q = 3$

10 **a** $\log_t A + 3\log_t B = 15$, $2\log_t A - \log_t B = 9$

b $\log_t A = 6$, $\log_t B = 3$ **c** $\log_t(B^5\sqrt{A}) = 18$

d $B = t^3$

EXERCISE 5D.1

1 **a** $\lg y = x \lg 2$ **b** $\lg y = 3 \lg x$
c $\lg M = 4 \lg d$ **d** $\lg T = x \lg 5$
e $\lg y = \frac{1}{2} \lg x$ **f** $\lg y = \lg 7 + x \lg 3$
g $\lg S = \lg 9 - \lg t$ **h** $\lg M = 2 + x \lg 7$
i $\lg T = \lg 5 + \frac{1}{2} \lg d$ **j** $\lg F = 3 - \frac{1}{2} \lg n$
k $\lg S = \lg 200 + t \lg 2$ **l** $\lg y = \frac{1}{2} \lg 15 - \frac{1}{2} \lg x$

2 **a** $y = 7^x$ **b** $D = 2x$ **c** $F = \dfrac{5}{t}$ **d** $y = 6 \times 2^x$
e $P = \sqrt{x}$ **f** $N = \dfrac{1}{\sqrt[3]{p}}$ **g** $P = 10x^3$ **h** $y = \dfrac{10^x}{2}$
i $y = \dfrac{x^2}{10}$ **j** $T = 2k^5$ **k** $P = \dfrac{n^4}{9}$ **l** $y = 8 \times 16^x$

3 **a** $y = \dfrac{x^3}{2}$ **b** **i** $y = 4$ **ii** $y = 32$

4 **a** $y = 100(10^{\frac{1}{3}x})$ **b** **i** $y = 100$ **ii** $y = 1000$

5 **a** If there is a *power* relationship between y and x, for example $y = 5x^3$, then there is a *linear* relationship between $\lg y$ and $\lg x$.
b If there is an *exponential* relationship between y and x, for example $y = 4 \times 2^x$, then there is a *linear* relationship between $\lg y$ and x.

EXERCISE 5D.2

1 **a** $x = 25$ **b** $x = 67$ **c** $x = 20$ **d** $x = \frac{125}{64}$
e $x = 5$ **f** no solution **g** $x = \frac{9}{8}$ **h** no solution

2 **a** $x = 5$ **b** $x = 3$ or 6 **c** $x = 2$ or 4 **d** $x = 2$
e $x = 1$ **f** no solution **g** $x = 2$ **h** $x = 4$

3 **a** $x = 8$ **b** $x = 3$ **c** $x = 6$ **d** $x = 4$

EXERCISE 5E.1

1 **a** 2 **b** 3 **c** $\frac{1}{2}$ **d** 0 **e** -1 **f** $\frac{1}{3}$ **g** -2
h $-\frac{1}{2}$

2 **a** 3 **b** 9 **c** $\frac{1}{5}$ **d** $\frac{1}{4}$

3 x does not exist such that $e^x = -2$ or 0

4 **a** a **b** $a + 1$ **c** $a + b$ **d** ab **e** $a - b$

5 **a** $e^{1.7918}$ **b** $e^{4.0943}$ **c** $e^{8.6995}$ **d** $e^{-0.5108}$
e $e^{-5.1160}$ **f** $e^{2.7081}$ **g** $e^{7.3132}$ **h** $e^{0.4055}$
i $e^{-1.8971}$ **j** $e^{-8.8049}$

6 **a** $x \approx 20.1$ **b** $x = e \approx 2.72$ **c** $x = 1$
d $x = \dfrac{1}{e} \approx 0.368$ **e** $x \approx 0.006\,74$
f $x \approx 2.30$ **g** $x \approx 8.54$ **h** $x \approx 0.0370$

EXERCISE 5E.2

1 **a** $\ln 45$ **b** $\ln 5$ **c** $\ln 4$ **d** $\ln 24$
e $\ln 1 = 0$ **f** $\ln 30$ **g** $\ln(4e)$ **h** $\ln\left(\dfrac{6}{e}\right)$
i $\ln 20$ **j** $\ln(4e^2)$ **k** $\ln\left(\dfrac{20}{e^2}\right)$ **l** $\ln 1 = 0$

2 **a** $\ln 972$ **b** $\ln 200$ **c** $\ln 1 = 0$ **d** $\ln 16$ **e** $\ln 6$
f $\ln\left(\frac{1}{3}\right)$ **g** $\ln\left(\frac{1}{2}\right)$ **h** $\ln 2$ **i** $\ln 16$

3 For example, for **a**, $\ln 27 = \ln 3^3 = 3 \ln 3$

4 **Hint:** In **d**, $\ln\left(\dfrac{e^2}{8}\right) = \ln e^2 - \ln 2^3$

5 **a** $D = ex$ **b** $F = \dfrac{e^2}{p}$ **c** $P = 5e^{2x}$
d $M = e^3 y^2$ **e** $B = \frac{1}{4}e^{3t}$ **f** $N = \dfrac{1}{\sqrt[3]{g}}$
g $Q \approx 8.66x^3$ **h** $D \approx 0.518n^{0.4}$ **i** $T \approx \dfrac{4.85}{e^x}$

EXERCISE 5F

1 **a** $x \approx 3.32$ **b** $x \approx 2.73$ **c** $x \approx 3.32$
d $x = 4$ **e** $x \approx 8.00$ **f** $x = -5$

2 **a** $x \approx 1.43$ **b** $x \approx 1.56$ **c** $x \approx 3.44$
d $x \approx 5.82$ **e** $x \approx -1.34$ **f** $x \approx 2.37$
g $x \approx 0.275$ **h** $x \approx 1.81$ **i** $x \approx 9.64$

3 **a** $x = \ln 10$ **b** $x = \ln 1000$ **c** $x = \ln 0.15$
d $x = 2 \ln 5$ **e** $x = \frac{1}{2} \ln 18$ **f** $x = 0$

4 **a** $x = \frac{1}{2} \ln 300$ **b** $x \approx 2.85$

5 **a** $x = -\dfrac{\lg(0.03)}{\lg 2}$ **b** $x = \dfrac{10 \lg\left(\frac{10}{3}\right)}{\lg 5}$ **c** $x = \dfrac{-4 \lg\left(\frac{1}{8}\right)}{\lg 3}$

6 **a** 3.90 hours **b** 15.5 hours **7** **b** $t \approx 6.93$ hours

8 **a** 50 g **b** $\approx 13\,200$ years

9 **a** $x = \ln 2$ **b** $x = 0$ **c** $x = \ln 2$ or $\ln 3$ **d** $x = 0$
e $x = \ln 4$ **f** $x = \ln\left(\frac{3+\sqrt{5}}{2}\right)$ or $\ln\left(\frac{3-\sqrt{5}}{2}\right)$

10 **a** $(\ln 3, 3)$ **b** $(\ln 2, 5)$ **c** $(0, 2)$ and $(\ln 5, -2)$

EXERCISE 5G

1 **a** ≈ 2.26 **b** ≈ -10.3 **c** ≈ -2.46 **d** ≈ 5.42

2 **a** $x \approx -4.29$ **b** $x \approx 3.87$ **c** $x \approx 0.139$

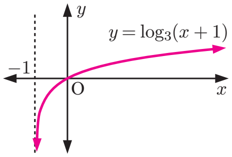
3 **a** $\log_9 26 = \frac{1}{2} \log_3 26$ **b** $\log_2 11 = 2 \log_4 11$
c $\dfrac{6}{\log_7 25} = 3 \log_5 7$

4 **a** $x = \sqrt[3]{50}$ **b** $x = \sqrt{13}$ **c** $x = 49$
d $x = 5$ **e** $x = 8$ **f** $x = 16$

5 **b** **i** $x = \frac{1}{9}$ or 9 **ii** $x = \frac{1}{2}$ or 32 **iii** $x = 2$ or 64

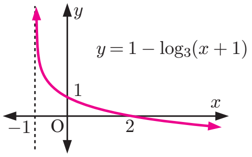
EXERCISE 5H

1 **a** **i** Domain is $\{x : x > -1\}$, Range is $\{y : y \in \mathbb{R}\}$
ii VA is $x = -1$, x and y-intercepts 0
iii



iv $x = -\frac{2}{3}$
v $f^{-1}(x) = 3^x - 1$

b **i** Domain is $\{x : x > -1\}$, Range is $\{y : y \in \mathbb{R}\}$
ii VA is $x = -1$, x-intercept 2, y-intercept 1
iii

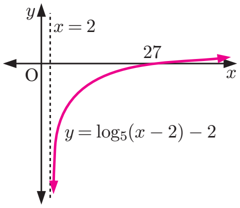


iv $x = 8$
v $f^{-1}(x) = 3^{1-x} - 1$

c **i** Domain is $\{x : x > 2\}$, Range is $\{y : y \in \mathbb{R}\}$

ii VA is $x = 2$, x-intercept 27, no y-intercept

iii



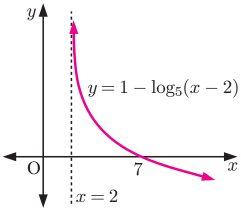
iv $x = 7$

v $f^{-1}(x) = 5^{2+x} + 2$

d **i** Domain is $\{x : x > 2\}$, Range is $\{y : y \in \mathbb{R}\}$

ii VA is $x = 2$, x-intercept 7, no y-intercept

iii



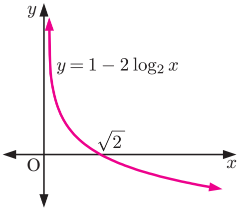
iv $x = 27$

v $f^{-1}(x) = 5^{1-x} + 2$

e **i** Domain is $\{x : x > 0\}$, Range is $\{y : y \in \mathbb{R}\}$

ii VA is $x = 0$, x-intercept $\sqrt{2}$, no y-intercept

iii

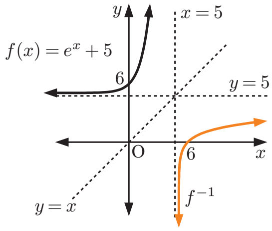


iv $x = 2$

v $f^{-1}(x) = 2^{\frac{1-x}{2}}$

2 **a** **i** $f^{-1}(x) = \ln(x-5)$

ii

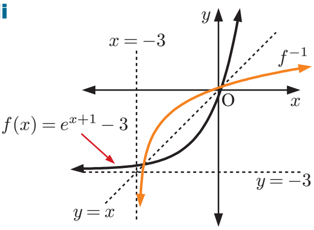


iii Domain of f is $\{x : x \in \mathbb{R}\}$, Range is $\{y : y > 5\}$
Domain of f^{-1} is $\{x : x > 5\}$, Range is $\{y : y \in \mathbb{R}\}$

iv f has a HA $y = 5$, f has y-int 6
f^{-1} has a VA $x = 5$, f^{-1} has x-int 6

b **i** $f^{-1}(x) = \ln(x+3) - 1$

ii

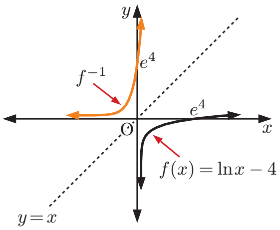


iii Domain of f is $\{x : x \in \mathbb{R}\}$, Range is $\{y : y > -3\}$
Domain of f^{-1} is $\{x : x > -3\}$, Range is $\{y : y \in \mathbb{R}\}$

iv f has a HA $y = -3$, x-int $\ln 3 - 1$, y-int $e - 3$
f^{-1} has a VA $x = -3$, x-int $e - 3$, y-int $\ln 3 - 1$

c **i** $f^{-1}(x) = e^{x+4}$

ii

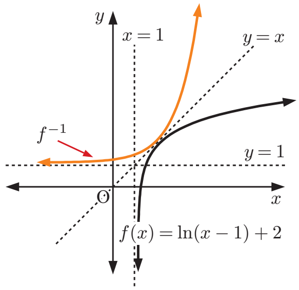


iii Domain of f is $\{x : x > 0\}$, Range of f is $\{y : y \in \mathbb{R}\}$
Domain of f^{-1} is $\{x : x \in \mathbb{R}\}$, Range is $\{y : y > 0\}$

iv f has a VA $x = 0$, x-int e^4
f^{-1} has a HA $y = 0$, y-int e^4

d **i** $f^{-1}(x) = 1 + e^{x-2}$

ii

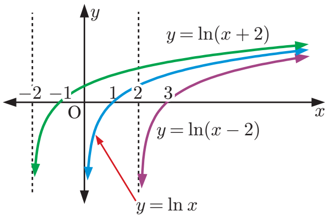


iii Domain of f is $\{x : x > 1\}$, Range is $\{y : y \in \mathbb{R}\}$
Domain of f^{-1} is $\{x : x \in \mathbb{R}\}$, Range is $\{y : y > 1\}$

iv f has a VA $x = 1$, x-int $1 + e^{-2}$
f^{-1} has a HA $y = 1$, y-int $1 + e^{-2}$

3 **a** A is $y = \ln x$ as its x-intercept is 1

b



c $y = \ln x$ has VA $x = 0$
$y = \ln(x-2)$ has VA $x = 2$
$y = \ln(x+2)$ has VA $x = -2$

4 $y = \ln(x^2) = 2\ln x$, so she is correct.
This is because the y-values are twice as large for $y = \ln(x^2)$ as they are for $y = \ln x$.

5 **a** $f^{-1} : x \mapsto \ln(x-2) - 3$

b **i** $x < -5.30$ **ii** $x < -7.61$ **iii** $x < -9.91$
iv $x < -12.2$ Conjecture HA is $y = 2$

c as $x \to \infty$, $f(x) \to \infty$,
as $x \to -\infty$, $e^{x+3} \to 0$ and $f(x) \to 2$
$\therefore$ HA is $y = 2$

d VA of f^{-1} is $x = 2$, Domain of f^{-1} is $\{x : x > 2\}$

6 **a** **i** $f(5) = 3$ **ii** $f(x^2) = \log_2(x^2 + 3)$
iii $f(2x-1) = 1 + \log_2(x+1)$

b Domain of $f(x)$ is $\{x : x > -3\}$ **c** $x = \pm 5$

7 **a** Range is $\{y : y > 1\}$ **b** $f^{-1}(x) = \frac{1}{3}\ln(x-1)$

c $f^{-1}(10) = \frac{1}{3}\ln 9$

d Domain of $f^{-1}(x)$ is $\{x : x > 1\}$

e $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$

8 **a** $f^{-1}(x) = \frac{1}{2}\ln x$

i $(f^{-1} \circ g)(x) = \frac{1}{2}\ln(2x-1)$

ii $(g \circ f)^{-1}(x) = \frac{1}{2}\ln\left(\dfrac{x+1}{2}\right)$

b $x = 13$

9 **a** $f(1) = \dfrac{10}{e}$, $g(6) = \ln 3$ **b** x-intercept of $g(x)$ is 4

c $fg(x) = \dfrac{10}{x-3}$ **d** $x = \ln 2$

10 **a** Domain of $f(x)$ is $\{x : x > -6\}$

b $f^{-1}(x) = e^x - 6$

c x-intercept is -5, y-intercept is $\ln 6$ **d** $x = -\frac{8}{3}$ or 3

REVIEW SET 5A

1 **a** 3 **b** 8 **c** -2 **d** $\frac{1}{2}$ **e** 0
f $\frac{1}{4}$ **g** -1 **h** $\frac{1}{2}$, $k > 0$

2 **a** $\frac{1}{2}$ **b** $-\frac{1}{3}$ **c** $a+b+1$

3 **a** $\ln 144$ **b** $\ln\left(\frac{3}{2}\right)$ **c** $\ln\left(\dfrac{25}{e}\right)$ **d** $\ln 3$

4 **a** $\frac{3}{2}$ **b** -3 **c** $2x$ **d** $1-x$

5 **a** $\lg 144$ **b** $\log_2\left(\frac{16}{9}\right)$ **c** $\log_4 80$

6 **a** $\lg P = \lg 3 + x\lg 7$ **b** $\lg m = 3\lg n - \lg 5$

7 **a** $x = 3$ **b** $x = 5$

8 **Hint:** Use change of base rule.

9 **a** $T = \dfrac{x^2}{5}$ **b** $K = 3 \times 2^x$

10 **a** $5\ln 2$ **b** $3\ln 5$ **c** $6\ln 3$

11

Function	$y = \log_2 x$	$y = \ln(x+5)$
Domain	$x > 0$	$x > -5$
Range	$y \in \mathbb{R}$	$y \in \mathbb{R}$

12 **a** $2A + 2B$ **b** $A + 3B$ **c** $3A + \frac{1}{2}B$
d $4B - 2A$ **e** $3A - 2B$

13 **a** $x = 0$ or $\ln\left(\frac{2}{3}\right)$ **b** $x = e^2$

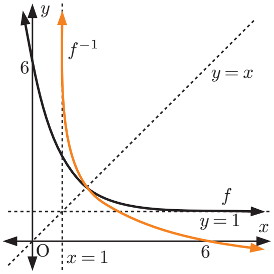
14 **a** $x \approx 2.46$ **b** $x \approx 1.88$ **15** ≈ 6.97 years

16 **a** Range of f is $\{y : y > 1\}$

b **i** $f^{-1}(x) = \ln\left(\dfrac{5}{x-1}\right)$ **ii** $f^{-1}(2) = \ln 5$

c Domain of f^{-1} is $\{x : x > 1\}$ **d** $x = 6$

e



REVIEW SET 5B

1 **a** $\frac{3}{2}$ **b** $\frac{2}{3}$ **c** $a + b$

2 **a** $\approx 10^{1.5051}$ **b** $\approx 10^{-2.8861}$ **c** $\approx 10^{-4.0475}$

3 **a** $x = \frac{1}{8}$ **b** $x \approx 82.7$ **c** $x \approx 0.0316$

4 **a** $k \approx 3.25 \times 2^x$ **b** $Q = 5P^3$ **c** $A = 6 \times 2^x$

5 **a** $x = \dfrac{\lg 7}{\lg 5}$ **b** $x = 2$

6 -1 **7** $\log_8 30 = \frac{1}{3}\log_2 30$

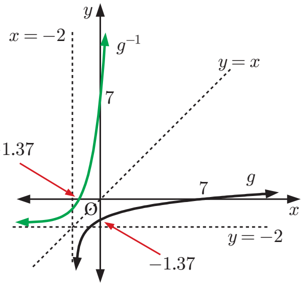
8 **a** $x = 8$ **b** $x = 3$ **9** **a** 9 **b** $\ln 5$

10 **a** $\ln 3$ **b** $\ln 4$ **c** $\ln 125$

11 **a** $\lg M = \lg 5 + x\lg 6$ **b** $\lg T = \lg 5 - \frac{1}{2}\lg l$
c $\lg G = \lg 4 - \lg c$

12 **a** $x = \ln 3$ **b** $x = \ln 3$ or $\ln 4$

13 **a** Domain is $\{x : x > -2\}$, Range is $\{y : y \in \mathbb{R}\}$
b VA is $x = -2$, x-intercept is 7, y-intercept is ≈ -1.37
c $g^{-1}(x) = 3^{x+2} - 2$
d



14 13.9 weeks

15 **a** $x = 5$ **b** $x = 32$ or $\frac{1}{32}$ **c** $x = 9$ or 81

16 **a** Domain is $\{x : x > 4\}$, Range is $\{y : y \in \mathbb{R}\}$
b x-intercept is 5, no y-intercept **c** $x = 4 + \sqrt{6}$
d $x = 0$

EXERCISE 6A.1

1 **a** $3x^2 + 6x + 9$ **b** $5x^2 + 7x + 9$ **c** $-7x^2 - 8x - 9$
d $4x^4 + 13x^3 + 28x^2 + 27x + 18$

2 **a** $x^3 + x^2 - 4x + 7$ **b** $x^3 - x^2 - 2x + 3$
c $3x^3 + 2x^2 - 11x + 19$ **d** $2x^3 - x^2 - x + 5$
e $x^5 - x^4 - x^3 + 8x^2 - 11x + 10$
f $x^4 - 2x^3 + 5x^2 - 4x + 4$

3 **a** $2x^3 - 3x^2 + 4x + 3$ **b** $x^4 + x^3 - 7x^2 + 7x - 2$
c $x^3 + 6x^2 + 12x + 8$ **d** $4x^4 - 4x^3 + 13x^2 - 6x + 9$
e $16x^4 - 32x^3 + 24x^2 - 8x + 1$
f $18x^4 - 87x^3 + 56x^2 + 20x - 16$

4 **a** $6x^3 - 11x^2 + 18x - 5$ **b** $8x^3 + 18x^2 - x + 10$
c $-2x^3 + 7x^2 + 13x + 10$ **d** $2x^3 - 7x^2 + 4x + 4$
e $2x^4 - 2x^3 - 9x^2 + 11x - 2$
f $15x^4 + x^3 - x^2 + 7x - 6$
g $x^4 - 2x^3 + 7x^2 - 6x + 9$
h $4x^4 + 4x^3 - 15x^2 - 8x + 16$
i $8x^3 + 60x^2 + 150x + 125$
j $x^6 + 2x^5 + x^4 - 4x^3 - 4x^2 + 4$

EXERCISE 6A.2

1 **a** $Q(x) = x$, $R = -3$, $x^2 + 2x - 3 = x(x+2) - 3$
b $Q(x) = x - 4$, $R = -3$,
$x^2 - 5x + 1 = (x-4)(x-1) - 3$
c $Q(x) = 2x^2 + 10x + 16$, $R = 35$,
$2x^3 + 6x^2 - 4x + 3 = (2x^2 + 10x + 16)(x-2) + 35$

2 **a** $x^2 - 3x + 6 = (x+1)(x-4) + 10$
b $x^2 + 4x - 11 = (x+1)(x+3) - 14$
c $2x^2 - 7x + 2 = (2x-3)(x-2) - 4$
d $2x^3 + 3x^2 - 3x - 2 = (x^2 + x - 2)(2x+1)$
e $3x^3 + 11x^2 + 8x + 7 = (x^2 + 4x + 4)(3x-1) + 11$
f $2x^4 - x^3 - x^2 + 7x + 4$
$= (x^3 - 2x^2 + \frac{5}{2}x - \frac{1}{4})(2x+3) + \frac{19}{4}$

3 **a** $x + 2 + \dfrac{9}{x-2}$ **b** $2x + 1 - \dfrac{1}{x+1}$
c $3x - 4 + \dfrac{3}{x+2}$ **d** $x^2 + 3x - 2$
e $2x^2 - 8x + 31 - \dfrac{124}{x+4}$ **f** $x^2 + 3x + 6 + \dfrac{7}{x-2}$

EXERCISE 6A.3

1 **a** quotient is $x+1$, remainder is $-x-4$

b quotient is 3, remainder is $-x+3$

c quotient is $3x$, remainder is $-2x-1$

d quotient is 0, remainder is $x-4$

2 **a** $1-\dfrac{2x}{x^2+x+1}$, $x^2-x+1=1(x^2+x+1)-2x$

b $x-\dfrac{2x}{x^2+2}$, $x^3=x(x^2+2)-2x$

c $x^2+x+3+\dfrac{3x-4}{x^2-x+1}$,

$x^4+3x^2+x-1=(x^2+x+3)(x^2-x+1)+3x-4$

d $2x+4+\dfrac{5x+2}{(x-1)^2}$,

$2x^3-x+6=(2x+4)(x-1)^2+5x+2$

e $x^2-2x+3-\dfrac{4x+3}{(x+1)^2}$,

$x^4=(x^2-2x+3)(x+1)^2-4x-3$

f $x^2-3x+5+\dfrac{15-10x}{(x-1)(x+2)}$,

$x^4-2x^3+x+5=(x^2-3x+5)(x-1)(x+2)+15-10x$

3 quotient is x^2+2x+3, remainder is 7

4 quotient is x^2-3x+5, remainder is $15-10x$

EXERCISE 6B.1

1 **a** $4, -\frac{3}{2}$ **b** $-3\pm\sqrt{10}$ **c** $5\pm\sqrt{19}$

d $0, \pm 2$ **e** $0, \pm\sqrt{11}$ **f** $\pm 2, \pm\sqrt{2}$

2 **a** $1, -\frac{2}{5}$ **b** $-\frac{1}{2}, \pm\sqrt{3}$ **c** $-3, \frac{1}{3}, 2$

d $0, 1\pm\sqrt{3}$ **e** $0, \pm\sqrt{7}$ **f** $\pm\sqrt{2}, \pm\sqrt{5}$

3 **a** $(2x+3)(x-5)$ **b** $x(x-7)(x-4)$

c $(x-3-\sqrt{6})(x-3+\sqrt{6})$

d $x(x+1+\sqrt{5})(x+1-\sqrt{5})$ **e** $x(3x-2)(2x+1)$

f $(x+1)(x-1)(x+\sqrt{5})(x-\sqrt{5})$

4 $P(\alpha)=0$, $P(\beta)=0$, $P(\gamma)=0$

5 **a** $P(x)=a(x+3)(x-4)(x-5)$, $a\neq 0$

b $P(x)=a(x+2)(x-2)(x-3)$, $a\neq 0$

c $P(x)=a(x-3)(x^2-2x-4)$, $a\neq 0$

d $P(x)=a(x+1)(x^2+4x+2)$, $a\neq 0$

6 **a** $P(x)=a(x^2-1)(x^2-2)$, $a\neq 0$

b $P(x)=a(x-2)(5x+1)(x^2-3)$, $a\neq 0$

c $P(x)=a(x+3)(4x-1)(x^2-2x-1)$, $a\neq 0$

d $P(x)=a(x^2-4x-1)(x^2+4x-3)$, $a\neq 0$

EXERCISE 6B.2

1 **a** $a=2$, $b=5$, $c=5$ **b** $a=3$, $b=4$, $c=3$

c $a=2$, $b=-5$, $c=4$

2 **a** $a=2$, $b=-2$ or $a=-2$, $b=2$

b $a=3$, $b=-1$

3 **a** $a=1$, $b=6$, $c=-7$ **b** $(x+3)(x+7)(x-1)$

4 **a** $p=2$, $q=7$, $r=5$ **b** $x=\frac{1}{2}, -1, -\frac{5}{2}$

5 **a** $a=3$, $b=-2$, $c=1$

b $3x^3+10x^2-7x+4=(x+4)(3x^2-2x+1)$

Δ of $3x^2-2x+1$ is -8,

$\therefore$ the only real zero is -4.

6 **a** $a=1$, $b=-2$, $c=-1$, $k=-4$

b $-\frac{2}{3}, 1\pm\sqrt{2}$

7 **a** $a=-2$, $b=2$ **b** $-1\pm\sqrt{3}$

8 $a=-11$, zeros are $\frac{3}{2}$, $\dfrac{-3\pm\sqrt{13}}{2}$

9 **a** $a=-9$, $b=-1$

b $P(x)=0$ when $x=-1, -\frac{1}{2}, 2, 4$

10 **Hint:** Let $x^3+3x^2-9x+c=(x+a)^2(x+b)$

When $c=5$, the cubic is $(x-1)^2(x+5)$.

When $c=-27$, the cubic is $(x+3)^2(x-3)$.

EXERCISE 6C

1 **a** $P(x)=Q(x)(x-2)+7$, $P(x)$ divided by $x-2$ leaves a remainder of 7.

b $P(-3)=-8$, $P(x)$ divided by $x+3$ leaves a remainder of -8.

c $P(5)=11$, $P(x)=Q(x)(x-5)+11$

2 **a** 4 **b** -19 **c** 1 **3** 4

4 **a** $a=3$ **b** $a=2$ **5** $a=-5$, $b=6$

6 $a=-5$, $b=6$ **7** -7

8 **a** $P(x)=Q(x)(2x-1)+R$

$$\begin{aligned}P(\tfrac{1}{2})&=Q(\tfrac{1}{2})(2\times\tfrac{1}{2}-1)+R\\&=Q(\tfrac{1}{2})\times 0+R\\&=R\end{aligned}$$

b **i** -3 **ii** 7 **iii** -7

9 $a=3$, $b=10$ **10** **a** -3 **b** 1

EXERCISE 6D

1 **a** factor **b** not a factor **c** factor **d** not a factor

2 **a** $c=2$ **b** $c=-2$ **c** $b=3$

3 $k=-8$, $P(x)=(x+2)(x-2)(2x+1)$

4 **a** $k=-8$ **b** $P(x)=(x-3)(3x^2+x-2)$

c $x=-1, \frac{2}{3}, 3$

5 $a=7$, $b=-14$ **6** $a=3$, $b=2$

7 **a** $a=7$, $b=-6$ **b** 60

c $P(x)=(x+3)(2x^2+3x-2)$ **d** $-3, -2, \frac{1}{2}$

8 **a** $a=7$, $b=2$ **b** $x=-2\pm\sqrt{6}$

9 **a** **i** $P(a)=0$, $x-a$ is a factor

ii $(x-a)(x^2+ax+a^2)$

b **i** $P(-a)=0$, $x+a$ is a factor

ii $(x+a)(x^2-ax+a^2)$

10 $a=2$

EXERCISE 6E

1 **a** $x=1, 2, 3$ **b** $x=-1, 2$ {2 is a double root}

c $x=1, -1, -2$ **d** $x=-1, 3, 4$ **e** $x=-5, -4, 4$

f $x=-3, -5$ {-5 is a double root}

2 **a** $x=-2, 2, 3$ **b** $x=-3, -2, 6$ **c** $x=-3, 4, 7$

REVIEW SET 6A

1 **a** $8x^2+6x+3$ **b** $7x^2-9x+9$

c $15x^4+32x^3+29x-4$

2 **a** quotient $=2x+5$, remainder $=3$

b quotient $=x^2-4x+2$, remainder $=-5$

3 **a** $\frac{4}{3}, -2$ **b** $-4\pm\sqrt{5}$

4 **a** $a = 1,\ b = -2,\ c = 3$
b Δ of $x^2 - 2x + 3$ is -8
$\therefore$ the only real root is $x = -3$.

5 **a** 1 **b** -53 **6** **a** not a factor **b** factor

7 $k = 6$ **8** $a = 4,\ b = -1$ **9** $c = 3$

10 **a** $a = -19,\ b = -20$ **b** $-5, -1, 4$

11 $x = -3, -1, 5$

REVIEW SET 6B

1 **a** $12x^4 - 9x^3 + 8x^2 - 26x + 15$
b $4x^4 - 4x^3 + 13x^2 - 6x + 9$

2 **a** $x^2 - 2x + 4 - \dfrac{8}{x+2}$ **b** $x - 5 + \dfrac{19x + 30}{(x+2)(x+3)}$

3 $P(x) = a(4x - 1)(x^2 - 2x - 4),\quad a \neq 0$

4 For $k = 3,\ b = 27,\ x = 3$ or -3.
For $k = -1,\ b = -5,\ x = -1$ or 5.

5 **a** -3 **b** -7 **6** **a** $a = 5$ **b** -12

7 **b** $(x - 2)(x^2 + 2x - 9)$ **c** $2, -1 \pm \sqrt{10}$

8 $a = \frac{8}{7},\ b = \frac{174}{7}$

9 $k = 8$, the zeros are $-1, -2$ {-2 is a double root}

10 **a** $a = -20,\ b = 12$ **b** $f(x) = (2x - 1)(x - 6)(x + 2)$

11 $x = -4, 2, 3$

EXERCISE 7A.1

1 **a** gradient $= 3$, y-intercept is 5
b gradient $= 4$, y-intercept is -2
c gradient $= \frac{1}{5}$, y-intercept is $\frac{3}{5}$
d gradient $= -7$, y-intercept is -3
e gradient $= \frac{1}{6}$, y-intercept is $\frac{1}{3}$
f gradient $= -\frac{5}{3}$, y-intercept is $\frac{8}{3}$

2 **a** $y = x - 2$ **b** $y = -x + 4$ **c** $y = 2x$
d $y = -\frac{1}{2}x + 3$

3 **a** $y = 4x - 13$ **b** $y = -3x - 5$ **c** $y = -5x + 32$
d $y = \frac{1}{2}x + \frac{7}{2}$ **e** $y = -\frac{1}{3}x + \frac{8}{3}$ **f** $y = 6$

4 **a** $2x - 3y = -11$ **b** $3x - 5y = -23$ **c** $x + 3y = 5$
d $2x + 7y = -2$ **e** $4x - y = -11$ **f** $2x + y = 7$
g $7x + 2y = 18$ **h** $6x - y = -40$

5 **a** $y = \frac{5}{2}x - 2$ **b** $y = -2x + 3$ **c** $y = -2$
d $y = -\frac{1}{5}x + \frac{2}{5}$ **e** $y = \frac{1}{6}x - \frac{11}{6}$ **f** $y = -\frac{2}{3}x - \frac{11}{3}$

6 **a** $x - 3y = -3$ **b** $5x - y = 1$ **c** $x - y = 3$
d $4x - 5y = 10$ **e** $x - 2y = -1$ **f** $2x + 3y = -5$

7 **a** $\sqrt{45}$ units **b** $(-1, \frac{7}{2})$ **c** $\frac{1}{2}$ **d** $y = \frac{1}{2}x + 4$

8 **a** $y = \frac{4}{3}x - 1$ **b** $2x - 3y = -13$ **c** $y = x + 1$
d $2x + y = -2$ **e** $y = -\frac{2}{3}x + 2$ **f** $3x + 7y = -9$

9 **a** $M = \frac{1}{3}p + 2$ **b** $R = -\frac{5}{4}n + 2$ **c** $T = \frac{1}{2}x - 1$
d $F = \frac{1}{10}x + 1$ **e** $H = -\frac{1}{2}z + 2$ **f** $W = -\frac{1}{6}t - 2$

10 **a** $x + 2y = 13$ **b** $(13, 0)$

11 **a** $3x + 5y = 10$ **b** $(0, 2)$ **12** 54 units^2

EXERCISE 7A.2

1 **a** $\sqrt{160}$ units **b** $(-1, 1)$ **c** -3 **d** $x - 3y = -4$

2 **a** $y = x - 4$ **b** $y = 2x + 6$ **c** $y = \frac{6}{5}x + \frac{7}{2}$ **d** $y = 1$

3 15 units^2

EXERCISE 7B

1 **a** $(1, 3)$ **b** $(6, -3)$ **c** $(-5, 3)$ **d** $(-1, -2)$

2 **a** $3x + 5y = 9$ **b** $(-2, 3)$ **3** $(4, 2)$

4 **a** $x - 3y = -8$ **b** $y = -3x - 4$ **c** $(-2, 2)$

5 **a** $(0, -1)$ **b** 25 units^2

6 **a** $(-1, 0)$ **b** 26 units^2 **7** 30 units^2

8 **a** **i** $(5, 0)$ **ii** $(7, -4)$ **iii** $(6, -2)$
b **Hint:** Find the gradients of MN and AC.
c **i** 15 units^2 **ii** 20 units^2

EXERCISE 7C

1 $(-1, -2)$ and $(\frac{11}{5}, -\frac{2}{5})$ **2** $\sqrt{18}$ units

3 $x - 2y = 0$ **4** $(-\frac{4}{3}, -\frac{8}{3})$ and $(2, -1)$

5 $\sqrt{125}$ units **6** $x - 3y = -13$

7 $(3, -\frac{3}{2})$ and $(4, -1)$ **8** $(\frac{7}{3}, \frac{5}{2})$

EXERCISE 7D

1 **a** $y = \frac{1}{2}x^3 + 2$ **b** $y = 3\sqrt{x} - 1,\quad x \geqslant 0$
c $y = 3 - x^4$ **d** $y = \frac{1}{3} \times 2^x$
e $y = \dfrac{2}{x} + 1$ **f** $y = -\frac{3}{2} \times 3^x + 11$

2 **a** **i** $y = x^2 + 3x$ **ii** $y = 18$
b **i** $y = -\frac{1}{2}\sqrt{x} + \dfrac{10}{\sqrt{x}},\quad x > 0$ **ii** $y = \frac{17\sqrt{3}}{6}$
c **i** $y = \dfrac{5}{3x} \times 2^x$ **ii** $y = \frac{40}{9}$
d **i** $y = 2x^3 - 9x$ **ii** $y = 27$
e **i** $y = \dfrac{1}{x^2} - \dfrac{12}{x} + 36$ **ii** $y = 32\frac{1}{9}$
f **i** $y = (x + 2)^2 + 3$ **ii** $y = 28$

3 **a** $\lg y = 2x - 1$ **b** $y = \frac{1}{10} \times 10^{2x}$

4 $y = 1000 \times 10^{-\frac{3}{2}x}$

5 **a** $y = \frac{1}{10\,000} \times 10^x$ **b** $y = 10\,000 \times (\frac{1}{10})^x$
c $y = 5 \times 4^x$

6 **a** $y = 10 \times 10^{\frac{1}{3}x}$ **b** $y = 1000$

7 **a** $\lg y = -\frac{1}{2}\lg x + 2$ **b** $y = \dfrac{100}{\sqrt{x}}$

8 **a** $y = x^{\frac{1}{4}}$ **b** $y = \dfrac{1000}{x}$ **c** $y = x^2\sqrt{1000}$

9 **a** $K = 7\sqrt{t}$ **b** $K = 21$ **10** **a** 3 **b** $\lg 4$

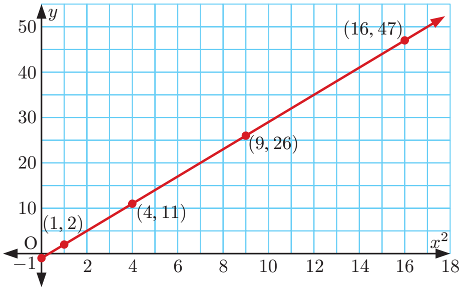
EXERCISE 7E

1 **a**

x^2	1	4	9	16
y	2	11	26	47

c $y = 3x^2 - 1$

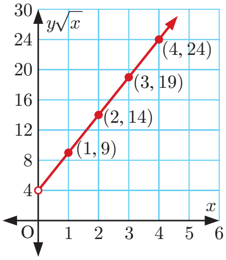
b



2 **a**

x	1	2	3	4
$y\sqrt{x}$	9	14	19	24

b



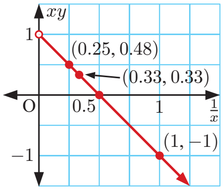
c $y = 5\sqrt{x} + \dfrac{4}{\sqrt{x}}, \quad x > 0$

d $y = 21$

3 **a**

$\dfrac{1}{x}$	1	0.5	0.33	0.25
xy	-1	0	0.33	0.48

b



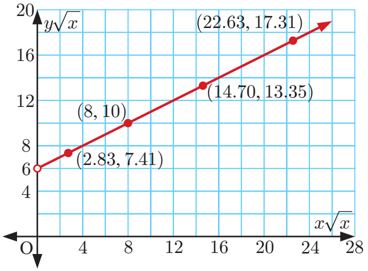
c $a = 1, \quad b = -2$

d $y = 0.08$

4 **a**

$x\sqrt{x}$	2.83	8	14.70	22.63
$y\sqrt{x}$	7.41	10	13.35	17.31

b



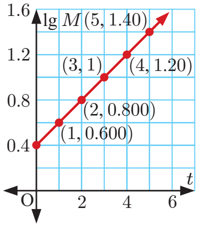
c $y = \frac{1}{2}x + \dfrac{6}{\sqrt{x}}, \quad x > 0$

d $y = 6.5$

5 **a**

t	1	2	3	4	5
$\lg M$	0.600	0.800	1	1.20	1.40

b

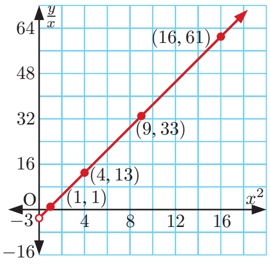


c $M \approx 2.51 \times 1.58^t$

d ≈ 2.51 g

6 **a** x^2

b



c $a = 4$,
$b = -3$

d $y = 485$

7 Plot xy against $\sqrt{x}$. $\quad y = \dfrac{8}{x} - \dfrac{4}{\sqrt{x}} \quad \{a = 8, \ b = -4\}$

8 **a** $a \approx 4.90, \ b \approx 2.00$ **b** ≈ 44.1 m **c** ≈ 4.04 seconds

REVIEW SET 7A

1 **a** $\sqrt{40}$ units **b** $(2, 5)$ **c** $x + 3y = 17$

2 $y = -2x + 6$

3 The gradient of a vertical line is undefined.

4 **a** $x + 2y = 7$ **b** $(7, 0)$ **5** $(3, -1)$

6 **a** $(-1, 4)$ **b** $32\frac{1}{2}$ units^2

7 $(-\frac{7}{5}, \frac{26}{5})$ and $(2, -5)$ **8** $y = x - 5$

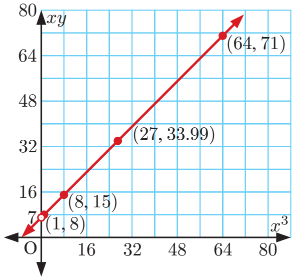
10 **a** $y = \dfrac{3}{\sqrt{x}} - \dfrac{2}{x}, \quad x > 0$ **b** $y = 1$

11 **a** $\lg y = \frac{1}{2}\lg x + 1$ **b** $y = 10\sqrt{x}$

12 **a**

x^3	1	8	27	64
xy	8	15	33.99	71

b



c $y = x^2 + \dfrac{7}{x}$

d $y = 50$

REVIEW SET 7B

1 $y = -3x + 7$ **2** $\sqrt{80}$ units

3 **a** $y = 5x - \dfrac{2}{x}$ **b** $39\frac{3}{4}$

4 **a** $r = \frac{5}{7}a + 2$ **b** $K = \frac{3}{5}s + 3$

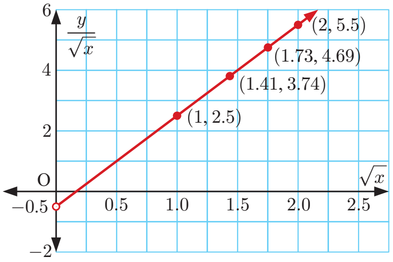
5 $(3, 2)$ **6** $5x - 8y = 31$

7 **a**

$\sqrt{x}$	1	1.41	1.73	2
$\dfrac{y}{\sqrt{x}}$	2.5	3.74	4.69	5.5

c $y = 3x - \dfrac{\sqrt{x}}{2}$

b



8 **a** $bx + ay = ab$

b **Hint:** $\cos\theta = \dfrac{a}{\sqrt{a^2 + b^2}}, \quad \sin\theta = \dfrac{b}{\sqrt{a^2 + b^2}}$

9 **a** **i** $(2, 12)$ **ii** $(11, 0)$ **b** 75 units^2

10 $(\frac{7}{8}, \frac{1}{2})$

11 **a** Plot $\lg y$ against x.

$y = 100 \times (10^{-\frac{1}{3}})^x \quad \{a = 100, \ b = 10^{-\frac{1}{3}}\}$

b $y \approx 46.4$

EXERCISE 8A

1 **a** $\frac{\pi}{2}^c$ **b** $\frac{\pi}{3}^c$ **c** $\frac{\pi}{6}^c$ **d** $\frac{\pi}{10}^c$ **e** $\frac{\pi}{20}^c$
f $\frac{3\pi}{4}^c$ **g** $\frac{5\pi}{4}^c$ **h** $\frac{3\pi}{2}^c$ **i** $2\pi^c$ **j** $4\pi^c$
k $\frac{7\pi}{4}^c$ **l** $3\pi^c$ **m** $\frac{\pi}{5}^c$ **n** $\frac{4\pi}{9}^c$ **o** $\frac{23\pi}{18}^c$

2 **a** 0.641^c **b** 2.39^c **c** 5.55^c **d** 3.83^c **e** 6.92^c

3 **a** 36° **b** 108° **c** 135° **d** 10° **e** 20°
f 140° **g** 18° **h** 27° **i** 210° **j** 22.5°

4 **a** 114.59° **b** 87.66° **c** 49.68° **d** 182.14°
e 301.78°

5 **a**

Degrees	0	45	90	135	180	225	270	315	360
Radians	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π

b

Deg.	0	30	60	90	120	150	180	210	240	270	300	330	360
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π

EXERCISE 8B

1 **a** 49.5 cm, 223 cm^2 **b** 23.0 cm, 56.8 cm^2

2 **a** 3.14 m **b** 9.30 m^2 **3** **a** 5.91 cm **b** 18.9 cm

4 **a** 0.686^c **b** 0.6^c

5 **a** $\theta = 0.75^c$, area $= 24\ \text{cm}^2$
b $\theta = 1.68^c$, area $= 21\ \text{cm}^2$
c $\theta \approx 2.32^c$, area $= 126.8\ \text{cm}^2$

6 10 cm, 25 cm^2

8 **a** 11.7 cm **b** $r \approx 11.7$ **c** 37.7 cm **d** 3.23^c

9 **a** $\alpha \approx 18.43$ **b** $\theta \approx 143.1$ **c** 387 m^2

10 25.9 cm **11** **b** 2 h 49 min **12** 227 m^2

EXERCISE 8C

1 **a** **i** A(cos 26°, sin 26°), B(cos 146°, sin 146°), C(cos 199°, sin 199°)
ii A(0.899, 0.438), B(−0.829, 0.559), C(−0.946, −0.326)
b **i** A(cos 123°, sin 123°), B(cos 251°, sin 251°), C(cos(−35°), sin(−35°))
ii A(−0.545, 0.839), B(−0.326, −0.946), C(0.819, −0.574)

2

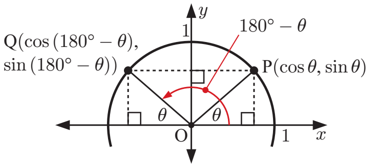
θ (degrees)	0°	90°	180°	270°	360°	450°
θ (radians)	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π	$\frac{5\pi}{2}$
sine	0	1	0	−1	0	1
cosine	1	0	−1	0	1	0
tangent	0	undef	0	undef	0	undef

3 **a** **i** $\frac{1}{\sqrt{2}} \approx 0.707$ **ii** $\frac{\sqrt{3}}{2} \approx 0.866$

b

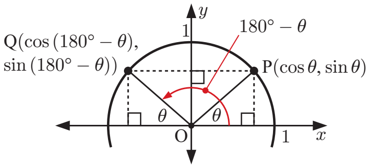
θ (degrees)	30°	45°	60°	135°	150°	240°	315°
θ (radians)	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{4\pi}{3}$	$\frac{7\pi}{4}$
sine	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}$
cosine	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
tangent	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	−1	$-\frac{1}{\sqrt{3}}$	$\sqrt{3}$	−1

4 **a** **i** 0.985 **ii** 0.985 **iii** 0.866 **iv** 0.866
v 0.5 **vi** 0.5 **vii** 0.707 **viii** 0.707
b $\sin(180° - \theta) = \sin\theta$
c $\sin\theta$ and $\sin(180° - \theta)$ have the same value, as P and Q have the same y-coordinate.



d **i** 135° **ii** 129° **iii** $\frac{2\pi}{3}$ **iv** $\frac{5\pi}{6}$

5 **a** **i** 0.342 **ii** −0.342 **iii** 0.5 **iv** −0.5
v 0.906 **vi** −0.906 **vii** 0.174 **viii** −0.174
b $\cos(180° - \theta) = -\cos\theta$
c $\cos(180° - \theta) = -\cos\theta$, as the x-coordinates of P and Q are negatives of each other.



d **i** 140° **ii** 161° **iii** $\frac{4\pi}{5}$ **iv** $\frac{3\pi}{5}$

6 **a** ≈ 0.6820 **b** ≈ 0.8572 **c** ≈ -0.7986
d ≈ 0.9135 **e** ≈ 0.9063 **f** ≈ -0.6691

7 **a**

Quadrant	*Degree measure*	*Radian measure*	$\cos\theta$	$\sin\theta$	$\tan\theta$
1	$0° < \theta < 90°$	$0 < \theta < \frac{\pi}{2}$	+ve	+ve	+ve
2	$90° < \theta < 180°$	$\frac{\pi}{2} < \theta < \pi$	−ve	+ve	−ve
3	$180° < \theta < 270°$	$\pi < \theta < \frac{3\pi}{2}$	−ve	−ve	+ve
4	$270° < \theta < 360°$	$\frac{3\pi}{2} < \theta < 2\pi$	+ve	−ve	−ve

b **i** 1 and 4 **ii** 2 and 3 **iii** 3 **iv** 2

8 **a** $\mathrm{A}\widehat{\mathrm{O}}\mathrm{Q} = 180° - \theta$ or $\pi - \theta$ radians
b [OQ] is a reflection of [OP] in the y-axis and so Q has coordinates $(-\cos\theta, \sin\theta)$.
c $\cos(180° - \theta) = -\cos\theta$, $\sin(180° - \theta) = \sin\theta$

9 **a**

θ^c	$\sin\theta$	$\sin(-\theta)$	$\cos\theta$	$\cos(-\theta)$
0.75	0.682	−0.682	0.732	0.732
1.772	0.980	−0.980	−0.200	−0.200
3.414	−0.269	0.269	−0.963	−0.963
6.25	−0.0332	0.0332	0.999	0.999
−1.17	−0.921	0.921	0.390	0.390

b $\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$

EXERCISE 8D.1

1 **a** $\cos\theta = \pm\frac{\sqrt{3}}{2}$ **b** $\cos\theta = \pm\frac{2\sqrt{2}}{3}$ **c** $\cos\theta = \pm 1$
d $\cos\theta = 0$

2 **a** $\sin\theta = \pm\frac{3}{5}$ **b** $\sin\theta = \pm\frac{\sqrt{7}}{4}$ **c** $\sin\theta = 0$
d $\sin\theta = \pm 1$

3 **a** $\sin\theta = \frac{\sqrt{5}}{3}$ **b** $\cos\theta = -\frac{\sqrt{21}}{5}$ **c** $\cos\theta = \frac{4}{5}$
d $\sin\theta = -\frac{12}{13}$

4 **a** $\tan\theta = -\frac{1}{2\sqrt{2}}$ **b** $\tan\theta = -2\sqrt{6}$ **c** $\tan\theta = \frac{1}{\sqrt{2}}$
d $\tan\theta = -\frac{\sqrt{7}}{3}$

5 **a** $\sin x = \frac{2}{\sqrt{13}}$, $\cos x = \frac{3}{\sqrt{13}}$
b $\sin x = \frac{4}{5}$, $\cos x = -\frac{3}{5}$
c $\sin x = -\sqrt{\frac{5}{14}}$, $\cos x = -\frac{3}{\sqrt{14}}$
d $\sin x = -\frac{12}{13}$, $\cos x = \frac{5}{13}$

6 $\sin\theta = \dfrac{-k}{\sqrt{k^2+1}}$, $\cos\theta = \dfrac{-1}{\sqrt{k^2+1}}$

EXERCISE 8D.2

1 **a** $\theta \approx 1.33$ or 4.47 **b** $\theta \approx 0.592$ or 5.69
c $\theta \approx 0.644$ or 2.50 **d** $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$
e $\theta \approx 0.876$ or 4.02 **f** $\theta \approx 0.674$ or 5.61
g $\theta \approx 0.0910$ or 3.05 **h** $\theta \approx 1.52$ or 4.66
i $\theta \approx 1.35$ or 1.79

2 **a** $\theta \approx 1.82$ or 4.46 **b** $\theta = 0, \pi$, or 2π
c $\theta \approx 1.88$ or 5.02 **d** $\theta \approx 3.58$ or 5.85
e $\theta \approx 1.72$ or 4.86 **f** $\theta \approx 1.69$ or 4.59
g $\theta \approx 1.99$ or 5.13 **h** $\theta \approx 2.19$ or 4.10
i $\theta \approx 3.83$ or 5.60

EXERCISE 8E

1

	a	b	c	d	e
$\sin\theta$	$\frac{1}{\sqrt{2}}$	1	$-\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$
$\cos\theta$	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$
$\tan\theta$	1	undef	-1	0	1

2

	a	b	c	d	e
$\sin\beta$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\cos\beta$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\tan\beta$	$\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$

3 **a** $\cos 120° = -\frac{1}{2}$, $\sin 120° = \frac{\sqrt{3}}{2}$, $\tan 120° = -\sqrt{3}$
b $\cos(-45°) = \frac{1}{\sqrt{2}}$, $\sin(-45°) = -\frac{1}{\sqrt{2}}$, $\tan(-45°) = -1$

4 **a** $\cos 270° = 0$, $\sin 270° = -1$
b $\tan 270°$ is undefined

5 **a** $\frac{3}{4}$ **b** $\frac{1}{4}$ **c** 3 **d** $\frac{1}{4}$ **e** $-\frac{1}{4}$ **f** 1
g $\sqrt{2}$ **h** $\frac{1}{2}$ **i** $\frac{1}{2}$ **j** 2 **k** -1 **l** $-\sqrt{3}$

6 **a** $30°, 150°$ **b** $60°, 120°$ **c** $45°, 315°$
d $120°, 240°$ **e** $135°, 225°$ **f** $240°, 300°$

7 **a** $\frac{\pi}{4}, \frac{5\pi}{4}$ **b** $\frac{3\pi}{4}, \frac{7\pi}{4}$ **c** $\frac{\pi}{3}, \frac{4\pi}{3}$
d $0, \pi, 2\pi$ **e** $\frac{\pi}{6}, \frac{7\pi}{6}$ **f** $\frac{2\pi}{3}, \frac{5\pi}{3}$

8 **a** $\frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$ **b** $\frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$ **c** $\frac{3\pi}{2}, \frac{7\pi}{2}$

9 **a** $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$ **b** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$ **c** $\theta = \pi$
d $\theta = \frac{\pi}{2}$ **e** $\theta = \frac{3\pi}{4}, \frac{5\pi}{4}$ **f** $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$
g $\theta = 0, \pi, 2\pi$ **h** $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
i $\theta = \frac{5\pi}{6}, \frac{11\pi}{6}$ **j** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

10 **a** $\theta = k\pi,\ k \in \mathbb{Z}$ **b** $\theta = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$

EXERCISE 8F

1 **a** $\frac{2}{\sqrt{3}}$ **b** $-\frac{1}{\sqrt{3}}$ **c** $-\frac{2}{\sqrt{3}}$ **d** undefined
e $-\frac{2}{\sqrt{3}}$ **f** $\sqrt{2}$

2 **a** $\operatorname{cosec} x = \frac{5}{3}$, $\sec x = \frac{5}{4}$, $\cot x = \frac{4}{3}$
b $\operatorname{cosec} x = -\frac{3}{\sqrt{5}}$, $\sec x = \frac{3}{2}$, $\cot x = -\frac{2}{\sqrt{5}}$

3 **a** $\sin\theta = -\frac{\sqrt{7}}{4}$, $\tan\theta = -\frac{\sqrt{7}}{3}$, $\operatorname{cosec}\theta = -\frac{4}{\sqrt{7}}$,
$\sec\theta = \frac{4}{3}$, $\cot\theta = -\frac{3}{\sqrt{7}}$
b $\cos x = -\frac{\sqrt{5}}{3}$, $\tan x = \frac{2}{\sqrt{5}}$, $\operatorname{cosec} x = -\frac{3}{2}$,
$\sec x = -\frac{3}{\sqrt{5}}$, $\cot x = \frac{\sqrt{5}}{2}$
c $\sin x = \frac{\sqrt{21}}{5}$, $\cos x = \frac{2}{5}$, $\tan x = \frac{\sqrt{21}}{2}$,
$\operatorname{cosec} x = \frac{5}{\sqrt{21}}$, $\cot x = \frac{2}{\sqrt{21}}$
d $\sin\theta = \frac{1}{2}$, $\cos\theta = -\frac{\sqrt{3}}{2}$, $\tan\theta = -\frac{1}{\sqrt{3}}$,
$\sec\theta = -\frac{2}{\sqrt{3}}$, $\cot\theta = -\sqrt{3}$
e $\sin\beta = -\frac{1}{\sqrt{5}}$, $\cos\beta = -\frac{2}{\sqrt{5}}$, $\operatorname{cosec}\beta = -\sqrt{5}$,
$\sec\beta = -\frac{\sqrt{5}}{2}$, $\cot\beta = 2$
f $\sin\theta = -\frac{3}{5}$, $\cos\theta = -\frac{4}{5}$, $\tan\theta = \frac{3}{4}$,
$\operatorname{cosec}\theta = -\frac{5}{3}$, $\sec\theta = -\frac{5}{4}$

4 **a** $\theta = k\pi,\ k \in \mathbb{Z}$ **b** $\theta = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$
c $\theta = \frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}$ **d** $\theta = k\pi,\ k \in \mathbb{Z}$

REVIEW SET 8A

1 **a** $\frac{2\pi}{3}$ **b** $\frac{5\pi}{4}$ **c** $\frac{5\pi}{6}$ **d** 3π

2 **a** $\frac{\pi}{3}$ **b** $15°$ **c** $84°$

3 **a** 0.358 **b** -0.035 **c** 0.259 **d** -0.731

4 111 cm^2 **5** $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

6

	a	b	c	d
$\sin\theta$	0	$\frac{\sqrt{3}}{2}$	0	$\frac{\sqrt{3}}{2}$
$\cos\theta$	1	$-\frac{1}{2}$	-1	$-\frac{1}{2}$
$\tan\theta$	0	$-\sqrt{3}$	0	$-\sqrt{3}$

7 $\sin\theta = \pm\frac{\sqrt{7}}{4}$ **8** **a** $\frac{\sqrt{3}}{2}$ **b** 0 **c** $\frac{1}{2}$

9 **a** $\frac{2}{\sqrt{13}}$ **b** $-\frac{3}{\sqrt{13}}$

10 perimeter $= 12$ units, area $= 8\text{ units}^2$ **11** $\frac{\sqrt{6}}{\sqrt{11}}$

12 **a** $150°, 210°$ **b** $45°, 315°$ **c** $120°, 300°$

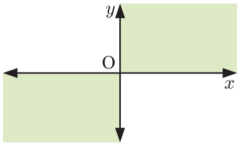
13 **a** $\theta = \pi$ **b** $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

14 $\cos x = -\frac{\sqrt{15}}{4}$, $\tan x = \frac{1}{\sqrt{15}}$, $\sec x = -\frac{4}{\sqrt{15}}$,
$\operatorname{cosec} x = -4$, $\cot x = \sqrt{15}$

REVIEW SET 8B

1 **a** 72° **b** 225° **c** 140° **d** 330°

2



3 **a** $\cos(\frac{3\pi}{2}) = 0$, $\sin(\frac{3\pi}{2}) = -1$

b $\cos(-\frac{\pi}{2}) = 0$, $\sin(-\frac{\pi}{2}) = -1$

4 **a** $\sin(\pi - p) = m$ **b** $\sin(p + 2\pi) = m$

c $\cos p = \sqrt{1 - m^2}$ **d** $\tan p = \dfrac{m}{\sqrt{1 - m^2}}$

5 **a** **i** 60° **ii** $\frac{\pi}{3}$ **b** $\frac{\pi}{3}$ units **c** $\frac{\pi}{6}$ units^2

7 $\sin\theta = \frac{\sqrt{21}}{5}$, $\tan\theta = -\frac{\sqrt{21}}{2}$, $\sec\theta = -\frac{5}{2}$,

$\text{cosec}\,\theta = \frac{5}{\sqrt{21}}$, $\cot\theta = -\frac{2}{\sqrt{21}}$

8 **a** $2\frac{1}{2}$ **b** $1\frac{1}{2}$ **c** $-\frac{1}{2}$

9 **a** $\theta \approx 0.841$ or 5.44 **b** $\theta \approx 3.39$ or 6.03

c $\theta \approx 1.25$ or 4.39

10 perimeter ≈ 34.1 cm, area ≈ 66.5 cm^2

11 $r \approx 8.79$ cm, area ≈ 81.0 cm^2 **12** **a** 0 **b** $\sin\theta$

13 $\sin\alpha = \frac{\sqrt{91}}{10}$, $\cos\alpha = -\frac{3}{10}$, $\tan\alpha = -\frac{\sqrt{91}}{3}$,

$\text{cosec}\,\alpha = \frac{10}{\sqrt{91}}$, $\cot\alpha = -\frac{3}{\sqrt{91}}$

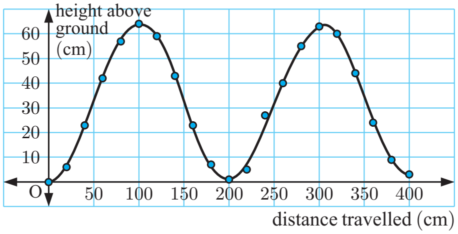
EXERCISE 9A

1 **a** periodic **b** periodic **c** periodic

d not periodic **e** periodic **f** periodic

g not periodic **h** not periodic

2 **a**



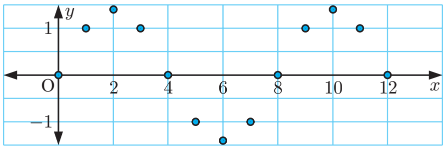
b A curve can be fitted to the data.

c The data is periodic.

i $y = 32$ (approx.) **ii** ≈ 64 cm

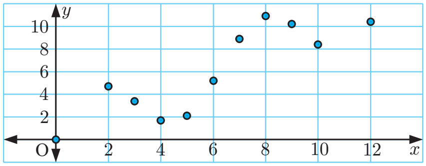
iii ≈ 200 cm **iv** ≈ 32 cm

3 **a**



Data exhibits periodic behaviour.

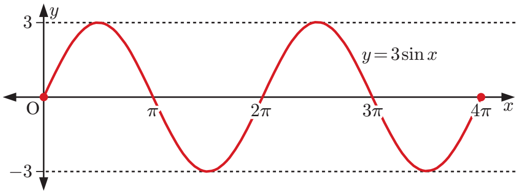
b



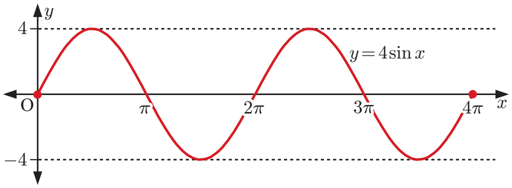
Not enough information to say data is periodic.

EXERCISE 9B

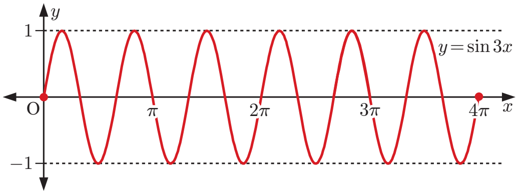
1 **a**



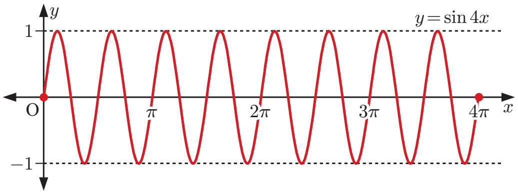
b



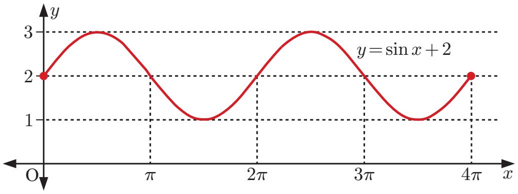
c



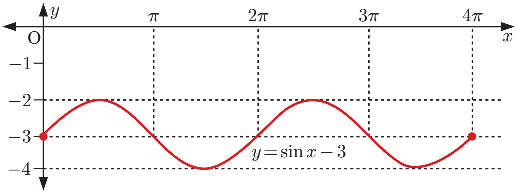
d



e



f

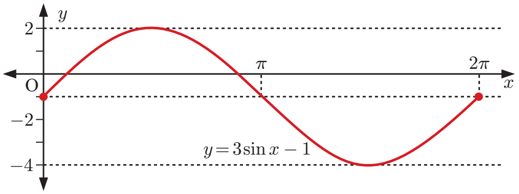


2 **a** $a = 2$ **b** $a = 5$ **c** $a = 11$

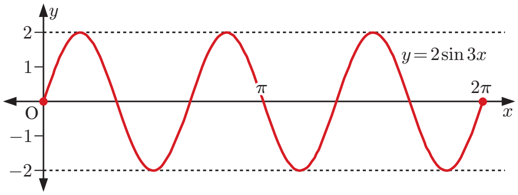
3 **a** $b = 3$ **b** $b = 5$ **c** $b = 6$ **d** $b = 4$

4 **a** $c = 3$ **b** $c = -1$ **c** $c = 5$

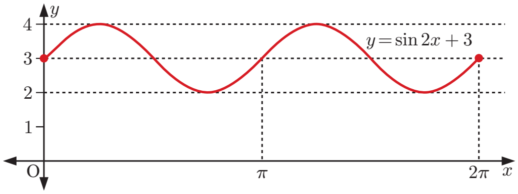
5 **a**



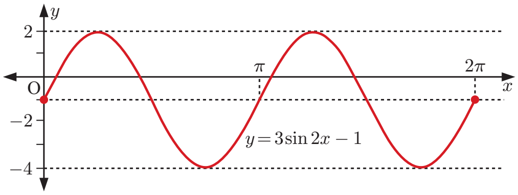
b



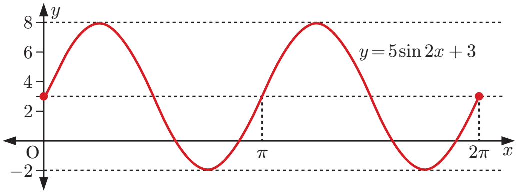
c



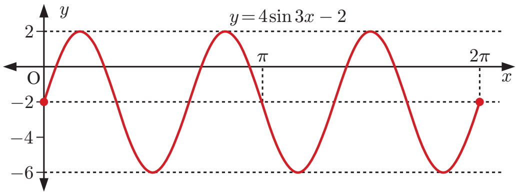
d



e



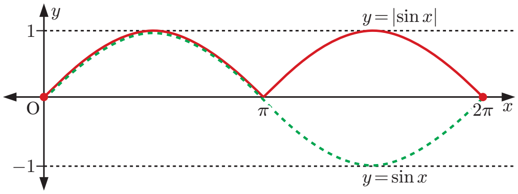
f



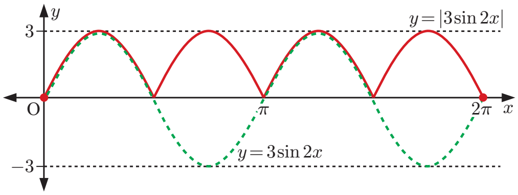
6 **a** $a = 3,\ b = 1,\ c = 0$ **b** $a = 2,\ b = 5,\ c = 6$
c $a = 5,\ b = 3,\ c = -2$

7 $m = 2,\ n = -3$

8 **a**

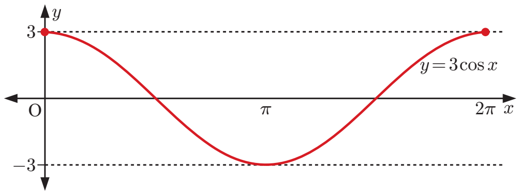


b

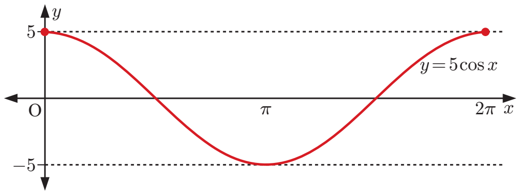


EXERCISE 9C

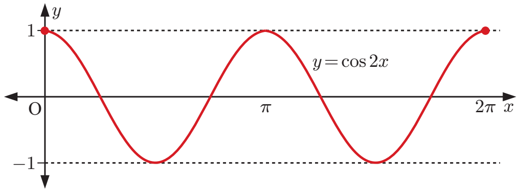
1 **a**



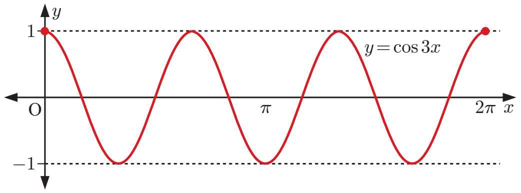
b



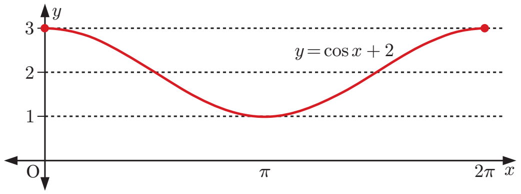
c



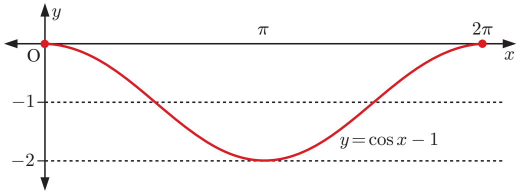
d



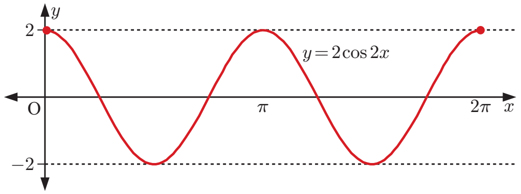
e



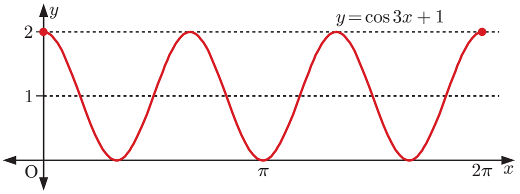
f



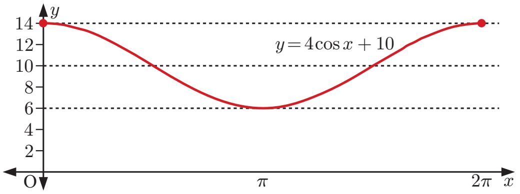
g



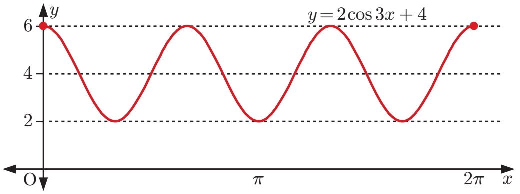
h



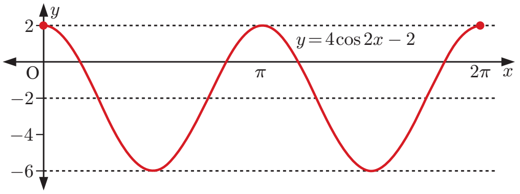
i



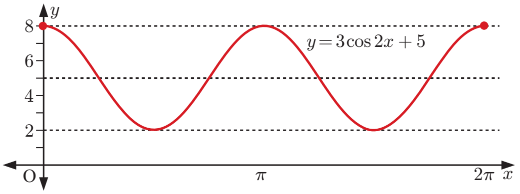
j



k



l

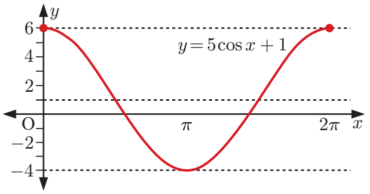


2 **a** $a = 4,\ b = 3,\ c = -1$ **b** $a = 3,\ b = 5,\ c = 3$

3 **a** $y = 2\cos 2x$ **b** $y = \cos\left(\frac{x}{2}\right) + 2$

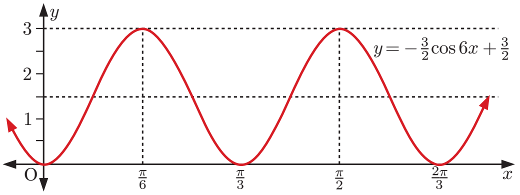
4 **a** $a = 5,$
$b = 1,$
$c = 1$

b



5 **a** $a = \frac{3}{2},\ b = 6,\ c = -\frac{3}{2}$

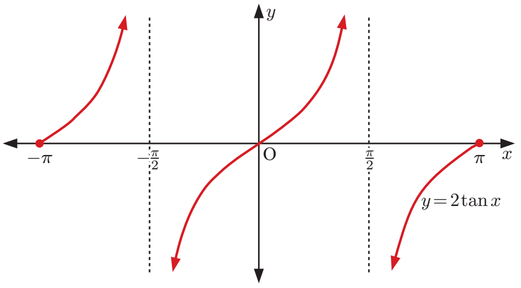
b



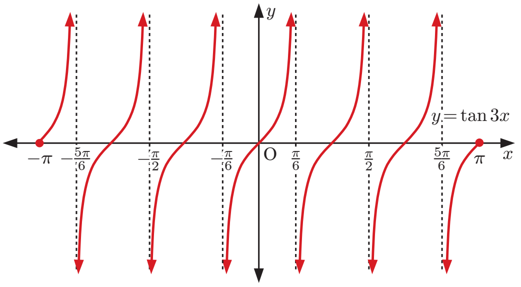
c $y = -\frac{3}{2}\cos 6x + \frac{3}{2}$

EXERCISE 9D

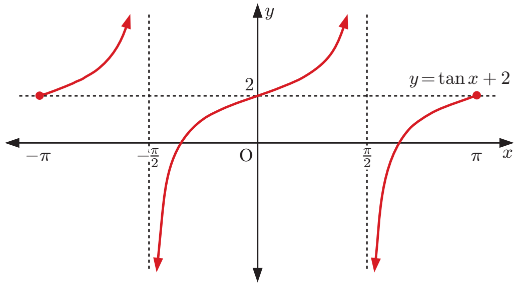
1 **a**



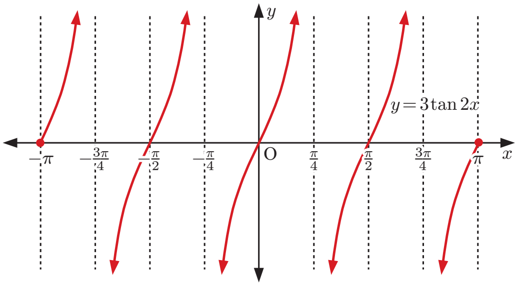
b



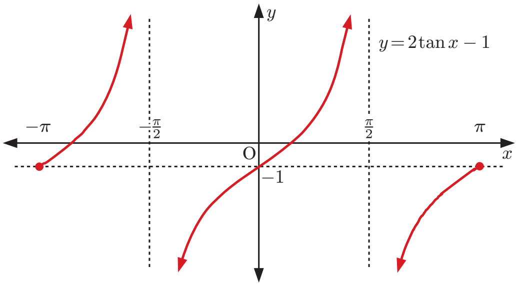
c



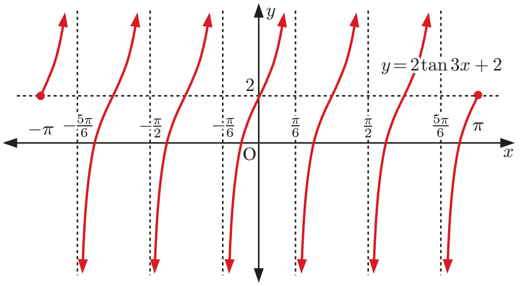
d



e



f



2 **a** $b = \frac{3}{2},\ c = 2$ **b** $b = 2,\ c = -3$

3 $p = \frac{1}{2},\ q = 1$

EXERCISE 9E.1

1 **a** $x \approx 0.3, 2.8, 6.6, 9.1, 12.9$ **b** $x \approx 5.9, 9.8, 12.2$

2 **a** $x \approx 0.9, 5.4, 7.2$ **b** $x \approx 4.4, 8.2, 10.7$

3 **a** $x \approx 0.4, 1.2, 3.5, 4.3, 6.7, 7.5, 9.8, 10.6, 13.0, 13.7$
b $x \approx 1.7, 3.0, 4.9, 6.1, 8.0, 9.3, 11.1, 12.4, 14.3, 15.6$

4 **a** **i** ≈ 1.6 **ii** ≈ -1.1
b **i** $x \approx 1.1, 4.2, 7.4$ **ii** $x \approx 2.2, 5.3$

5 **a** $x \approx 0.446, 2.70, 6.73, 8.98$
b $x \approx 2.52, 3.76, 8.80, 10.0$
c $x \approx 0.588, 3.73, 6.87, 10.0$

6 **a** $x \approx -0.644, 0.644$
b $x \approx -4.56, -1.42, 1.72, 4.87$
c $x \approx -2.76, -0.384, 3.53$

7 **a** $x \approx 1.57$ **b** $m = -2$ or $m = 1$
c $-1 \leqslant \sin x \leqslant 1$, so $m = \sin x = -2$ is not a valid solution.

EXERCISE 9E.2

1 **a** $x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$ **b** $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$
c $x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

2 **a** $x = -\frac{5\pi}{3}, -\frac{4\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$ **b** $x = -\frac{5\pi}{4}, -\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$
c $x = -\frac{5\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4}$

3 **a** $x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$ **b** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$

4 **a** $x = \frac{5\pi}{4}, \frac{7\pi}{4}$ **b** $x = \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{23\pi}{12}$

5 **a** $x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}$
b $x = -330°, -210°, 30°, 150°$
c $x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}$ **d** $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$
e $x = -\frac{8\pi}{9}, -\frac{4\pi}{9}, -\frac{2\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$

6 $x = \frac{\pi}{3}, \frac{4\pi}{3}$
a $x = \frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}, \frac{5\pi}{6}, \frac{13\pi}{12}, \frac{4\pi}{3}, \frac{19\pi}{12}, \frac{11\pi}{6}$
b $x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

7 $x = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}$

8 **a** $x = -\frac{2\pi}{3}, \frac{2\pi}{3}$ **b** $x = -\frac{5\pi}{6}, -\frac{2\pi}{3}, \frac{\pi}{6}, \frac{\pi}{3}$
c $x = -\frac{\pi}{2}, \frac{\pi}{2}$

9 **a** $x = \frac{\pi}{4}, \frac{5\pi}{4}$ **b** $x = \frac{3\pi}{4}, \frac{7\pi}{4}$
c $x = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}, \frac{13\pi}{12}, \frac{17\pi}{12}, \frac{7\pi}{4}$
d $x = \frac{\pi}{6}, \frac{2\pi}{3}, \frac{7\pi}{6}, \frac{5\pi}{3}$

10 $x = \frac{\pi}{2}$

EXERCISE 9F.1

1 **a** $2\sin\theta$ **b** $3\cos\theta$ **c** $2\sin\theta$ **d** $\sin\theta$
e $-2\tan\theta$ **f** $-3\cos^2\theta$

2 **a** 3 **b** -2 **c** -1 **d** $3\cos^2\theta$
e $4\sin^2\theta$ **f** $\cos\theta$ **g** $-\sin^2\theta$ **h** $-\cos^2\theta$
i $-2\sin^2\theta$ **j** 1 **k** $\sin\theta$ **l** $\sin\theta$

3 **a** $2\tan x$ **b** $\tan^2 x$ **c** $\sin x$ **d** $\cos x$
e $5\sin x$ **f** $2\sec x$ **g** 1 **h** 1
i $\operatorname{cosec} x$ **j** $\cos x$ **k** $\cos x$ **l** $5\sin x$

4 **a** $1 + 2\sin\theta + \sin^2\theta$ **b** $\sin^2\alpha - 4\sin\alpha + 4$
c $\tan^2\alpha - 2\tan\alpha + 1$ **d** $1 + 2\sin\alpha\cos\alpha$
e $1 - 2\sin\beta\cos\beta$ **f** $-4 + 4\cos\alpha - \cos^2\alpha$

5 **a** $-\tan^2\beta$ **b** 1 **c** $\sin^2\alpha$
d $\sin^2 x - \tan^2 x$ **e** 13 **f** $\cos^2\theta$ **g** 0

EXERCISE 9F.2

1 **a** $(1 - \sin\theta)(1 + \sin\theta)$
b $(\sin\alpha + \cos\alpha)(\sin\alpha - \cos\alpha)$
c $(\tan\alpha + 1)(\tan\alpha - 1)$ **d** $\sin\beta(2\sin\beta - 1)$
e $\cos\phi(2 + 3\cos\phi)$ **f** $3\sin\theta(\sin\theta - 2)$
g $(\tan\theta + 3)(\tan\theta + 2)$ **h** $(2\cos\theta + 1)(\cos\theta + 3)$
i $(3\cos\alpha + 1)(2\cos\alpha - 1)$ **j** $\tan\alpha(3\tan\alpha - 2)$
k $(\sec\beta + \operatorname{cosec}\beta)(\sec\beta - \operatorname{cosec}\beta)$
l $(2\cot x - 1)(\cot x - 1)$
m $(2\sin x + \cos x)(\sin x + 3\cos x)$

2 **a** $1 + \sin\alpha$ **b** $\tan\beta - 1$ **c** $\cos\phi - \sin\phi$
d $\cos\phi + \sin\phi$ **e** $\dfrac{1}{\sin\alpha - \cos\alpha}$ **f** $\dfrac{\cos\theta}{2}$
g $\sin\theta$ **h** $\cos\theta$ **i** $\sec\theta + 1$

EXERCISE 9G

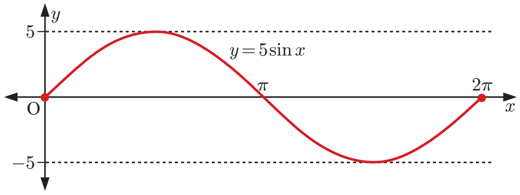
1 **a** $x = 0, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi$ **b** $x = \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{3}$
c $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ **d** $x = \frac{7\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}$
e no solutions **f** $x = 0, 2\pi$

2 **a** $x = \pi$ **b** $x = \frac{\pi}{6}, \frac{5\pi}{6}$

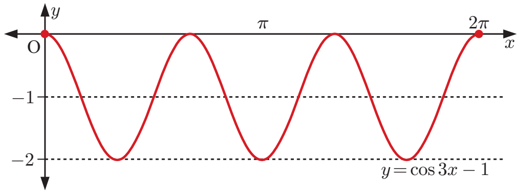
REVIEW SET 9A

1 **a** no **b** yes

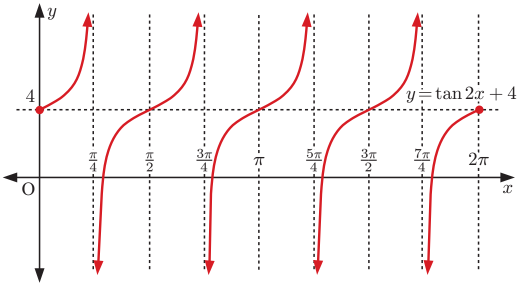
2 **a**



b



c



3 **a** minimum $= 0$, maximum $= 2$
b minimum $= -2$, maximum $= 2$
c minimum $= -3$, maximum $= 3$
d minimum $= -2$, maximum $= 0$

4 **a** 2π **b** $\frac{\pi}{2}$ **c** π **d** $\frac{\pi}{3}$

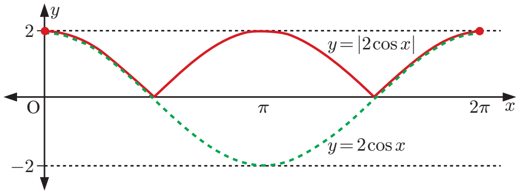
5

Function	*Period*	*Amplitude*
$y = 3\sin 2x + 1$	π	3
$y = \tan 2x$	$\frac{\pi}{2}$	undefined
$y = 2\cos 3x - 3$	$\frac{2\pi}{3}$	2

Function	*Domain*	*Range*
$y = 3\sin 2x + 1$	$x \in \mathbb{R}$	$-2 \leqslant y \leqslant 4$
$y = \tan 2x$	$x \neq \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \ldots$	$y \in \mathbb{R}$
$y = 2\cos 3x - 3$	$x \in \mathbb{R}$	$-5 \leqslant y \leqslant -1$

6 $y = 4\cos 2x$

7



8 **a** $x \approx 115°, 245°, 475°, 605°$ **b** $x \approx 25°, 335°, 385°$

9 **a** $x = \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{19\pi}{6}, \frac{23\pi}{6}$ **b** $x = -\frac{7\pi}{4}, -\frac{5\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$
c $x = \frac{4\pi}{9}, \frac{5\pi}{9}, \frac{10\pi}{9}, \frac{11\pi}{9}, \frac{16\pi}{9}, \frac{17\pi}{9}$
d $x = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}$

10 **a** $1 - \cos\theta$ **b** $\dfrac{1}{\sin\alpha + \cos\alpha}$ **c** $-\dfrac{\cos\alpha}{2}$
d $\operatorname{cosec}\theta + 1$

12 **a** $x = -\frac{2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}$ **b** $x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$

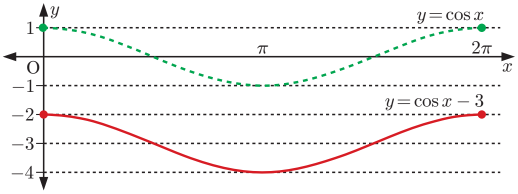
REVIEW SET 9B

1 **a** The function repeats itself over and over in a horizontal direction, in intervals of length 8 units.
b **i** 8 **ii** 5 **iii** -1

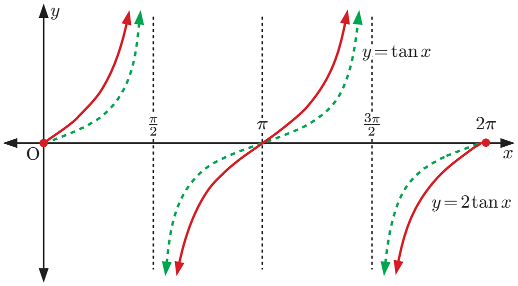
2 **a** $b = 6$ **b** $b = 24$

3 **a** minimum $= -8$, maximum $= 2$
b minimum $= -2$, maximum $= 4$
c minimum $= 5$, maximum $= 13$

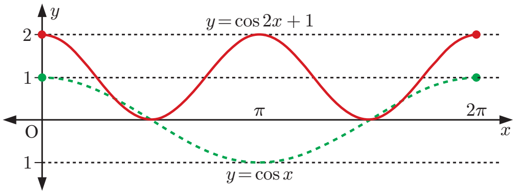
4 **a**



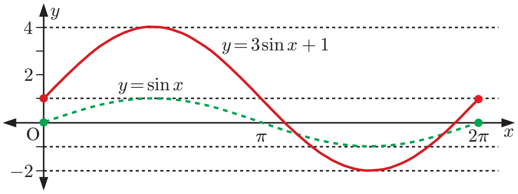
b



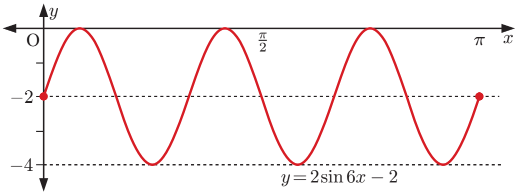
c



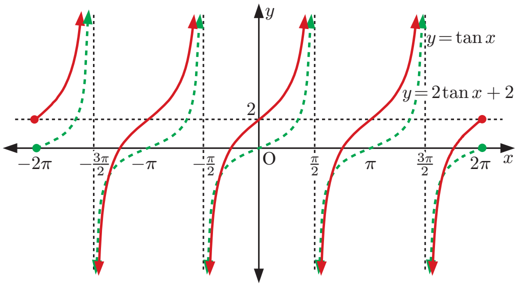
d



5 **a** $a = 2$, $b = 6$, $c = -2$
b



6 **a** $y = 2\tan x + 2$
b



7 **a** $x \approx -6.1, -3.4$ **b** $x \approx 0.8$

8 $m = 3$, $n = -1$

9 **a** $x = \frac{3\pi}{2}$ **b** $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

10 **a** $\cos\theta$ **b** $-\sin\theta$ **c** $5\cos^2\theta$ **d** $-\cos\theta$

11 **a** $4\sin^2\alpha - 4\sin\alpha + 1$ **b** $1 - 2\sin\alpha\cos\alpha$

EXERCISE 10A

1 18 **2** **a** 4 **b** 8 **c** 24 **3** 6

4 42 **5** 1680 **6** **a** 125 **b** 60

7 17 576 000 **8** **a** 4 **b** 9 **c** 81

EXERCISE 10B

1 **a** 13 **b** 20 **c** 19 **d** 32 **2** 13

EXERCISE 10C.1

1 1, 1, 2, 6, 24, 120, 720, 5040, 40 320, 362 880, 3 628 800

2 **a** 6 **b** 30 **c** $\frac{1}{7}$ **d** $\frac{1}{30}$ **e** 100 **f** 21

3 **a** $n,\ n \geqslant 1$ **b** $(n+2)(n+1),\ n \geqslant 0$
c $(n+1)n,\ n \geqslant 1$

4 **a** $\dfrac{7!}{4!}$ **b** $\dfrac{10!}{8!}$ **c** $\dfrac{11!}{6!}$ **d** $\dfrac{13!}{10!3!}$ **e** $\dfrac{3!}{6!}$ **f** $\dfrac{4!16!}{20!}$

5 **a** $6 \times 4!$ **b** $10 \times 10!$ **c** $73 \times 7!$ **d** $131 \times 10!$
e $81 \times 7!$ **f** $62 \times 6!$ **g** $10 \times 11!$ **h** $32 \times 8!$

6 **a** $11!$ **b** $9!$ **c** $8!$ **d** 9
e 34 **f** $n+1$ **g** $(n-1)!$ **h** $(n+1)!$

EXERCISE 10C.2

1 **a** 3 **b** 6 **c** 35 **d** 210

2 **a** **i** 28 **ii** 28 **3** $k = 3$ or 6

EXERCISE 10D

1 **a** W, X, Y, Z
b WX, WY, WZ, XW, XY, XZ, YW, YX, YZ, ZW, ZX, ZY
c WXY, WXZ, WYX, WYZ, WZX, WZY, XWY, XWZ, XYW, XYZ, XZW, XZY, YWX, YWZ, YXW, YXZ, YZX, YZW, ZWX, ZWY, ZXW, ZXY, ZYW, ZYX

2 **a** AB, AC, AD, AE, BA, BC, BD, BE, CA, CB, CD, CE, DA, DB, DC, DE, EA, EB, EC, ED
b ABC, ABD, ABE, ACB, ACD, ACE, ADB, ADC, ADE, AEB, AEC, AED, BAC, BAD, BAE, BCA, BCD, BCE, BDA, BDC, BDE, BEA, BEC, BED, CAB, CAD, CAE, CBA, CBD, CBE, CDA, CDB, CDE, CEA, CEB, CED, DAB, DAC, DAE, DBA, DBC, DBE, DCA, DCB, DCE, DEA, DEB, DEC, EAB, EAC, EAD, EBA, EBC, EBD, ECA, ECB, ECD, EDA, EDB, EDC

3 **a** 120 **b** 336 **c** 5040 **4** 110

5 **a** 12 **b** 24 **c** 36 **6** **a** 15 120 **b** 720

7 **a** 720 **b** **i** 24 **ii** 24 **iii** 48

8 **a** 343 **b** 210 **c** 120

9 **a** 648 **b** 64 **c** 72 **d** 136

10 **a** 6720 **b** 240 **c** 4200

11 **a** 120 **b** 48 **c** 72

12 **a** 3 628 800 **b** 241 920

13 **a** 720 **b** 144 **c** 72 **d** 144

14 **a** 48 **b** 24 **c** 15 **15** **a** 360 **b** 336 **c** 288

16 **a** 3 628 800 **b** **i** 151 200 **ii** 33 600

EXERCISE 10E

1 **a** permutation **b** combination
c permutation **d** combination

2 ABCD, ABCE, ABCF, ABDE, ABDF, ABEF, ACDE, ACDF, ACEF, ADEF, BCDE, BCDF, BCEF, BDEF, CDEF, $\binom{6}{4} = 15$

3 $\binom{17}{11} = 12\,376$ **4** **a** $\binom{9}{5} = 126$ **b** $\binom{1}{1}\binom{8}{4} = 70$

5 **a** $\binom{13}{3} = 286$ **b** $\binom{1}{1}\binom{12}{2} = 66$

6 **a** $\binom{12}{5} = 792$
b **i** $\binom{2}{2}\binom{10}{3} = 120$ **ii** $\binom{2}{1}\binom{10}{4} = 420$

7 $\binom{3}{3}\binom{1}{0}\binom{11}{6} = 462$

8 **a** $\binom{1}{1}\binom{9}{3} = 84$ **b** $\binom{2}{0}\binom{8}{4} = 70$
c $\binom{2}{0}\binom{1}{1}\binom{7}{3} = 35$

9 **a** $\binom{16}{5} = 4368$ **b** $\binom{10}{3}\binom{6}{2} = 1800$
c $\binom{10}{5}\binom{6}{0} = 252$
d $\binom{10}{3}\binom{6}{2} + \binom{10}{4}\binom{6}{1} + \binom{10}{5}\binom{6}{0} = 3312$
e $\binom{16}{5} - \binom{10}{5}\binom{6}{0} - \binom{10}{0}\binom{6}{5} = 4110$

10 **a** 6435 **b** 2520 **c** 36 **d** 4005 **11** 1050

12 **a** $\binom{6}{2}\binom{3}{1}\binom{7}{2} = 945$ **b** $\binom{6}{2}\binom{10}{3} = 1800$
c $\binom{16}{5} - \binom{9}{0}\binom{7}{5} = 4347$

13 $\binom{20}{2} - 20 = 170$

14 **a** **i** $\binom{12}{2} = 66$ **ii** $\binom{11}{1} = 11$
b **i** $\binom{12}{3} = 220$ **ii** $\binom{11}{2} = 55$

15 $\binom{9}{4} = 126$

16 **a** Selecting the different committees of 4 from 5 men and 6 women in all possible ways.
b $\binom{m+n}{r}$

17 **a** $\dfrac{\binom{12}{6}}{2} = 462$ **b** $\dfrac{\binom{12}{4}\binom{8}{4}\binom{4}{4}}{3!} = 5775$

18 **a** 45, yes **b** 37 128 **c** 3 628 800

EXERCISE 10F

1 **a** $p^3 + 3p^2q + 3pq^2 + q^3$ **b** $x^3 + 3x^2 + 3x + 1$
c $x^3 - 9x^2 + 27x - 27$ **d** $8 + 12x + 6x^2 + x^3$
e $27x^3 - 27x^2 + 9x - 1$ **f** $8x^3 + 60x^2 + 150x + 125$
g $8a^3 - 12a^2b + 6ab^2 - b^3$ **h** $27x^3 - 9x^2 + x - \frac{1}{27}$
i $8x^3 + 12x + \dfrac{6}{x} + \dfrac{1}{x^3}$

2 **a** $1 + 4x + 6x^2 + 4x^3 + x^4$
b $p^4 - 4p^3q + 6p^2q^2 - 4pq^3 + q^4$
c $x^4 - 8x^3 + 24x^2 - 32x + 16$
d $81 - 108x + 54x^2 - 12x^3 + x^4$
e $1 + 8x + 24x^2 + 32x^3 + 16x^4$
f $16x^4 - 96x^3 + 216x^2 - 216x + 81$
g $16x^4 + 32x^3b + 24x^2b^2 + 8xb^3 + b^4$
h $x^4 + 4x^2 + 6 + \dfrac{4}{x^2} + \dfrac{1}{x^4}$
i $16x^4 - 32x^2 + 24 - \dfrac{8}{x^2} + \dfrac{1}{x^4}$

3 **a** $x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$
b $x^5 - 10x^4y + 40x^3y^2 - 80x^2y^3 + 80xy^4 - 32y^5$
c $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$
d $x^5 - 5x^3 + 10x - \dfrac{10}{x} + \dfrac{5}{x^3} - \dfrac{1}{x^5}$

4 $64 + 160x^2 + 20x^4$

5 **a** 1 6 15 20 15 6 1

b **i** $x^6 + 12x^5 + 60x^4 + 160x^3 + 240x^2 + 192x + 64$

ii $64x^6 - 192x^5 + 240x^4 - 160x^3 + 60x^2 - 12x + 1$

iii $x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$

6 **a** $7 + 5\sqrt{2}$ **b** $161 + 72\sqrt{5}$ **c** $232 - 164\sqrt{2}$

7 $\frac{59 + 34\sqrt{3}}{13}$

8 **a** $64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$

b 65.944 160 601 201

9 **a** $a = 2$ and $b = e^x$ **b** $T_3 = 6e^{2x}$ and $T_4 = e^{3x}$

10 $2x^5 + 11x^4 + 24x^3 + 26x^2 + 14x + 3$

11 **a** 270 **b** 4320

EXERCISE 10G

1 **a** $1^{11} + \binom{11}{1}(2x)^1 + \binom{11}{2}(2x)^2 + ... + \binom{11}{10}(2x)^{10} + (2x)^{11}$

b $(3x)^{15} + \binom{15}{1}(3x)^{14}\left(\frac{2}{x}\right)^1 + \binom{15}{2}(3x)^{13}\left(\frac{2}{x}\right)^2 + ...$

$.... + \binom{15}{14}(3x)^1\left(\frac{2}{x}\right)^{14} + \left(\frac{2}{x}\right)^{15}$

c $(2x)^{20} + \binom{20}{1}(2x)^{19}\left(-\frac{3}{x}\right)^1 + \binom{20}{2}(2x)^{18}\left(-\frac{3}{x}\right)^2 + ...$

$.... + \binom{20}{19}(2x)^1\left(-\frac{3}{x}\right)^{19} + \left(-\frac{3}{x}\right)^{20}$

2 **a** $T_6 = \binom{15}{5}(2x)^{10}5^5$ **b** $T_4 = \binom{9}{3}(x^2)^6y^3$

c $T_{10} = \binom{17}{9}x^8\left(-\frac{2}{x}\right)^9$ **d** $T_9 = \binom{21}{8}(2x^2)^{13}\left(-\frac{1}{x}\right)^8$

3 **a** $\binom{12}{4}2^8 3^4 = 10\,264\,320$ **b** $\binom{12}{7}2^5 3^7 = 55\,427\,328$

4 **a** $\binom{10}{3}1^7(-3)^3 = -3240$ **b** $\binom{10}{7}1^3(-3)^7 = -262\,440$

5 **a** 144 **b** 5376 **c** 2304

6 **a** $T_{r+1} = \binom{7}{r}x^{7-r}b^r$ **b** $b = -2$

7 **a** $\binom{15}{5}2^5 = 96\,096$ **b** $\binom{9}{3}(-3)^3 = -2268$

8 **a** $\binom{10}{5}3^5 2^5 = 1\,959\,552$ **b** $\binom{6}{3}2^3(-3)^3 = -4320$

c $\binom{6}{3}2^3(-3)^3 = -4320$ **d** $\binom{12}{4}2^8(-1)^4 = 126\,720$

9 $k = 5$ **10** $a = 3$ **11** **b** $a = 5,\ b = 2$

12 $\binom{8}{6} = 28$ **13** $2\binom{9}{3}3^6x^6 - \binom{9}{4}3^5x^6 = 91\,854x^6$

14 **a** $\binom{7}{4}3^3(-2)^4 = 15\,120$

b $\binom{7}{4}3^3(-2)^4 + 3\binom{7}{3}3^4(-2)^3 = -52\,920$

15 **a** $\binom{8}{3}2^5(-5)^3 - 3\binom{8}{1}2^7(-5)^1 = -208\,640$

b $\binom{6}{3}2^3 - \binom{6}{4}2^4 = -80$

16 $a = 3,\ b = -2,\ c = 57$ **17** $n = 8$

18 $n = 6$ **19** $84x^3$ **20** $k = -2,\ n = 6$

REVIEW SET 10A

1 **a** $n(n-1),\ n \geqslant 2$ **b** $n + 2$ **2** 28

3 **a** 24 **b** 6 **4** **a** 900 **b** 180

5 **a** $a = e^x$ and $b = -e^{-x}$

b $(e^x - e^{-x})^4 = e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x}$

6 $362 + 209\sqrt{3}$ **7** It does not have one. **8** $c = 3$

9 **a** 720 **b** 72 **c** 504 **10** 2500

11 **a** 252 **b** 246 **12** $\binom{12}{6}2^6(-3)^6 = 43\,110\,144$

13 $8\binom{6}{2} - 6\binom{6}{1} = 84$ **14** $a = \pm 4$ **15** $k = 0$ or ± 2

REVIEW SET 10B

1 **a** $26^2 \times 10^4 = 6\,760\,000$

b $5 \times 26 \times 10^4 = 1\,300\,000$

c $26 \times 25 \times 10 \times 9 \times 8 \times 7 = 3\,276\,000$

2 **a** 3003 **b** 980 **c** 2982

3 **a** $x^3 - 6x^2y + 12xy^2 - 8y^3$

b $81x^4 + 216x^3 + 216x^2 + 96x + 16$

4 20 000 **5** 60 **6** $-103 + 74\sqrt{2}$ **7** 4200

8 $\binom{5}{2}2^3 - 3\binom{5}{1}2^4 = -160$

9 **a** 3024 **b** 840 **c** 42

10 $q = 0$ or $\pm\sqrt{\frac{3}{35}}$ **11** 4320 **12** $k = 180$

13 **a** 43 758 teams **b** 11 550 teams **c** 41 283 teams

d 3861 teams

14 $n = 7$ **15** $k = -\frac{1}{4},\ n = 16$

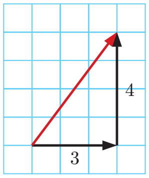
EXERCISE 11A

1 **a** $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$, $7\mathbf{i} + 3\mathbf{j}$ **b** $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$, $-6\mathbf{i}$

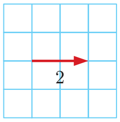
c $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$, $2\mathbf{i} - 5\mathbf{j}$ **d** $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$, $6\mathbf{j}$

e $\begin{pmatrix} -6 \\ 3 \end{pmatrix}$, $-6\mathbf{i} + 3\mathbf{j}$ **f** $\begin{pmatrix} -5 \\ -5 \end{pmatrix}$, $-5\mathbf{i} - 5\mathbf{j}$

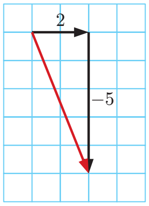
2 **a** $3\mathbf{i} + 4\mathbf{j}$



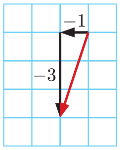
b $2\mathbf{i}$



c $2\mathbf{i} - 5\mathbf{j}$



d $-\mathbf{i} - 3\mathbf{j}$



3 **a** **i** $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $4\mathbf{i} + \mathbf{j}$ **ii** $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$, $-4\mathbf{i} - \mathbf{j}$

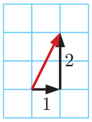
iii $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$, $-\mathbf{i} - 5\mathbf{j}$ **iv** $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, $2\mathbf{i}$

v $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$, $3\mathbf{i} - 4\mathbf{j}$ **vi** $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $4\mathbf{i} + \mathbf{j}$

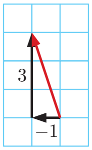
b $\overrightarrow{AB}$ and $\overrightarrow{DE}$. They have the same magnitude and direction.

c $\overrightarrow{BA}$ is the negative of both $\overrightarrow{AB}$ and $\overrightarrow{DE}$. They have the same magnitude but opposite direction.

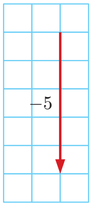
4 **a** $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$



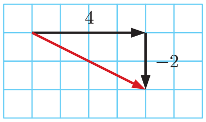
b $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$



c $\begin{pmatrix}0\\-5\end{pmatrix}$



d $\begin{pmatrix}4\\-2\end{pmatrix}$



5 **a** $\begin{pmatrix}-1\\-4\end{pmatrix}$ **b** $\begin{pmatrix}2\\-3\end{pmatrix}$ **c** $\begin{pmatrix}-5\\2\end{pmatrix}$ **d** $\begin{pmatrix}0\\6\end{pmatrix}$

EXERCISE 11B

1 **a** 5 units **b** 5 units **c** 2 units
d $\sqrt{8}$ units **e** 3 units

2 **a** $\sqrt{2}$ units **b** 13 units **c** $\sqrt{17}$ units
d 3 units **e** $|k|$ units

3 **a** unit vector **b** unit vector **c** not a unit vector
d unit vector **e** not a unit vector

4 **a** $k = \pm 1$ **b** $k = \pm 1$ **c** $k = 0$
d $k = \pm\frac{1}{\sqrt{2}}$ **e** $k = \pm\frac{\sqrt{3}}{2}$

5 $p = \pm 3$

EXERCISE 11C

1 **a** $\begin{pmatrix}-2\\6\end{pmatrix}$ **b** $\begin{pmatrix}-2\\6\end{pmatrix}$ **c** $\begin{pmatrix}-1\\-1\end{pmatrix}$ **d** $\begin{pmatrix}-1\\-1\end{pmatrix}$
e $\begin{pmatrix}-5\\-3\end{pmatrix}$ **f** $\begin{pmatrix}-5\\-3\end{pmatrix}$ **g** $\begin{pmatrix}-6\\4\end{pmatrix}$ **h** $\begin{pmatrix}-4\\1\end{pmatrix}$

2 **a** $\begin{pmatrix}-3\\7\end{pmatrix}$ **b** $\begin{pmatrix}-4\\-3\end{pmatrix}$ **c** $\begin{pmatrix}-8\\-1\end{pmatrix}$ **d** $\begin{pmatrix}-6\\9\end{pmatrix}$
e $\begin{pmatrix}0\\-5\end{pmatrix}$ **f** $\begin{pmatrix}6\\-9\end{pmatrix}$

3 **a** $\mathbf{a} + \mathbf{0} = \begin{pmatrix}a_1\\a_2\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}a_1+0\\a_2+0\end{pmatrix} = \begin{pmatrix}a_1\\a_2\end{pmatrix} = \mathbf{a}$

b $\mathbf{a} - \mathbf{a} = \begin{pmatrix}a_1\\a_2\end{pmatrix} - \begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}a_1-a_1\\a_2-a_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} = \mathbf{0}$

4 **a** $\begin{pmatrix}-3\\-15\end{pmatrix}$ **b** $\begin{pmatrix}-1\\2\end{pmatrix}$ **c** $\begin{pmatrix}0\\14\end{pmatrix}$ **d** $\begin{pmatrix}5\\-3\end{pmatrix}$
e $\begin{pmatrix}\frac{5}{2}\\\frac{11}{2}\end{pmatrix}$ **f** $\begin{pmatrix}-7\\7\end{pmatrix}$ **g** $\begin{pmatrix}5\\11\end{pmatrix}$ **h** $\begin{pmatrix}3\\\frac{17}{3}\end{pmatrix}$

5 **a** $\begin{pmatrix}8\\-1\end{pmatrix}$ **b** $\begin{pmatrix}8\\-1\end{pmatrix}$ **c** $\begin{pmatrix}8\\-1\end{pmatrix}$

In each case, the result is $2\mathbf{p} + 3\mathbf{q} = \begin{pmatrix}8\\-1\end{pmatrix}$.

6 **a** $\sqrt{13}$ units **b** $\sqrt{17}$ units **c** $5\sqrt{2}$ units
d $\sqrt{10}$ units **e** $\sqrt{29}$ units

7 **a** $\sqrt{10}$ units **b** $2\sqrt{10}$ units **c** $2\sqrt{10}$ units
d $3\sqrt{10}$ units **e** $3\sqrt{10}$ units **f** $2\sqrt{5}$ units
g $8\sqrt{5}$ units **h** $8\sqrt{5}$ units **i** $\sqrt{5}$ units
j $\sqrt{5}$ units

8 **a** $3\mathbf{i} + 2\mathbf{j}$ **b** $-\mathbf{i} + 9\mathbf{j}$ **c** $6\mathbf{i} - \mathbf{j}$ **d** $7\mathbf{j}$
e 2 units **f** $2\sqrt{10}$ units

EXERCISE 11D

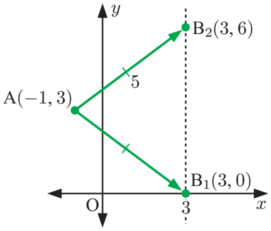
1 **a** $\begin{pmatrix}2\\4\end{pmatrix}$ **b** $\begin{pmatrix}-2\\5\end{pmatrix}$ **c** $\begin{pmatrix}3\\-3\end{pmatrix}$ **d** $\begin{pmatrix}1\\-5\end{pmatrix}$
e $\begin{pmatrix}6\\-5\end{pmatrix}$ **f** $\begin{pmatrix}1\\3\end{pmatrix}$

2 **a** B(4, 2) **b** C(2, 2) **3** **a** $\begin{pmatrix}2\\1\end{pmatrix}$ **b** Q(3, 3)

4 **a** $\begin{pmatrix}5\\1\end{pmatrix}$ **b** $\begin{pmatrix}-5\\-1\end{pmatrix}$ **c** D(−1, −2)

5 **a** $\overrightarrow{AB} = \begin{pmatrix}4\\k-3\end{pmatrix}$, $|\overrightarrow{AB}| = \sqrt{16+(k-3)^2} = 5$ units

b $k = 0$ or 6 **c**



6 **a** $\overrightarrow{AB} = \begin{pmatrix}2\\3\end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix}3\\-3\end{pmatrix}$

b $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC} = -\overrightarrow{AB} + \overrightarrow{AC}$ **c** $\overrightarrow{BC} = \begin{pmatrix}1\\-6\end{pmatrix}$

7 **a** $\begin{pmatrix}-5\\4\end{pmatrix}$ **b** $\begin{pmatrix}1\\2\end{pmatrix}$ **c** $\begin{pmatrix}6\\-5\end{pmatrix}$

8 **a** M(1, 4) **b** $\overrightarrow{CA} = \begin{pmatrix}7\\5\end{pmatrix}$, $\overrightarrow{CM} = \begin{pmatrix}5\\3\end{pmatrix}$, $\overrightarrow{CB} = \begin{pmatrix}3\\1\end{pmatrix}$

EXERCISE 11E

1 $r = 3$ **2** $a = -6$

3 **a** $\overrightarrow{AB}$ is parallel and in the same direction as $\overrightarrow{CD}$, and 3 times its length.

b $\overrightarrow{RS}$ is parallel and in the opposite direction to $\overrightarrow{KL}$, and half its length.

c A, B, and C are collinear. $\overrightarrow{AB}$ is parallel and in the same direction as $\overrightarrow{BC}$, and twice its length.

4 **a** $\begin{pmatrix}4\\8\end{pmatrix}$ **b** $\begin{pmatrix}-1\\-2\end{pmatrix}$

5 **a** $\frac{1}{\sqrt{5}}\mathbf{i} + \frac{2}{\sqrt{5}}\mathbf{j}$ **b** $\frac{1}{\sqrt{10}}\mathbf{i} - \frac{3}{\sqrt{10}}\mathbf{j}$ **c** $\frac{2}{\sqrt{5}}\mathbf{i} - \frac{1}{\sqrt{5}}\mathbf{j}$

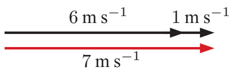
6 **a** $\mathbf{v} = \frac{3}{\sqrt{5}}\begin{pmatrix}2\\-1\end{pmatrix}$ **b** $\mathbf{v} = \frac{2}{\sqrt{17}}\begin{pmatrix}1\\4\end{pmatrix}$

7 **a** $\overrightarrow{AB} = \begin{pmatrix}2\sqrt{2}\\-2\sqrt{2}\end{pmatrix}$ **b** $\overrightarrow{OB} = \begin{pmatrix}3+2\sqrt{2}\\2-2\sqrt{2}\end{pmatrix}$

c $B(3+2\sqrt{2},\ 2-2\sqrt{2})$

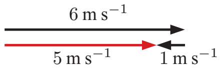
EXERCISE 11F

1 **a**



∴ $7\ \text{m s}^{-1}$

b



∴ $5\ \text{m s}^{-1}$

2 **a** $1.34\ \text{m s}^{-1}$ in the direction 26.6° to the right of intended line

b **i** 30° to the left of Q **ii** $1.04\ \text{m s}^{-1}$

3 **a** 24.6 km h^{-1} **b** $\approx 9.93°$ east of south

4 **a** 82.5 m **b** 23.3° to the left of directly across **c** 48.4 s

5 **a** The plane's speed in still air would be $\approx 437\ \text{km h}^{-1}$.
The wind slows the plane down to 400 km h^{-1}.
b 4.64° north of due east

EXERCISE 11G

1 **a** **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $t \in \mathbb{R}$
ii $x = 3 + t$, $y = -4 + 4t$, $t \in \mathbb{R}$ **iii** $4x - y = 16$

b **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 7 \end{pmatrix}$, $t \in \mathbb{R}$
ii $x = -6 + 3t$, $y = 7t$, $t \in \mathbb{R}$ **iii** $7x - 3y = -42$

c **i** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 11 \end{pmatrix} + t\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, $t \in \mathbb{R}$
ii $x = -1 - 2t$, $y = 11 + t$, $t \in \mathbb{R}$ **iii** $x + 2y = 21$

2 **a** $x = -1 + 2t$, $y = 4 - t$, $t \in \mathbb{R}$

b

t	0	1	3	-1	-4
Point	$(-1, 4)$	$(1, 3)$	$(5, 1)$	$(-3, 5)$	$(-9, 8)$

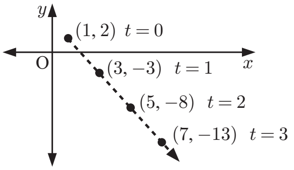
3 **a** When $t = 1$, $x = 3$, $y = -2$, $\therefore$ yes **b** $k = -5$

4 **a** (0, 8) **b** It is a non-zero scalar multiple of $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

c $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} + s\begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $s \in \mathbb{R}$

EXERCISE 11H

1 **a** (1, 2) **b**

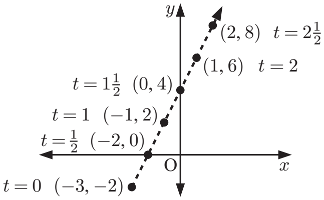


c $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$

d $\sqrt{29}\ \text{cm s}^{-1}$

2 **a** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ -5 \end{pmatrix}$, $t \geqslant 0$ **b** (8, −4.5)
c 45 minutes

3 **a** $\begin{pmatrix} -3 + 2t \\ -2 + 4t \end{pmatrix}$ **d**



b $\begin{pmatrix} 2 \\ 8 \end{pmatrix}$

c **i** $t = 1.5$ s
ii $t = 0.5$ s

4 **a** **i** (−4, 3) **ii** $\begin{pmatrix} 12 \\ 5 \end{pmatrix}$ **iii** 13 m s^{-1}

b **i** (3, 0) **ii** $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ **iii** $\sqrt{5}\ \text{m s}^{-1}$

5 **a** $\begin{pmatrix} 120 \\ -90 \end{pmatrix}$ **b** $\begin{pmatrix} 20\sqrt{5} \\ 10\sqrt{5} \end{pmatrix}$ **6** $\begin{pmatrix} 35 \\ -84 \end{pmatrix}$

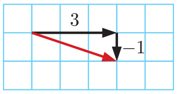
7 **a** A is at (4, 5), B is at (1, −8)
b For A it is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. For B it is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.
c For A, speed is $\sqrt{5}\ \text{km h}^{-1}$. For B, speed is $\sqrt{5}\ \text{km h}^{-1}$.
d Yacht A: $2x + y = 13$, Yacht B: $x - 2y = 17$
e Yacht A moves with gradient −2; Yacht B with gradient $\frac{1}{2}$.
So, their paths are perpendicular.
f no

8 **a** $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -5 \\ 4 \end{pmatrix} + t\begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $t \geqslant 0$
$\therefore$ $x_1(t) = -5 + 3t$, $y_1(t) = 4 - t$, $t \geqslant 0$
b speed $= \sqrt{10}\ \text{km min}^{-1}$
c a minutes later, $(t - a)$ min have elapsed.
$\therefore$ $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 15 \\ 7 \end{pmatrix} + (t - a)\begin{pmatrix} -4 \\ -3 \end{pmatrix}$, $t \geqslant 0$
$\therefore$ $x_2(t) = 15 - 4(t - a)$, $y_2(t) = 7 - 3(t - a)$, $t \geqslant 0$
d Torpedo is fired at 1:35:28 pm and the explosion occurs at 1:37:42 pm.

REVIEW SET 11A

1 **a** $\mathbf{x} = \begin{pmatrix} 5 \\ 1 \end{pmatrix} = 5\mathbf{i} + \mathbf{j}$, $\mathbf{y} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \mathbf{i} - 2\mathbf{j}$
b **i** $6\mathbf{i} - \mathbf{j}$ **ii** $-9\mathbf{i} - 4\mathbf{j}$

2 **a** $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ **b**



c $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$

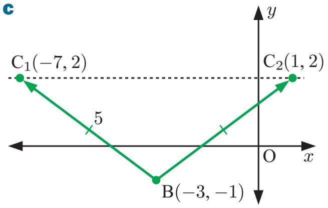
d $\sqrt{10}$ units

3 **a** $k = \pm\frac{1}{\sqrt{2}}$ **b** $\begin{pmatrix} -2\sqrt{5} \\ \sqrt{5} \end{pmatrix}$

4 **a** $\begin{pmatrix} 3 \\ -3 \end{pmatrix}$ **b** $\begin{pmatrix} 7 \\ -3 \end{pmatrix}$ **c** 5 units

5 **a** $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ **b** $\begin{pmatrix} -3 \\ -6 \end{pmatrix}$ **c** $\sqrt{34}$ units

6 **a** $\overrightarrow{BC} = \begin{pmatrix} k + 3 \\ 3 \end{pmatrix}$
$|\overrightarrow{BC}| = 5$
b $k = -7$ or 1
c



7 **a** 11.5° east of due north **b** $\approx 343\ \text{km h}^{-1}$

8 **a** $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix} + t\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, $t \in \mathbb{R}$
b $x = -6 + 4t$, $y = 3 - 3t$, $t \in \mathbb{R}$ **c** $3x + 4y = -6$

9 $m = 10$ **10** $\begin{pmatrix} 6\sqrt{10} \\ -2\sqrt{10} \end{pmatrix}$

11 **a** (5, 2) **b** $\begin{pmatrix} 4 \\ 10 \end{pmatrix}$ is a non-zero scalar multiple of $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$
c $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + s\begin{pmatrix} 4 \\ 10 \end{pmatrix}$, $s \in \mathbb{R}$

12 **a** (−4, 3) **b** (28, 27) **c** $\begin{pmatrix} 8 \\ 6 \end{pmatrix}$ **d** 10 m s^{-1}

REVIEW SET 11B

1 **a** **i** $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} = 4\mathbf{i}$ **ii** $\overrightarrow{BC} = \begin{pmatrix} -2 \\ -4 \end{pmatrix} = -2\mathbf{i} - 4\mathbf{j}$
iii $\overrightarrow{CA} = \begin{pmatrix} -2 \\ 4 \end{pmatrix} = -2\mathbf{i} + 4\mathbf{j}$

b $\overrightarrow{BC}$ and $\overrightarrow{CA}$. Each vector has components of the same magnitude, but differing signs (which do not affect the length of $\overrightarrow{BC}$ or $\overrightarrow{CA}$).

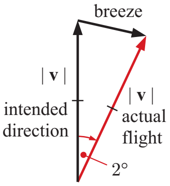
c $\overrightarrow{AC}$, $\begin{pmatrix}2\\-4\end{pmatrix}$, $2\mathbf{i}-4\mathbf{j}$

2 **a** $\sqrt{13}$ units **b** $\sqrt{10}$ units **c** $\sqrt{109}$ units

3 **a** $k=\pm\frac{12}{13}$ **b** $k=\pm\frac{1}{\sqrt{2}}$ **4** $\begin{pmatrix}1\\4\end{pmatrix}$

5 **a** M$(-2,\ 4)$ **b** 5 units **6** $m=5$

7 **a**



b **i** isosceles triangle ∴ 2 remaining angles = 89° each, breeze makes angle of $180-89=91°$ to intended direction of the arrow.

ii bisect angle 2° and use $\sin 1° = \dfrac{\frac{1}{2}\text{ speed}}{|\mathbf{v}|}$

∴ speed $=2\,|\mathbf{v}|\sin 1°$

8 $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\8\end{pmatrix}+t\begin{pmatrix}5\\4\end{pmatrix},\ t\in\mathbb{R}$

9 **a** **i** $-6\mathbf{i}+10\mathbf{j}$ **ii** $-5\mathbf{i}-15\mathbf{j}$

iii $(-6-5t)\mathbf{i}+(10-15t)\mathbf{j},\ \ t\geqslant 0$

b $t=40$ min ($\frac{2}{3}$ h), yacht is ≈ 9.33 km away from the beacon.

10 **a** **i** $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}2\\-3\end{pmatrix}+t\begin{pmatrix}4\\-1\end{pmatrix},\ t\in\mathbb{R}$

ii $x=2+4t,\ \ y=-3-t,\ \ t\in\mathbb{R}$

b **i** $\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\6\end{pmatrix}+t\begin{pmatrix}3\\-4\end{pmatrix},\ t\in\mathbb{R}$

ii $x=-1+3t,\ \ y=6-4t,\ \ t\in\mathbb{R}$

11 **a** $x_1(t)=2+t,\ \ y_1(t)=4-3t,\ \ t\geqslant 0$

b $x_2(t)=13-t,\ \ y_2(t)=[3-2a]+at,\ \ t\geqslant 2$

c interception occurred at 2:22:30 pm

d bearing ≈ 12.7° west of due south, at ≈ 4.54 km min^{-1}

EXERCISE 12A

1 **a** 1×4 **b** 2×1 **c** 2×2 **d** 3×3

2 **a** $(2\quad 1\quad 6\quad 1)$ **b** $\begin{pmatrix}1.95\\2.35\\0.45\\2.95\end{pmatrix}$

c total cost of groceries

3 $\begin{pmatrix}1000&1500&1250\\1500&1000&1000\\800&2300&1300\\1200&1200&1200\end{pmatrix}$ **4** $\begin{pmatrix}40&50&55&40\\25&65&44&30\\35&40&40&35\\35&40&35&50\end{pmatrix}$

EXERCISE 12B.1

1 **a** $\begin{pmatrix}9&1\\3&3\end{pmatrix}$ **b** $\begin{pmatrix}6&8\\-1&1\end{pmatrix}$

c $\begin{pmatrix}3&4\\-6&-1\end{pmatrix}$ **d** $\begin{pmatrix}0&0\\-11&-3\end{pmatrix}$

2 **a** $\begin{pmatrix}20&1&-8\\8&10&-2\\1&-5&18\end{pmatrix}$ **b** $\begin{pmatrix}-14&9&-14\\12&-6&14\\-5&3&-4\end{pmatrix}$

c $\begin{pmatrix}14&-9&14\\-12&6&-14\\5&-3&4\end{pmatrix}$

3 **a** $\begin{array}{cc}\text{Friday}&\text{Saturday}\\ \begin{pmatrix}85\\92\\52\end{pmatrix}&\begin{pmatrix}102\\137\\49\end{pmatrix}\end{array}$ **b** $\begin{pmatrix}187\\229\\101\end{pmatrix}$

4 **a** **i** $\begin{pmatrix}1.72\\27.85\\0.92\\2.53\\3.56\end{pmatrix}$ **ii** $\begin{pmatrix}1.79\\28.75\\1.33\\2.25\\3.51\end{pmatrix}$

b subtract cost price from selling price **c** $\begin{pmatrix}0.07\\0.90\\0.41\\-0.28\\-0.05\end{pmatrix}$

5 **a** $\begin{array}{c}\begin{array}{cc}\text{L}&\text{R}\end{array}\\ \begin{pmatrix}23&19\\17&29\\31&24\end{pmatrix}\end{array}\begin{array}{l}\\ \text{fr}\\ \text{st}\\ \text{mi}\end{array}$ **b** $\begin{array}{c}\begin{array}{cc}\text{L}&\text{R}\end{array}\\ \begin{pmatrix}18&25\\7&13\\36&19\end{pmatrix}\end{array}\begin{array}{l}\\ \text{fr}\\ \text{st}\\ \text{mi}\end{array}$

c $\begin{array}{c}\begin{array}{cc}\text{L}&\text{R}\end{array}\\ \begin{pmatrix}41&44\\24&42\\67&43\end{pmatrix}\end{array}\begin{array}{l}\\ \text{fr}\\ \text{st}\\ \text{mi}\end{array}$

6 **a** $x=-2,\ \ y=-2$ **b** $x=0,\ \ y=0$

7 **a** $\mathbf{A}+\mathbf{B}=\begin{pmatrix}1&3\\5&2\end{pmatrix}$, $\mathbf{B}+\mathbf{A}=\begin{pmatrix}1&3\\5&2\end{pmatrix}$

8 **a** $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\begin{pmatrix}6&3\\-1&6\end{pmatrix}$, $\mathbf{A}+(\mathbf{B}+\mathbf{C})=\begin{pmatrix}6&3\\-1&6\end{pmatrix}$

EXERCISE 12B.2

1 **a** $\begin{pmatrix}12&24\\48&12\end{pmatrix}$ **b** $\begin{pmatrix}2&4\\8&2\end{pmatrix}$

c $\begin{pmatrix}\frac{1}{2}&1\\2&\frac{1}{2}\end{pmatrix}$ **d** $\begin{pmatrix}-3&-6\\-12&-3\end{pmatrix}$

2 **a** $\begin{pmatrix}3&5&6\\2&8&7\end{pmatrix}$ **b** $\begin{pmatrix}1&1&4\\0&4&1\end{pmatrix}$

c $\begin{pmatrix}5&8&11\\3&14&11\end{pmatrix}$ **d** $\begin{pmatrix}5&7&14\\2&16&9\end{pmatrix}$

3 $12\mathbf{F}=\begin{pmatrix}12\\48\\24\\12\end{pmatrix}$

4 **a** $\mathbf{A}=\begin{pmatrix}75\\27\\102\end{pmatrix}\begin{array}{l}\longleftarrow\text{DVD}\\ \longleftarrow\text{Blu-ray}\\ \longleftarrow\text{games}\end{array}$ $\mathbf{B}=\begin{pmatrix}136\\43\\129\end{pmatrix}\begin{array}{l}\longleftarrow\text{DVD}\\ \longleftarrow\text{Blu-ray}\\ \longleftarrow\text{games}\end{array}$

b $5\mathbf{A}+2\mathbf{B}=\begin{pmatrix}647\\221\\768\end{pmatrix}\begin{array}{l}\longleftarrow\text{DVD}\\ \longleftarrow\text{Blu-ray}\\ \longleftarrow\text{games}\end{array}$

c total weekly average hirings

5 **a** $\begin{array}{c}\begin{array}{cccc}\text{A}&\text{B}&\text{C}&\text{D}\end{array}\\ \begin{pmatrix}35&46&46&69\\58&46&35&86\\46&46&58&58\\12&23&23&17\end{pmatrix}\end{array}$ **b** $\begin{array}{c}\begin{array}{cccc}\text{A}&\text{B}&\text{C}&\text{D}\end{array}\\ \begin{pmatrix}26&34&34&51\\43&34&26&64\\34&34&43&43\\9&17&17&13\end{pmatrix}\end{array}$

EXERCISE 12B.3

1 **a** $3\mathbf{A}$ **b** $\mathbf{O}$ **c** $-\mathbf{C}$ **d** $\mathbf{O}$ **e** $2\mathbf{A}+2\mathbf{B}$
f $-\mathbf{A}-\mathbf{B}$ **g** $-2\mathbf{A}+\mathbf{C}$ **h** $4\mathbf{A}-\mathbf{B}$ **i** $3\mathbf{B}$

2 **a** $\mathbf{X}=\mathbf{A}-\mathbf{B}$ **b** $\mathbf{X}=\mathbf{C}-\mathbf{B}$ **c** $\mathbf{X}=2\mathbf{C}-4\mathbf{B}$
d $\mathbf{X}=\frac{1}{2}\mathbf{A}$ **e** $\mathbf{X}=\frac{1}{3}\mathbf{B}$ **f** $\mathbf{X}=\mathbf{A}-\mathbf{B}$
g $\mathbf{X}=2\mathbf{C}$ **h** $\mathbf{X}=\frac{1}{2}\mathbf{B}-\mathbf{A}$ **i** $\mathbf{X}=\frac{1}{4}(\mathbf{A}-\mathbf{C})$

3 **a** $\mathbf{X}=\begin{pmatrix}3 & 6\\ 9 & 18\end{pmatrix}$ **b** $\mathbf{X}=\begin{pmatrix}\frac{1}{2} & -\frac{1}{4}\\ \frac{3}{4} & \frac{5}{4}\end{pmatrix}$
c $\mathbf{X}=\begin{pmatrix}-1 & -6\\ 1 & -\frac{1}{2}\end{pmatrix}$

EXERCISE 12C.1

1 **a** (11) **b** (22) **c** (16) **2** **b** $\begin{pmatrix}w & x & y & z\end{pmatrix}\begin{pmatrix}\frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{4}\\ \frac{1}{4}\end{pmatrix}$

3 **a** $\mathbf{P}=\begin{pmatrix}27 & 35 & 39\end{pmatrix}$, $\mathbf{Q}=\begin{pmatrix}4\\ 3\\ 2\end{pmatrix}$
b total cost $=\begin{pmatrix}27 & 35 & 39\end{pmatrix}\begin{pmatrix}4\\ 3\\ 2\end{pmatrix}=\$291$

4 **a** $\mathbf{P}=\begin{pmatrix}10 & 6 & 3 & 1\end{pmatrix}$, $\mathbf{N}=\begin{pmatrix}3\\ 2\\ 4\\ 2\end{pmatrix}$
b total points $=\begin{pmatrix}10 & 6 & 3 & 1\end{pmatrix}\begin{pmatrix}3\\ 2\\ 4\\ 2\end{pmatrix}=56$ points

EXERCISE 12C.2

1 Number of columns in **A** does not equal number of rows in **B**.
2 **a** $m=n$ **b** 2×3 **c** **B** has 3 columns, **A** has 2 rows
3 **a** does not exist **b** $\begin{pmatrix}28 & 29\end{pmatrix}$
4 **a** (8) **b** $\begin{pmatrix}2 & 0 & 3\\ 8 & 0 & 12\\ 4 & 0 & 6\end{pmatrix}$
5 **a** $\begin{pmatrix}3 & 5 & 3\end{pmatrix}$ **b** $\begin{pmatrix}-2\\ 1\\ 1\end{pmatrix}$
6 **a** $\mathbf{Q}=\begin{pmatrix}32 & 24\\ 25 & 16\\ 13 & 9\end{pmatrix}$ **b** $\mathbf{P}=\begin{pmatrix}1.19\\ 1.55\end{pmatrix}$
c $\mathbf{QP}=\begin{pmatrix}32 & 24\\ 25 & 16\\ 13 & 9\end{pmatrix}\begin{pmatrix}1.19\\ 1.55\end{pmatrix}=\begin{pmatrix}75.28\\ 54.55\\ 29.42\end{pmatrix}$
It represents the total value of sales for each pen colour.
d $\$75.28+\$54.55+\$29.42=\$159.25$
7 **a** $\mathbf{C}=\begin{pmatrix}12.5\\ 9.5\end{pmatrix}$ $\mathbf{N}=\begin{pmatrix}2375 & 5156\\ 2502 & 3612\end{pmatrix}$
b $\begin{pmatrix}78\,669.5\\ 65\,589\end{pmatrix}$ income from day 1 / income from day 2 **c** $144 258.50
8 **a** $\mathbf{R}=\begin{pmatrix}1 & 1\\ 1 & 2\\ 2 & 3\end{pmatrix}$ **b** $\mathbf{P}=\begin{pmatrix}7 & 3 & 19\\ 6 & 2 & 22\end{pmatrix}$
c $\begin{pmatrix}48 & 70\\ 52 & 76\end{pmatrix}$ **d** **i** $48 **ii** $76
e The elements of **PR** tell us that, if all the items are to be bought at one store, it is cheapest to do so at store A for both you and your friend. However, the cheapest way is to buy paint from store A, and hammers and screwdrivers from store B.

EXERCISE 12C.3

1 **a** $\mathbf{A}^2+\mathbf{A}$ **b** $\mathbf{B}^2+2\mathbf{B}$ **c** $\mathbf{A}^3-2\mathbf{A}^2+\mathbf{A}$
d $\mathbf{A}^3+\mathbf{A}^2-2\mathbf{A}$ **e** $\mathbf{AC}+\mathbf{AD}+\mathbf{BC}+\mathbf{BD}$
f $\mathbf{A}^2+\mathbf{AB}+\mathbf{BA}+\mathbf{B}^2$ **g** $\mathbf{A}^2-\mathbf{AB}+\mathbf{BA}-\mathbf{B}^2$
h $\mathbf{A}^2+2\mathbf{A}+\mathbf{I}$ **i** $9\mathbf{I}-6\mathbf{B}+\mathbf{B}^2$

2 **a** $\mathbf{A}^3=3\mathbf{A}-2\mathbf{I}$, $\mathbf{A}^4=4\mathbf{A}-3\mathbf{I}$
b $\mathbf{B}^3=3\mathbf{B}-2\mathbf{I}$, $\mathbf{B}^4=6\mathbf{I}-5\mathbf{B}$, $\mathbf{B}^5=11\mathbf{B}-10\mathbf{I}$
c $\mathbf{C}^3=13\mathbf{C}-12\mathbf{I}$, $\mathbf{C}^5=121\mathbf{C}-120\mathbf{I}$

3 **a** **i** $\mathbf{I}+2\mathbf{A}$ **ii** $2\mathbf{I}-2\mathbf{A}$ **iii** $10\mathbf{A}+6\mathbf{I}$
b $\mathbf{A}^2+\mathbf{A}+2\mathbf{I}$ **c** **i** $-3\mathbf{A}$ **ii** $-2\mathbf{A}$ **iii** $\mathbf{A}$

4 **a** $\mathbf{A}^2=\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}$
b false as $\mathbf{A}(\mathbf{A}-\mathbf{I})=\mathbf{O}$ does not imply that $\mathbf{A}=\mathbf{O}$ or $\mathbf{A}-\mathbf{I}=\mathbf{O}$
c $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$, $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$, $\begin{pmatrix}a & b\\ \dfrac{a-a^2}{b} & 1-a\end{pmatrix}$, $b\neq 0$

5 For example, $\mathbf{A}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$ gives $\mathbf{A}^2=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$.

6 **a** $a=3,\ b=-4$ **b** $a=1,\ b=8$
7 **a** $p=-2,\ q=1$ **b** $\mathbf{A}^3=5\mathbf{A}-2\mathbf{I}$
c $\mathbf{A}^4=-12\mathbf{A}+5\mathbf{I}$

EXERCISE 12D.1

1 **a** $\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}=3\mathbf{I}$, $\begin{pmatrix}1 & -2\\ -\frac{2}{3} & \frac{5}{3}\end{pmatrix}$
b $\begin{pmatrix}10 & 0\\ 0 & 10\end{pmatrix}=10\mathbf{I}$, $\begin{pmatrix}0.2 & 0.4\\ -0.1 & 0.3\end{pmatrix}$

2 **a** -2 **b** -1 **c** 0 **d** 1
3 **a** 26 **b** 6 **c** -1 **d** a^2+a
4 **a** -3 **b** -3 **c** -12 **5** **Hint:** Let $\mathbf{A}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$

6 **a** **i** $\det\mathbf{A}=ad-bc$ **ii** $\det\mathbf{B}=wz-xy$
iii $\mathbf{AB}=\begin{pmatrix}aw+by & ax+bz\\ cw+dy & cx+dz\end{pmatrix}$
iv $\det\mathbf{AB}=(ad-bc)(wz-xy)$

7 **a** $\det\mathbf{A}=-2$, $\det\mathbf{B}=-1$
b **i** $\det(2\mathbf{A})=-8$ **ii** $\det(-\mathbf{A})=-2$
iii $\det(-3\mathbf{B})=-9$ **iv** $\det(\mathbf{AB})=2$

8 **a** $\frac{1}{14}\begin{pmatrix}5 & -4\\ 1 & 2\end{pmatrix}$ **b** $\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$ **c** does not exist
d $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ **e** $\frac{1}{10}\begin{pmatrix}2 & 0\\ 1 & 5\end{pmatrix}$ **f** does not exist
g $-\frac{1}{15}\begin{pmatrix}7 & -2\\ -4 & -1\end{pmatrix}$ **h** $\frac{1}{10}\begin{pmatrix}2 & -4\\ 1 & 3\end{pmatrix}$ **i** $\begin{pmatrix}-3 & -1\\ 2 & 1\end{pmatrix}$

9 **a** $\dfrac{1}{2k+6}\begin{pmatrix}2 & -1\\ 6 & k\end{pmatrix}$, $k\neq -3$ **b** $\dfrac{1}{3k}\begin{pmatrix}k & 1\\ 0 & 3\end{pmatrix}$, $k\neq 0$

c $\dfrac{1}{(k+2)(k-1)}\begin{pmatrix} k & -2 \\ -1 & k+1 \end{pmatrix}$, $k \neq -2$ or 1

d $\dfrac{1}{k(k+1)}\begin{pmatrix} k & -k \\ 3 & k-2 \end{pmatrix}$, $k \neq 0$ or -1

e $\dfrac{1}{k(2-k)}\begin{pmatrix} 1 & 1-k \\ -2k & k^2 \end{pmatrix}$, $k \neq 0$ or 2

f $\dfrac{1}{(k+4)(k-1)}\begin{pmatrix} 3k & -2 \\ -k^2-2 & k+1 \end{pmatrix}$, $k \neq -4$ or 1

EXERCISE 12D.2

1 $\mathbf{X} = \begin{pmatrix} \frac{1}{4} & \frac{3}{4} \\ 1 & 0 \end{pmatrix}$ **2** **a** $\mathbf{X} = \mathbf{ABZ}$ **b** $\mathbf{Z} = \mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{X}$

3 $\mathbf{A}^2 = 2\mathbf{A} - \mathbf{I}$, $\mathbf{A}^{-1} = 2\mathbf{I} - \mathbf{A}$

4 **a** $\mathbf{A}^{-1} = 4\mathbf{I} - \mathbf{A}$ **b** $\mathbf{A}^{-1} = 5\mathbf{I} + \mathbf{A}$
c $\mathbf{A}^{-1} = \frac{3}{2}\mathbf{A} - 2\mathbf{I}$

6 If $\mathbf{A}^{-1}$ exists, that is, $\det \mathbf{A} \neq 0$.

EXERCISE 12E

1 **a** $\begin{pmatrix} 3 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$

b $\begin{pmatrix} 4 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ -5 \end{pmatrix}$

c $\begin{pmatrix} 3 & -1 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \end{pmatrix}$

2 **a** $x = \frac{32}{7}$, $y = \frac{22}{7}$ **b** $x = -\frac{37}{23}$, $y = -\frac{75}{23}$
c $x = \frac{17}{13}$, $y = -\frac{37}{13}$ **d** $x = \frac{59}{13}$, $y = -\frac{25}{13}$
e $x = -40$, $y = -24$ **f** $x = \frac{1}{34}$, $y = \frac{55}{34}$

3 **b** **i** $\mathbf{X} = \begin{pmatrix} -\frac{4}{3} & \frac{13}{9} \\ -1 & \frac{4}{3} \end{pmatrix}$ **ii** $\mathbf{X} = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}$

iii $\mathbf{X} = \begin{pmatrix} \frac{13}{7} & \frac{3}{7} \\ -\frac{2}{7} & -\frac{8}{7} \end{pmatrix}$ **iv** $\mathbf{X} = \begin{pmatrix} \frac{19}{7} & \frac{6}{7} \\ \frac{20}{7} & -\frac{25}{7} \end{pmatrix}$

4 **a** **i** $\begin{pmatrix} 2 & -3 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$, $\det \mathbf{A} = 10$
ii Yes, $x = 2.5$, $y = -1$

b **i** $\begin{pmatrix} 2 & k \\ 4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix}$, $\det \mathbf{A} = -2 - 4k$
ii $k \neq -\frac{1}{2}$, $x = \dfrac{8+11k}{2+4k}$, $y = \dfrac{5}{1+2k}$
iii $k = -\frac{1}{2}$, no solutions

REVIEW SET 12A

1 **a** $\begin{pmatrix} 4 & 2 \\ -2 & 3 \end{pmatrix}$ **b** $\begin{pmatrix} 9 & 6 \\ 0 & -3 \end{pmatrix}$ **c** $\begin{pmatrix} -2 & 0 \\ 4 & -8 \end{pmatrix}$

d $\begin{pmatrix} 2 & 2 \\ 2 & -5 \end{pmatrix}$ **e** $\begin{pmatrix} -5 & -4 \\ -2 & 6 \end{pmatrix}$ **f** $\begin{pmatrix} 7 & 6 \\ 4 & -11 \end{pmatrix}$

g $\begin{pmatrix} -1 & 8 \\ 2 & -4 \end{pmatrix}$ **h** $\begin{pmatrix} 3 & 2 \\ -6 & -8 \end{pmatrix}$ **i** $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ 0 & -1 \end{pmatrix}$

j $\begin{pmatrix} 9 & 4 \\ 0 & 1 \end{pmatrix}$ **k** $\begin{pmatrix} -3 & -10 \\ 6 & 8 \end{pmatrix}$ **l** $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{6} & \frac{1}{12} \end{pmatrix}$

2 **a** $a = 0$, $b = 5$, $c = 1$, $d = -4$
b $a = 2$, $b = -1$, $c = 3$, $d = 8$

3 **a** $\mathbf{Y} = \mathbf{B} - \mathbf{A}$ **b** $\mathbf{Y} = \frac{1}{2}(\mathbf{A} - \mathbf{C})$ **c** $\mathbf{Y} = \mathbf{A}^{-1}\mathbf{B}$
d $\mathbf{Y} = \mathbf{CB}^{-1}$ **e** $\mathbf{Y} = \mathbf{A}^{-1}(\mathbf{C} - \mathbf{B})$ **f** $\mathbf{Y} = \mathbf{B}^{-1}\mathbf{A}$

4 **a** $4\mathbf{L}$ **b** $-2\mathbf{L}$

5 **a** $\begin{pmatrix} 10 & -12 \\ -10 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & 6 & -3 \\ -4 & -2 & 11 \end{pmatrix}$ **c** not possible

6 **a** $\mathbf{A} - \mathbf{A}^2$ **b** $\mathbf{AB} + \mathbf{A}^2 - \mathbf{B}^2 - \mathbf{BA}$
c $4\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I}$

7 $\mathbf{A}^3 = 27\mathbf{A} + 10\mathbf{I}$, $\mathbf{A}^4 = 145\mathbf{A} + 54\mathbf{I}$

8 $a = 4$, $b = -7$

9 **a** $\begin{pmatrix} \frac{7}{2} & -4 \\ -\frac{5}{2} & 3 \end{pmatrix}$ **b** does not exist **c** $\begin{pmatrix} 1 & \frac{5}{3} \\ -2 & -\frac{11}{3} \end{pmatrix}$

10 Unique solution if $k \neq \frac{3}{4}$.

11 **a** $x = 0$, $y = -\frac{1}{2}$ **b** $x = \frac{12}{7}$, $y = \frac{13}{7}$

12 **b** $(\mathbf{A} - \mathbf{I})(\mathbf{A} + 3\mathbf{I}) = 2\mathbf{A} - \mathbf{I}$

REVIEW SET 12B

1 **a** $\begin{pmatrix} 4 & 2 \\ 2 & 4 \\ 3 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 2 & -2 \\ 0 & 4 \\ -1 & -2 \end{pmatrix}$ **c** $\begin{pmatrix} -\frac{3}{2} & 3 \\ \frac{1}{2} & -4 \\ 2 & \frac{7}{2} \end{pmatrix}$

2 **a** $\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 & 2 \\ 4 & 1 \\ 3 & 1 \end{pmatrix}$ **b** **i** Book 2 (hard cover)
ii \$101

4 **a** $\mathbf{X} = \frac{1}{2}(\mathbf{B} - \mathbf{A})$ **b** $\mathbf{X} = \frac{1}{3}(2\mathbf{B} - 3\mathbf{A})$
c $\mathbf{X} = \frac{1}{4}(\mathbf{B} - \mathbf{A})$

5 $\mathbf{X} = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{3}{2} \end{pmatrix}$

6 **a** $\begin{pmatrix} 4 & 8 \\ 0 & 2 \\ 6 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 2 \\ 0 & \frac{1}{2} \\ \frac{3}{2} & 1 \end{pmatrix}$ **c** $\begin{pmatrix} 11 & 12 \end{pmatrix}$
d $\mathbf{BA}$ does not exist

7 **a** $\det \mathbf{B} \neq 0$ **b** $\mathbf{AB} = \mathbf{BA}$

8 **a** $\det \mathbf{A} = 5$ **b** $\det(-2\mathbf{A}) = 20$ **c** $\det(\mathbf{A}^2) = 25$

9 **a** $x = \frac{14}{3}$, $y = \frac{1}{3}$ **b** $x = -1$, $y = 3$

10 $k \in \mathbb{R}$, $k \neq 3, -2, 2$

11 Unique solution for $k \neq -3$ or 1. $x = \dfrac{-6}{k-1}$, $y = \dfrac{2}{k-1}$

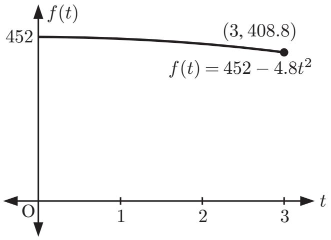
12 $\mathbf{A}(\frac{5}{3}\mathbf{A} - 2\mathbf{I}) = \mathbf{I}$, $\mathbf{A}^{-1} = \frac{5}{3}\mathbf{A} - 2\mathbf{I}$

EXERCISE 13A

1 **a** 7 **b** 7 **c** 11 **d** 16 **e** 0 **f** 5
2 **a** 5 **b** 7 **c** c
3 **a** -2 **b** 7 **c** -1 **d** 1
4 **a** -3 **b** 5 **c** -1 **d** 6 **e** -4 **f** -8
g 1 **h** 2 **i** 5

EXERCISE 13B

1 **a**



b no
c **i** $0\ \text{m s}^{-1}$
ii $9.6\ \text{m s}^{-1}$
iii $19.2\ \text{m s}^{-1}$
iv $28.8\ \text{m s}^{-1}$

2 **a**

x	Point B	Gradient of AB
0	(0, 0)	2
1	(1, 1)	3
1.5	(1.5, 2.25)	3.5
1.9	(1.9, 3.61)	3.9
1.99	(1.99, 3.9601)	3.99
1.999	(1.999, 3.996 001)	3.999

x	Point B	Gradient of AB
5	(5, 25)	7
3	(3, 9)	5
2.5	(2.5, 6.25)	4.5
2.1	(2.1, 4.41)	4.1
2.01	(2.01, 4.0401)	4.01
2.001	(2.001, 4.004 001)	4.001

b $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2} = 4$

The gradient of the tangent to $y = x^2$ at the point (2, 4) is 4.

EXERCISE 13C

1 **a** $f(2) = 3$ **b** $f'(2) = 0$

2 **a** $f(0) = 4$ **b** $f'(0) = -1$ **3** $f(2) = 3,\ f'(2) = 1$

EXERCISE 13D

1 **a** $f'(x) = 1$ **b** $f'(x) = 0$ **c** $f'(x) = 2$

2 **a** $\dfrac{dy}{dx} = -1$ **b** $\dfrac{dy}{dx} = 2x - 3$ **c** $\dfrac{dy}{dx} = 4x + 1$

3 **a** 3 **b** -12 **c** 9 **d** 10

EXERCISE 13E

1 **a** $f'(x) = 3x^2$ **b** $f'(x) = 6x^2$

c $f'(x) = 14x$ **d** $f'(x) = \dfrac{3}{\sqrt{x}}$

e $f'(x) = \dfrac{1}{\sqrt[3]{x^2}}$ **f** $f'(x) = 2x + 1$

g $f'(x) = -4x$ **h** $f'(x) = 2x + 3$

i $f'(x) = 2x^3 - 12x$ **j** $f'(x) = \dfrac{6}{x^2}$

k $f'(x) = -\dfrac{2}{x^2} + \dfrac{6}{x^3}$ **l** $f'(x) = 2x - \dfrac{5}{x^2}$

m $f'(x) = 2x + \dfrac{3}{x^2}$ **n** $f'(x) = -\dfrac{1}{2x\sqrt{x}}$

o $f'(x) = 8x - 4$ **p** $f'(x) = 3x^2 + 12x + 12$

2 **a** $\dfrac{dy}{dx} = 7.5x^2 - 2.8x$ **b** $\dfrac{dy}{dx} = 2\pi x$

c $\dfrac{dy}{dx} = -\dfrac{2}{5x^3}$ **d** $\dfrac{dy}{dx} = 100$

e $\dfrac{dy}{dx} = 10$ **f** $\dfrac{dy}{dx} = 12\pi x^2$

3 **a** 6 **b** $\dfrac{3\sqrt{x}}{2}$ **c** $2x - 10$ **d** $2 - 9x^2$ **e** $2x - 1$

f $-\dfrac{2}{x^3} + \dfrac{3}{\sqrt{x}}$ **g** $4 + \dfrac{1}{4x^2}$ **h** $6x^2 - 6x - 5$

4 **a** 4 **b** $-\frac{16}{729}$ **c** -7 **d** $\frac{13}{4}$ **e** $\frac{1}{8}$ **f** -11

5 $b = 3,\ c = -4$

6 **a** $f'(x) = \dfrac{2}{\sqrt{x}} + 1$ **b** $f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$

c $f'(x) = \dfrac{1}{x\sqrt{x}}$ **d** $f'(x) = 2 - \dfrac{1}{2\sqrt{x}}$

e $f'(x) = -\dfrac{2}{x\sqrt{x}}$ **f** $f'(x) = 6x - \frac{3}{2}\sqrt{x}$

g $f'(x) = \dfrac{-25}{2x^3\sqrt{x}}$ **h** $f'(x) = 2 + \dfrac{9}{2x^2\sqrt{x}}$

7 **a** $\dfrac{dy}{dx} = 4 + \dfrac{3}{x^2}$, $\dfrac{dy}{dx}$ is the gradient function of $y = 4x - \dfrac{3}{x}$ from which the gradient at any point can be found.

b $\dfrac{dS}{dt} = 4t + 4$ m s^{-1}, $\dfrac{dS}{dt}$ is the instantaneous rate of change in position at the time t, or the velocity function.

c $\dfrac{dC}{dx} = 3 + 0.004x$ \$ per toaster, $\dfrac{dC}{dx}$ is the instantaneous rate of change in cost as the number of toasters changes.

EXERCISE 13F.1

1 **a** $gf(x) = (2x + 7)^2$ **b** $gf(x) = 2x^2 + 7$

c $gf(x) = \sqrt{3 - 4x}$ **d** $gf(x) = 3 - 4\sqrt{x}$

e $gf(x) = \dfrac{2}{x^2 + 3}$ **f** $gf(x) = \dfrac{4}{x^2} + 3$

2 **Note:** There may be other answers.

a $g(x) = x^3,\quad f(x) = 3x + 10$

b $g(x) = \dfrac{1}{x},\quad f(x) = 2x + 4$

c $g(x) = \sqrt{x},\quad f(x) = x^2 - 3x$

d $g(x) = \dfrac{10}{x^3},\quad f(x) = 3x - x^2$

EXERCISE 13F.2

1 **a** $u^{-2},\quad u = 2x - 1$ **b** $u^{\frac{1}{2}},\quad u = x^2 - 3x$

c $2u^{-\frac{1}{2}},\quad u = 2 - x^2$ **d** $u^{\frac{1}{3}},\quad u = x^3 - x^2$

e $4u^{-3},\quad u = 3 - x$ **f** $10u^{-1},\quad u = x^2 - 3$

2 **a** $\dfrac{dy}{dx} = 8(4x - 5)$ **b** $\dfrac{dy}{dx} = 2(5 - 2x)^{-2}$

c $\dfrac{dy}{dx} = \frac{1}{2}(3x - x^2)^{-\frac{1}{2}} \times (3 - 2x)$

d $\dfrac{dy}{dx} = -12(1 - 3x)^3$ **e** $\dfrac{dy}{dx} = -18(5 - x)^2$

f $\dfrac{dy}{dx} = \frac{1}{3}(2x^3 - x^2)^{-\frac{2}{3}} \times (6x^2 - 2x)$

g $\dfrac{dy}{dx} = -60(5x - 4)^{-3}$

h $\dfrac{dy}{dx} = -4(3x - x^2)^{-2} \times (3 - 2x)$

i $\dfrac{dy}{dx} = 6\left(x^2 - \dfrac{2}{x}\right)^2 \times \left(2x + \dfrac{2}{x^2}\right)$

3 **a** $-\frac{1}{\sqrt{3}}$ **b** -18 **c** -8 **d** -4 **e** $-\frac{3}{32}$ **f** 0

4 $a = 3,\ b = 1$ **5** $a = 2,\ b = 1$

6 **a** $\dfrac{dy}{dx} = 3x^2,\quad \dfrac{dx}{dy} = \frac{1}{3}y^{-\frac{2}{3}}$ **Hint:** Substitute $y = x^3$

b $\dfrac{dy}{dx} \times \dfrac{dx}{dy} = \dfrac{dy}{dy}$ {chain rule} $= 1$

EXERCISE 13G

1 **a** $f'(x) = 2x - 1$ **b** $f'(x) = 4x + 2$

c $f'(x) = 2x(x+1)^{\frac{1}{2}} + \frac{1}{2}x^2(x+1)^{-\frac{1}{2}}$

2 **a** $\dfrac{dy}{dx} = 2x(2x-1) + 2x^2$

b $\dfrac{dy}{dx} = 4(2x+1)^3 + 24x(2x+1)^2$

c $\dfrac{dy}{dx} = 2x(3-x)^{\frac{1}{2}} - \frac{1}{2}x^2(3-x)^{-\frac{1}{2}}$

d $\dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x-3)^2 + 2\sqrt{x}(x-3)$

e $\dfrac{dy}{dx} = 10x(3x^2-1)^2 + 60x^3(3x^2-1)$

f $\dfrac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}(x-x^2)^3 + 3\sqrt{x}(x-x^2)^2(1-2x)$

3 **a** -48 **b** $406\frac{1}{4}$ **c** $\frac{13}{3}$ **d** $\frac{11}{2}$

4 **b** $x = 3$ or $\frac{3}{5}$ **c** $x \leqslant 0$ **5** $x = -1$ and $x = -\frac{5}{3}$

EXERCISE 13H

1 **a** $\dfrac{dy}{dx} = \dfrac{7}{(2-x)^2}$ **b** $\dfrac{dy}{dx} = \dfrac{2x(2x+1) - 2x^2}{(2x+1)^2}$

c $\dfrac{dy}{dx} = \dfrac{(x^2-3) - 2x^2}{(x^2-3)^2}$

d $\dfrac{dy}{dx} = \dfrac{\frac{1}{2}x^{-\frac{1}{2}}(1-2x) + 2\sqrt{x}}{(1-2x)^2}$

e $\dfrac{dy}{dx} = \dfrac{2x(3x-x^2) - (x^2-3)(3-2x)}{(3x-x^2)^2}$

f $\dfrac{dy}{dx} = \dfrac{(1-3x)^{\frac{1}{2}} + \frac{3}{2}x(1-3x)^{-\frac{1}{2}}}{1-3x}$

2 **a** 1 **b** 1 **c** $-\frac{7}{324}$ **d** $-\frac{28}{27}$

3 **b** **i** never $\{\dfrac{dy}{dx}$ is undefined at $x = -1\}$

ii $x \leqslant 0$ and $x = 1$

4 **b** **i** $x = -2 \pm \sqrt{11}$ **ii** $x = -2$

EXERCISE 13I

1 **a** $f'(x) = 4e^{4x}$ **b** $f'(x) = e^x$

c $f'(x) = -2e^{-2x}$ **d** $f'(x) = \frac{1}{2}e^{\frac{x}{2}}$

e $f'(x) = -e^{-\frac{x}{2}}$ **f** $f'(x) = 2e^{-x}$

g $f'(x) = 2e^{\frac{x}{2}} + 3e^{-x}$ **h** $f'(x) = \dfrac{e^x - e^{-x}}{2}$

i $f'(x) = -2xe^{-x^2}$ **j** $f'(x) = e^{\frac{1}{x}} \times \dfrac{-1}{x^2}$

k $f'(x) = 20e^{2x}$ **l** $f'(x) = 40e^{-2x}$

m $f'(x) = 2e^{2x+1}$ **n** $f'(x) = \frac{1}{4}e^{\frac{x}{4}}$

o $f'(x) = -4xe^{1-2x^2}$ **p** $f'(x) = -0.02e^{-0.02x}$

2 **a** $e^x + xe^x$ **b** $3x^2e^{-x} - x^3e^{-x}$

c $\dfrac{xe^x - e^x}{x^2}$ **d** $\dfrac{1-x}{e^x}$

e $2xe^{3x} + 3x^2e^{3x}$ **f** $\dfrac{xe^x - \frac{1}{2}e^x}{x\sqrt{x}}$

g $\frac{1}{2}x^{-\frac{1}{2}}e^{-x} - x^{\frac{1}{2}}e^{-x}$ **h** $\dfrac{e^x + 2 + 2e^{-x}}{(e^{-x}+1)^2}$

3 **a** 108 **b** -1 **c** $\frac{9}{\sqrt{19}}$ **4** $k = -9$

5 **a** $\dfrac{dy}{dx} = 2^x \ln 2$ **6** $\text{P} = (0, 0)$ or $(2, \dfrac{4}{e^2})$

EXERCISE 13J

1 **a** $\dfrac{dy}{dx} = \dfrac{1}{x}$ **b** $\dfrac{dy}{dx} = \dfrac{2}{2x+1}$ **c** $\dfrac{dy}{dx} = \dfrac{1-2x}{x-x^2}$

d $\dfrac{dy}{dx} = -\dfrac{2}{x}$ **e** $\dfrac{dy}{dx} = 2x\ln x + x$

f $\dfrac{dy}{dx} = \dfrac{1-\ln x}{2x^2}$ **g** $\dfrac{dy}{dx} = e^x \ln x + \dfrac{e^x}{x}$

h $\dfrac{dy}{dx} = \dfrac{2\ln x}{x}$ **i** $\dfrac{dy}{dx} = \dfrac{1}{2x\sqrt{\ln x}}$

j $\dfrac{dy}{dx} = \dfrac{e^{-x}}{x} - e^{-x}\ln x$ **k** $\dfrac{dy}{dx} = \dfrac{\ln(2x)}{2\sqrt{x}} + \dfrac{1}{\sqrt{x}}$

l $\dfrac{dy}{dx} = \dfrac{\ln x - 2}{\sqrt{x}(\ln x)^2}$ **m** $\dfrac{dy}{dx} = \dfrac{4}{1-x}$

n $\dfrac{dy}{dx} = \ln(x^2+1) + \dfrac{2x^2}{x^2+1}$

2 **a** $\dfrac{dy}{dx} = \ln 5$ **b** $\dfrac{dy}{dx} = \dfrac{3}{x}$ **c** $\dfrac{dy}{dx} = \dfrac{4x^3+1}{x^4+x}$

d $\dfrac{dy}{dx} = \dfrac{1}{x-2}$ **e** $\dfrac{dy}{dx} = \dfrac{6}{2x+1}[\ln(2x+1)]^2$

f $\dfrac{dy}{dx} = \dfrac{1-\ln(4x)}{x^2}$ **g** $\dfrac{dy}{dx} = -\dfrac{1}{x}$

h $\dfrac{dy}{dx} = \dfrac{1}{x\ln x}$ **i** $\dfrac{dy}{dx} = \dfrac{-1}{x(\ln x)^2}$

3 **a** $\dfrac{dy}{dx} = \dfrac{-1}{1-2x}$ **b** $\dfrac{dy}{dx} = \dfrac{-2}{2x+3}$ **c** $\dfrac{dy}{dx} = 1 + \dfrac{1}{2x}$

d $\dfrac{dy}{dx} = \dfrac{1}{x} - \dfrac{1}{2(2-x)}$ **e** $\dfrac{dy}{dx} = \dfrac{1}{x+3} - \dfrac{1}{x-1}$

f $\dfrac{dy}{dx} = \dfrac{2}{x} + \dfrac{1}{3-x}$ **g** $f'(x) = \dfrac{9}{3x-4}$

h $f'(x) = \dfrac{1}{x} + \dfrac{2x}{x^2+1}$ **i** $f'(x) = \dfrac{2x+2}{x^2+2x} - \dfrac{1}{x-5}$

4 **a** 2 **b** $-\frac{5}{3}$ **5** $a = 3,\ b = -e$

EXERCISE 13K

1 **a** $\dfrac{dy}{dx} = 2\cos(2x)$ **b** $\dfrac{dy}{dx} = \cos x - \sin x$

c $\dfrac{dy}{dx} = -3\sin(3x) - \cos x$ **d** $\dfrac{dy}{dx} = \cos(x+1)$

e $\dfrac{dy}{dx} = 2\sin(3-2x)$ **f** $\dfrac{dy}{dx} = \dfrac{5}{\cos^2(5x)}$

g $\dfrac{dy}{dx} = \frac{1}{2}\cos\left(\frac{x}{2}\right) + 3\sin x$ **h** $\dfrac{dy}{dx} = \dfrac{3\pi}{\cos^2(\pi x)}$

i $\dfrac{dy}{dx} = 4\cos x + 2\sin(2x)$

2 **a** $2x - \sin x$ **b** $\dfrac{1}{\cos^2 x} - 3\cos x$

c $e^x\cos x - e^x\sin x$ **d** $-e^{-x}\sin x + e^{-x}\cos x$

e $\dfrac{\cos x}{\sin x}$ **f** $2e^{2x}\tan x + \dfrac{e^{2x}}{\cos^2 x}$ **g** $3\cos(3x)$

h $-\frac{1}{2}\sin\left(\frac{x}{2}\right)$ **i** $\dfrac{6}{\cos^2(2x)}$ **j** $\cos x - x\sin x$

k $\dfrac{x\cos x - \sin x}{x^2}$ **l** $\tan x + \dfrac{x}{\cos^2 x}$

3 **a** $2x\cos(x^2)$ **b** $-\frac{1}{2\sqrt{x}}\sin(\sqrt{x})$ **c** $-\frac{\sin x}{2\sqrt{\cos x}}$

d $2\sin x\cos x$ **e** $-3\sin x\cos^2 x$

f $-\sin x\sin(2x)+2\cos x\cos(2x)$

g $\sin x\sin(\cos x)$ **h** $-12\sin(4x)\cos^2(4x)$

i $-\frac{\cos x}{\sin^2 x}$ **j** $\frac{2\sin(2x)}{\cos^2(2x)}$

k $-\frac{8\cos(2x)}{\sin^3(2x)}$ **l** $\frac{-12}{\cos^2(\frac{x}{2})\tan^4(\frac{x}{2})}$

4 **a** $-\frac{9}{8}$ **b** 0

EXERCISE 13L

1 **a** $f''(x)=6$ **b** $f''(x)=\frac{3}{2x^{\frac{5}{2}}}$

c $f''(x)=12x-6$ **d** $f''(x)=\frac{12-6x}{x^4}$

e $f''(x)=24-48x$ **f** $f''(x)=\frac{20}{(2x-1)^3}$

2 **a** $\frac{d^2y}{dx^2}=-6x$ **b** $\frac{d^2y}{dx^2}=2-\frac{30}{x^4}$

c $\frac{d^2y}{dx^2}=-\frac{9}{4}x^{-\frac{5}{2}}$ **d** $\frac{d^2y}{dx^2}=\frac{8}{x^3}$

e $\frac{d^2y}{dx^2}=6(x^2-3x)(5x^2-15x+9)$

f $\frac{d^2y}{dx^2}=2+\frac{2}{(1-x)^3}$

3 **a** $f(2)=9$ **b** $f'(2)=10$ **c** $f''(2)=12$

5 **a** $x=1$

b $x=0,\ \pm\sqrt{6}$

6

x	-1	0	1
$f(x)$	$-$	0	$+$
$f'(x)$	$+$	$-$	$+$
$f''(x)$	$-$	0	$+$

7 **b** $f''(x)=3\sin x\cos 2x+6\cos x\sin 2x$

8 **a** $\frac{d^2y}{dx^2}=\frac{1}{x^2}$ **b** $\frac{d^2y}{dx^2}=\frac{1}{x}$

c $\frac{d^2y}{dx^2}=\frac{2}{x^2}(1-\ln x)$

9 **a** $f(1)=0$ **b** $f'(1)=3$ **c** $f''(1)=0$

10 **Hint:** Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and substitute into the equation.

REVIEW SET 13A

1 **a** -1 **b** -1 **c** 8

2 **a** $f'(x)=2x+2$ **b** $\frac{dy}{dx}=-6x$

3 **a** $f'(t)=-9.6t\ \text{m s}^{-1}$

b $f'(2)=-19.2\ \text{m s}^{-1}$
(the negative sign indicates travelling downwards)

4 **a** $f(3)=-17$ **b** $f'(3)=-17$ **c** $f''(3)=-6$

5 **a** $\frac{dy}{dx}=6x-4x^3$ **b** $\frac{dy}{dx}=1+\frac{1}{x^2}$

6 $(0,\ 0)$ **7** **a** $\frac{dy}{dx}=3x^2e^{x^3+2}$ **b** $\frac{dy}{dx}=\frac{1}{x+3}-\frac{2}{x}$

9 **a** $5+3x^{-2}$ **b** $4(3x^2+x)^3(6x+1)$

c $2x(1-x^2)^3-6x(x^2+1)(1-x^2)^2$

10 $(-2,\ 19)$ and $(1,\ -2)$

11 **a** $\frac{dy}{dx}=-2(5-4x)^{-\frac{1}{2}}$ **b** $\frac{d^2y}{dx^2}=-4(5-4x)^{-\frac{3}{2}}$

12 **a** $5\cos(5x)\ln x+\frac{\sin(5x)}{x}$

b $\cos x\cos(2x)-2\sin x\sin(2x)$

c $-2e^{-2x}\tan x+\frac{e^{-2x}}{\cos^2 x}$

13 $\frac{\sqrt{3}}{2}$

14 **a** $f'(x)=8x(x^2+3)^3$

b $g'(x)=\frac{\frac{1}{2}x(x+5)^{-\frac{1}{2}}-2(x+5)^{\frac{1}{2}}}{x^3}$

15 **a** $f''(2)=\frac{23}{4}$ **b** $f''(2)=-\frac{1}{8\sqrt{2}}$

16 **a** $10-10\cos(10x)$ **b** $\tan x$

c $5\cos(5x)\ln(2x)+\frac{\sin(5x)}{x}$

REVIEW SET 13B

1 **a** -3 **b** 3 **c** -1 **2** $f'(1)=3$

3 **a** $\frac{dy}{dx}=4x$ **b** when $x=4$, gradient $=16$

c when gradient $=-12$, $x=-3$

4 **a** $\frac{dy}{dx}=3x^2(1-x^2)^{\frac{1}{2}}-x^4(1-x^2)^{-\frac{1}{2}}$

b $\frac{dy}{dx}=\frac{(2x-3)(x+1)^{\frac{1}{2}}-\frac{1}{2}(x^2-3x)(x+1)^{-\frac{1}{2}}}{x+1}$

5 **a** $\frac{d^2y}{dx^2}=36x^2-\frac{4}{x^3}$ **b** $\frac{d^2y}{dx^2}=6x+\frac{3}{4}x^{-\frac{5}{2}}$

6 $(1,\ e)$ **7** **a** $f'(x)=\frac{e^x}{e^x+3}$ **b** $f'(x)=\frac{3}{x+2}-\frac{1}{x}$

8 When $x=1$, $\frac{dy}{dx}=0$.

9 **a** $\frac{dy}{dx}=\frac{3x^2-3}{x^3-3x}$ **b** $\frac{dy}{dx}=\frac{e^x(x-2)}{x^3}$

10 $x=-\frac{1}{2},\ \frac{3}{2}$

11 **a** $f(\pi)=\pi+1$ **b** $f'(\frac{\pi}{2})=2$ **c** $f''(\frac{3\pi}{4})=-\frac{\sqrt{2}}{2}$

12 **a** $f'(x)=\frac{1}{2}x^{-\frac{1}{2}}\cos(4x)-4x^{\frac{1}{2}}\sin(4x)$,

$$f''(x)=-\frac{1}{4}x^{-\frac{3}{2}}\cos(4x)-4x^{-\frac{1}{2}}\sin(4x)$$
$$-16x^{\frac{1}{2}}\cos(4x)$$

b $f'(\frac{\pi}{16})\approx-0.455$, $f''(\frac{\pi}{8})\approx-6.38$

14 **a** $x=-6\pm\sqrt{33}$ **b** $x=\pm\sqrt{3}$ **c** $x=0,\ \pm3$

15 **a** $f(x)=-5\sin 4x$

b $f'(x)=0$ when $x=\frac{\pi}{8},\ \frac{3\pi}{8},\ \frac{5\pi}{8},\ \frac{7\pi}{8},\ 0\leqslant x\leqslant\pi$

16 $\frac{dy}{dx}=3b\cos(bx)+2a\sin(2x)$, $a=2$, $b=\pm1$

EXERCISE 14A

1 **a** $y=-7x+11$ **b** $x-4y=-8$ **c** $y=-2x-2$

d $y=-2x+6$ **e** $y=-5x-9$ **f** $y=-5x-1$

2 **a** $x+6y=57$ **b** $x+7y=26$ **c** $x-3y=-11$

d $x+6y=43$

3 $y = 21$ and $y = -6$

4 $(\frac{1}{2}, 2\sqrt{2})$

5 $k = -5$

6 $y = -3x + 1$

7 $a = -4,\ b = 7$

8 $a = 2,\ b = 4$

10 **a** $x - 3y = -5$ **b** $x - 9y = -4$
c $x - 16y = 3$ **d** $y = -4$

11 **a** $y = 2x - \frac{7}{4}$ **b** $y = -27x - \frac{242}{3}$
c $4x + 57y = 1042$ **d** $x - 2y = -1$

12 $a = 4,\ b = 3$

13 **a** $x + ey = 2$ **b** $x + 3y = 3\ln 3 - 1$
c $2x + e^2y = \dfrac{2}{e^2} - e^2$

15 **a** $y = x$ **b** $y = x$ **c** $2x - y = \frac{\pi}{3} - \frac{\sqrt{3}}{2}$ **d** $x = \frac{\pi}{4}$

16 **a** $(-4, -64)$ **b** $(4, -31)$

17 **a** $f'(x) = 2x - \dfrac{8}{x^3}$ **b** $x = \pm\sqrt{2}$ **c** tangent is $y = 4$

18 A is $(\frac{2}{3}, 0)$, B is $(0, -2e)$

19 **a** $y = (2a-1)x - a^2 + 9$
b $y = 5x$, contact at $(3, 15)$, $y = -7x$, contact at $(-3, 21)$

20 $y = 0,\ y = 27x + 54$

21 $y = -\sqrt{14}x + 4\sqrt{14}$

22 $y = e^a x + e^a(1-a)$ so $y = ex$ is the tangent to $y = e^x$ from the origin.

23 **a** **Hint:** They must have the same y-coordinate at $x = b$ and the same gradient.
c $a = \dfrac{1}{2e}$ **d** $y = e^{-\frac{1}{2}}x - \frac{1}{2}$

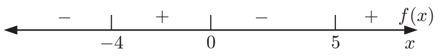
24 $\approx 63.43°$

25 **a** **Hint:** $y = f(a) + f'(a)(x - a)$
b **Hint:** Expand $f(x) = 4 - 8(x+1) - (x+1)^2 + 2(x+1)^3$
c Notice the first 2 terms in **b** are the same as the tangent line found in part **a**.

EXERCISE 14B

1 **a** A - local max, B - stationary inflection, C - local min.

b **i**



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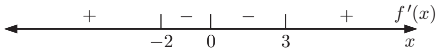
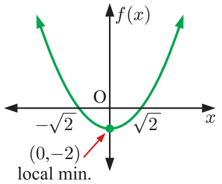
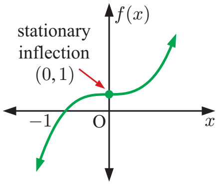


2 **a**



b



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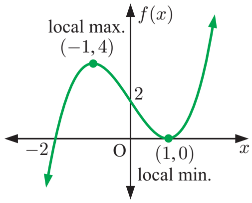

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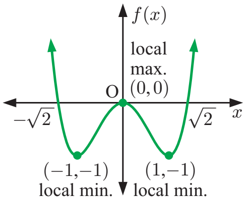

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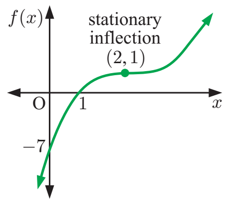

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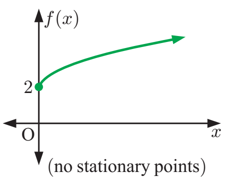

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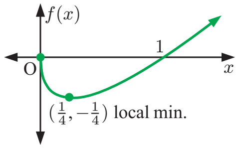

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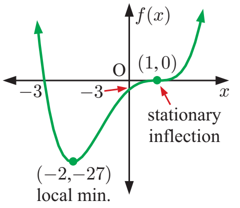

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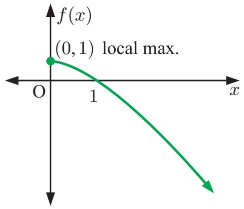

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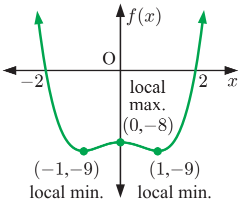


3 $x = -\dfrac{b}{2a}$, local min if $a > 0$, local max if $a < 0$

4 $a = 9$

5 **a** $a = -12,\ b = -13$
b $(-2, 3)$ local max., $(2, -29)$ local min.

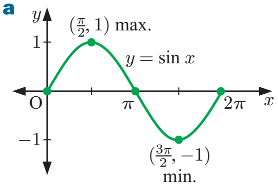
6 **a** local maximum at $(1, e^{-1})$
b local maximum at $(-2, 4e^{-2})$, local minimum at $(0, 0)$
c local minimum at $(1, e)$
d local maximum at $(-1, e)$

7 **a** $x > 0$

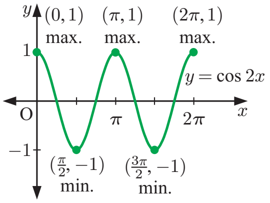
8 **a** Greatest value is 63 when $x = 5$, least value is -18 when $x = 2$.
b Greatest value is 4 when $x = 3$ and $x = 0$, least value is -16 when $x = -2$.

9 $P(x) = -9x^3 - 9x^2 + 9x + 2$

10 **a**



b



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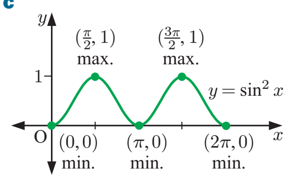

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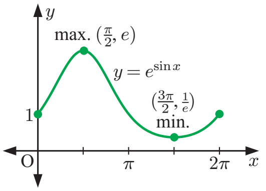
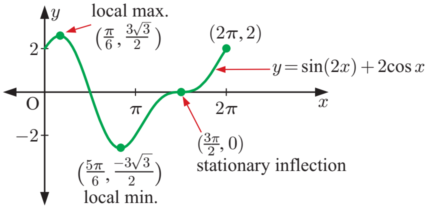


e



11 **Hint:** Find $\frac{dy}{dx}$, then determine the nature of the stationary points.

12 **Hint:** Show that as $x \to 0$, $f(x) \to -\infty$, and as $x \to \infty$, $f(x) \to 0$.

13 **a** **Hint:** Find $f'(x)$, then determine the nature of the stationary points.

b **Hint:** Show that $f(x) \geqslant 1$ for all $x > 0$.

EXERCISE 14C.1

1 **a** 7 m s^{-1} **b** $(h+5)$ m s^{-1}

c 5 m s^{-1} $= s'(1)$ is the instantaneous velocity at $t = 1$ s

d average velocity $= (2t + h + 3)$ m s^{-1}, $\lim\limits_{h \to 0} (2t + h + 3) = 2t + 3$ m s^{-1} is the instantaneous velocity at time t seconds.

2 **a** -14 cm s^{-1} **b** $(-8 - 2h)$ cm s^{-1}

c -8 cm s^{-1} $= s'(2)$

$\therefore$ instantaneous velocity $= -8$ cm s^{-1} at $t = 2$

d $-4t = s'(t) = v(t)$ is the instantaneous velocity at time t seconds.

3 **a** $\frac{2}{3}$ cm s^{-2} **b** $\dfrac{2\sqrt{1+h} - 2}{h}$ cm s^{-2}

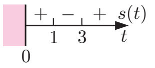
c 1 cm s^{-2} $= v'(1)$ is the instantaneous accn. at $t = 1$ s

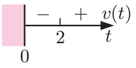
d $\dfrac{1}{\sqrt{t}}$ cm s^{-2} $= v'(t)$, the instantaneous accn. at time t

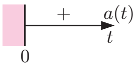
4 **a** velocity at $t = 4$ **b** acceleration at $t = 4$

EXERCISE 14C.2

1 **a** $v(t) = 2t - 4$ cm s^{-1}, $a(t) = 2$ cm s^{-2}





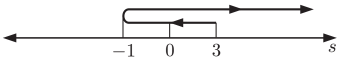


b $s(0) = 3$ cm, $v(0) = -4$ cm s^{-1}, $a(0) = 2$ cm s^{-2}
The object is initially 3 cm to the right of the origin and is moving to the left at 4 cm s^{-1}. It is accelerating at 2 cm s^{-2} to the right.

c $s(2) = -1$ cm, $v(2) = 0$ cm s^{-1}, $a(2) = 2$ cm s^{-2}
The object is instantaneously stationary, 1 cm to the left of the origin and is accelerating to the right at 2 cm s^{-2}.

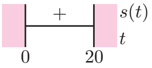
d At $t = 2$, $s(2) = 1$ cm to the left of the origin.

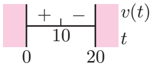
e

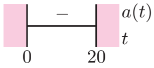


f $0 \leqslant t \leqslant 2$

2 **a** $v(t) = 98 - 9.8t$ m s^{-1}, $a(t) = -9.8$ m s^{-2}







b $s(0) = 0$ m above the ground, $v(0) = 98$ m s^{-1} skyward

c $t = 5$ s Stone is 367.5 m above the ground and moving skyward at 49 m s^{-1}. Its speed is decreasing.
$t = 12$ s Stone is 470.4 m above the ground and moving groundward at 19.6 m s^{-1}. Its speed is increasing.

d 490 m **e** 20 seconds

3 **a** 1.2 m

b $s'(t) = 28.1 - 9.8t$ represents the instantaneous velocity of the ball.

c $t = 2.87$ s. The ball has reached its maximum height and is instantaneously at rest.

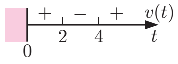
d 41.5 m

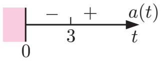
e **i** 28.1 m s^{-1} **ii** 8.5 m s^{-1} **iii** 20.9 m s^{-1}
$s'(t) \geqslant 0$ when the ball is travelling upwards.
$s'(t) \leqslant 0$ when the ball is travelling downwards.

f 5.78 s

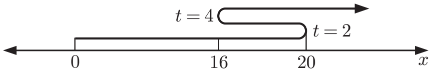
g $s''(t)$ is the rate of change of $s'(t)$, or the instantaneous acceleration.

4 **a** $v(t) = 3t^2 - 18t + 24$ m s^{-1} $a(t) = 6t - 18$ m s^{-2}





b $x(2) = 20$, $x(4) = 16$



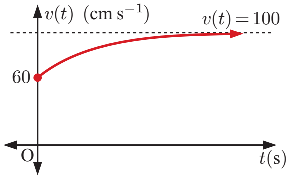
c **i** $0 \leqslant t \leqslant 2$ and $3 \leqslant t \leqslant 4$ **ii** $0 \leqslant t \leqslant 3$

d 28 m

5 **a** $v(t) = 100 - 40e^{-\frac{t}{5}}$ cm s^{-1}, $a(t) = 8e^{-\frac{t}{5}}$ cm s^{-2}

b $s(0) = 200$ cm on positive side of origin
$v(0) = 60$ cm s^{-1}, $a(0) = 8$ cm s^{-2}

c



d after 3.47 s

6 **a** $x(0) = -1$ cm, $v(0) = 0$ cm s^{-1}, $a(0) = 2$ cm s^{-2}

b At $t = \frac{\pi}{4}$ seconds, the particle is $(\sqrt{2} - 1)$ cm left of the origin, moving right at $\sqrt{2}$ cm s^{-1}, with increasing speed.

c changes direction when $t = \pi$, $x(\pi) = 3$ cm

d $0 \leqslant t \leqslant \frac{\pi}{2}$ and $\pi \leqslant t \leqslant \frac{3\pi}{2}$

7 **Hint:** Assume that $s(t) = at^2 + bt + c$
$s'(t) = v(t)$ and $s''(t) = a(t) = g$
Show that $a = \frac{1}{2}g$, $b = v(0)$, $c = 0$.

8 **a** 0.675 s

b **i** $S'(t) = u + at$ m s^{-1} **ii** $t = -\dfrac{u}{a}$ s

iii $a = -\frac{640}{99} \approx -6.46$ m s^{-2}

iv **Hint:** Substitute $t = -\dfrac{u}{a}$ into $S(t)$.

v If the speed u is doubled, then the braking distance is quadrupled ($2^2 = 4$ times).

EXERCISE 14D

1 **a** \$118 000 **b** $\dfrac{dP}{dt} = 4t - 12$, \$1000s per year

c $\dfrac{dP}{dt}$ is the rate of change in profit with time

d **i** $0 \leqslant t \leqslant 3$ years **ii** $t > 3$ years

e minimum profit is \$100 000 when $t = 3$

f $\left.\frac{dP}{dt}\right|_{t=4} = 4$ Profit is increasing at \$4000 per year after 4 years.

$\left.\frac{dP}{dt}\right|_{t=10} = 28$ Profit is increasing at \$28 000 per year after 10 years.

$\left.\frac{dP}{dt}\right|_{t=25} = 88$ Profit is increasing at \$88 000 per year after 25 years.

2 **a** **i** $Q(0) = 100$ **ii** $Q(25) = 50$ **iii** $Q(100) = 0$

b **i** decr. 1 unit per year **ii** decr. $\frac{1}{\sqrt{2}}$ units per year

c $Q'(t) = -\frac{5}{\sqrt{t}} < 0$

3 **a** 0.5 m

b $t = 4$: 9.17 m, $t = 8$: 12.5 m, $t = 12$: 14.3 m

c $t = 0$: 3.9 m year^{-1}, $t = 5$: 0.975 m year^{-1}, $t = 10$: 0.433 m year^{-1}

d As $\frac{dH}{dt} = \frac{97.5}{(t+5)^2} > 0$ for all $t \geqslant 0$, the tree is always growing.

4 **a** $C'(x) = 0.0009x^2 + 0.04x + 4$ dollars per pair

b $C'(220) = \$56.36$ per pair. This estimates the additional cost of making one more pair of jeans if 220 pairs are currently being made.

c \$56.58 This is the actual increase in cost to make an extra pair of jeans (221 rather than 220).

d $C''(x) = 0.0018x + 0.04$

$C''(x) = 0$ when $x = -22.2$. This is where the rate of change is a minimum, however it is out of the bounds of the model (you cannot make < 0 jeans!).

5 **a** **i** €4500 **ii** €4000

b **i** decrease of €210.22 per km h^{-1}

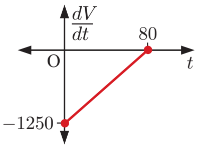
ii increase of €11.31 per km h^{-1}

c $\frac{dC}{dv} = 0$ at $v = \sqrt[3]{500\,000} \approx 79.4$ km h^{-1}

6 **a** $\frac{dV}{dt} = -1250\left(1 - \frac{t}{80}\right)$ L min^{-1}

b at $t = 0$ when the tap was first opened

c $\frac{d^2V}{dt^2} = \frac{125}{8}$ L min^{-2}



This shows that the rate of change of V is constantly increasing, so the outflow is decreasing at a constant rate.

7 **a** The near part of the lake is 2 km from the sea, the furthest part is 3 km.

b $\frac{dy}{dx} = \frac{3}{10}x^2 - x + \frac{3}{5}$

$\left.\frac{dy}{dx}\right|_{x=\frac{1}{2}} = 0.175$, height of hill is increasing as gradient is positive.

$\left.\frac{dy}{dx}\right|_{x=1\frac{1}{2}} = -0.225$, height of hill is decreasing as gradient is negative.

$\therefore$ top of the hill is between $x = \frac{1}{2}$ and $x = 1\frac{1}{2}$.

c 2.55 km from the sea, 63.1 m deep

8 **a** $k = \frac{1}{50}\ln 2 \approx 0.0139$

b **i** 20 grams **ii** 14.3 grams **iii** 1.95 grams

c 9 days and 6 minutes (216 hours)

d **i** -0.0693 g h^{-1} **ii** -2.64×10^{-7} g h^{-1}

e **Hint:** You should find $\frac{dW}{dt} = -\frac{1}{50}\ln 2 \times 20e^{-\frac{1}{50}\ln 2t}$

9 **a** $k = \frac{1}{15}\ln\left(\frac{19}{3}\right) \approx 0.123$ **b** 100°C

c $c = -k \approx -0.123$

d **i** decreasing at 11.7°C min^{-1}

ii decreasing at 3.42°C min^{-1}

iii decreasing at 0.998°C min^{-1}

10 **a** 43.9 cm **b** 10.4 years

c **i** growing at 5.45 cm per year

ii growing at 1.88 cm per year

11 **a** $A(0) = 0$

b **i** $k = \frac{\ln 2}{3}$ (≈ 0.231)

ii 0.728 litres of alcohol produced per hour

12 $\frac{21}{\sqrt{2}}$ cm^2 per radian

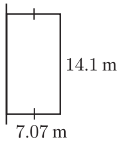
13 **a** rising at 2.73 m per hour **b** rising

14 **b** **i** 0 **ii** 1 **iii** ≈ 1.11

EXERCISE 14E

1 250 items

2 **b** $L_{\text{min}} \approx 28.3$ m, $x \approx 7.07$ m **c**

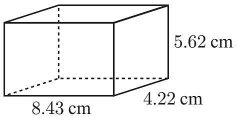


3 10 blankets **4** 14.8 km h^{-1} **5** at 4.41 months old

6 **a** **Hint:** $V = 200 = 2x \times x \times h$

b **Hint:** Show $h = \frac{100}{x^2}$ and substitute into the surface area equation.

c $SA_{\text{min}} \approx 213$ cm^2, $x \approx 4.22$ cm **d**



7 20 kettles **8** $C\left(\frac{1}{\sqrt{2}}, e^{-\frac{1}{2}}\right)$

9 **a** Recall that $V_{\text{cylinder}} = \pi r^2 h$ and that $1\text{ L} = 1000\text{ cm}^3$.

b Recall that $SA_{\text{cylinder}} = 2\pi r^2 + 2\pi rh$.

c radius ≈ 5.42 cm, height ≈ 10.8 cm

10 **b** $\theta \approx 1.91$, $A \approx 237$ cm^2 **11** **b** 6 cm × 6 cm

12 **a** $0 \leqslant x \leqslant 63.7$

b $l = 100$ m, $x = \frac{100}{\pi} \approx 31.83$ m, $A = \frac{20\,000}{\pi} \approx 6366$ m^2

13 after 13.8 weeks **14** after 40 minutes

15 **c** $\theta = 30°$, $A \approx 130$ cm^2

16 **a** **Hint:** Show that $AC = \frac{\theta}{360} \times 2\pi \times 10$

b **Hint:** Show that $2\pi r = AC$

c **Hint:** Use the result from **b** and Pythagoras' theorem.

d $V = \frac{1}{3}\pi\left(\frac{\theta}{36}\right)^2\sqrt{100 - \left(\frac{\theta}{36}\right)^2}$ **e** $\theta \approx 294°$

17 1 hour 34 min 53 s when $\theta \approx 36.9°$ **18** 9.87 m

EXERCISE 14F

1 a is decreasing at 7.5 units per second

2 increasing at 1 cm per minute

3 **a** 4π m^2 per second **b** 8π m^2 per second

4 increasing at 6π m^2 per minute

5 decreasing at $0.16\ \text{m}^3$ per minute **6** $\frac{20}{3}$ cm per minute

7 $\frac{25\sqrt{3}}{6} \approx 7.22$ cm per minute

8 decreasing at $\frac{250}{13} \approx 19.2\ \text{m s}^{-1}$

9 **a** $0.2\ \text{m s}^{-1}$ **b** $\frac{4}{45}\ \text{m s}^{-1}$

10 decreasing at $\frac{\sqrt{2}}{100}$ radians per second

11 increasing at 0.12 radians per minute

REVIEW SET 14A

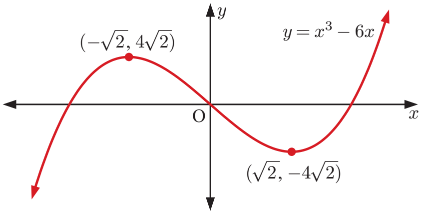
1 **a** $y = 4x + 2$ **b** $y = 4x + 4\ln 2 - 4$ **c** $y = e^2$

2 $a = \frac{5}{2},\ b = -\frac{3}{2}$

3 **a** $a = -6$

b local max. $(-\sqrt{2}, 4\sqrt{2})$, local min. $(\sqrt{2}, -4\sqrt{2})$

c



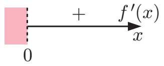
4 **a** $y = \frac{1}{5}x - \frac{11}{5}$ (or $x - 5y = 11$) **b** $y = -4x + 14$

5 $\frac{3267}{152}$ units^2 **6** $a = 64$ **7** P(0, 7.5), Q(3, 0)

9 $3x - 4y = -5$

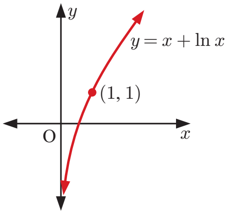
10 **a** $x > 0$

b Sign diagram of $f'(x)$



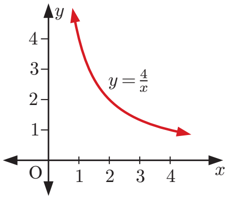
$f(x)$ is increasing for all $x > 0$.

c



d normal is $x + 2y = 3$

11 **a**



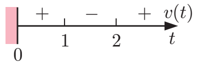
b $y = -\dfrac{4}{k^2}x + \dfrac{8}{k}$

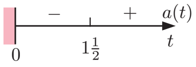
c A(2k, 0), $\text{B}\left(0, \dfrac{8}{k}\right)$

d Area = 8 units^2

e $k = 2$

12 **a** $v(t) = (6t^2 - 18t + 12)\ \text{cm s}^{-1}$, $a(t) = (12t - 18)\ \text{cm s}^{-2}$



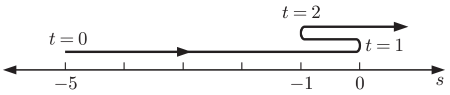


b $s(0) = -5$ cm (5 cm to the left of origin)
$v(0) = 12\ \text{cm s}^{-1}$ towards origin
$a(0) = -18\ \text{cm s}^{-2}$ (reducing speed)

c At $t = 2$, particle is 1 cm to the left of the origin, is stationary and is accelerating towards the origin.

d $t = 1,\ s = 0$ and $t = 2,\ s = -1$

e



f $1 \leqslant t \leqslant 1\frac{1}{2}$ and $t \geqslant 2$

13 **b** $k = 9$

14 **a** $x(0) = 3$ cm, $x'(0) = 2\ \text{cm s}^{-1}$, $x''(0) = 0\ \text{cm s}^{-2}$

b $t = \frac{\pi}{4}$ s and $\frac{3\pi}{4}$ s **c** 4 cm

15 6 cm from each end

16 **a** $y = \dfrac{1}{x^2},\ x > 0$

c base is 1.26 m square, height 0.630 m

17 **a** $v(t) = 15 + \dfrac{120}{(t+1)^3}\ \text{cm s}^{-1}$, $a(t) = \dfrac{-360}{(t+1)^4}\ \text{cm s}^{-2}$

b At $t = 3$, particle is 41.25 cm to the right of the origin, moving to the right at 16.88 cm s^{-1} and decelerating at 1.41 cm s^{-2}.

c speed is never increasing

18 $\text{A}\left(\dfrac{1}{2}, \dfrac{1}{e}\right)$ **19** $\frac{20\sqrt{10}}{3} \approx 21.1$ m per minute

20 **a** $V(r) = \frac{8}{9}\pi r^3\ \text{m}^3$

b $\dfrac{dr}{dt} = -\frac{8}{375\pi} \approx -0.006\,79\ \text{m min}^{-1}$

REVIEW SET 14B

1 **a** $x = 1$ **b** $ex - 2y = e - \dfrac{2}{e}$ **c** $y = 16x - \frac{127}{2}$

2 $a = -14,\ b = 21$

3 **a** $f(3) = 2,\ f'(3) = -1$ **b** $f(x) = x^2 - 7x + 14$

4 **a** $2x + 3y = \frac{2\pi}{3} + 2\sqrt{3}$ **b** $\sqrt{2}y - 4x = 1 - 2\pi$

5 $p = 1,\ q = -8$ **6** $(-2, -25)$ **7** $a = \frac{1}{2}$

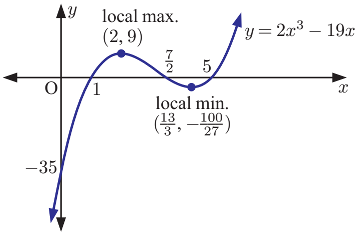
8 **a** local minimum at (0, 1) **c** $f''(x) = e^x$

9 $(0,\ \ln 4 - 1)$

10 **a** y-intercept $= -35$ **b** $x = 1,\ \frac{7}{2},\ 5$

c local maximum at (2, 9), local minimum at $(\frac{13}{3}, -\frac{100}{27})$

d



11 $\text{BC} = \frac{8\sqrt{10}}{3}$ units

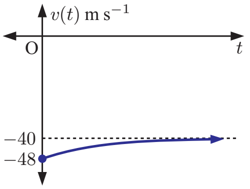
12 **a** 60 cm **b** **i** 4.24 years **ii** 201 years

c **i** 16 cm per year **ii** 1.95 cm per year

13 **a** $v(t) = -8e^{-\frac{t}{10}} - 40\ \text{m s}^{-1}$
$a(t) = \frac{4}{5}e^{-\frac{t}{10}}\ \text{m s}^{-2}\quad \{t \geqslant 0\}$

b $s(0) = 80$ m
$v(0) = -48\ \text{m s}^{-1}$
$a(0) = 0.8\ \text{m s}^{-2}$

c



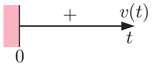
d $t = 10\ln 2$ seconds

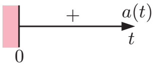
14 **a** **i** \$535 **ii** \$1385.79

b **i** $-\$0.267$ per km h^{-1} **ii** \$2.33 per km h^{-1}

c 51.3 km h^{-1}

15 **a** $v(t) = 3 - \dfrac{1}{2\sqrt{t+1}}$ $\qquad a(t) = \dfrac{1}{4(t+1)^{\frac{3}{2}}}$





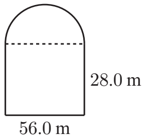
b $x(0) = -1$, $v(0) = 2.5$, $a(0) = 0.25$

Particle is 1 cm to the left of the origin, is travelling to the right at 2.5 cm s^{-1}, and accelerating at 0.25 cm s^{-2}.

c Particle is 21 cm to the right of the origin, is travelling to the right at 2.83 cm s^{-1}, and accelerating at 0.009 26 cm s^{-2}.

d never changes direction **e** never decreasing

16 **b** $A = 200x - 2x^2 - \frac{1}{2}\pi x^2$ **c**



17 **a** $v(0) = 0$ cm s^{-1}, $v(\frac{1}{2}) = -\pi$ cm s^{-1}, $v(1) = 0$ cm s^{-1}, $v(\frac{3}{2}) = \pi$ cm s^{-1}, $v(2) = 0$ cm s^{-1}

b $0 \leqslant t \leqslant 1$, $2 \leqslant t \leqslant 3$, $4 \leqslant t \leqslant 5$, etc.

So, for $2n \leqslant t \leqslant 2n+1$, $n \in \{0, 1, 2, 3, ...\}$

18 $x = \dfrac{k}{2}\left(1 - \dfrac{1}{\sqrt{3}}\right)$ **19** 3.60 m s^{-1}

20 increasing at 0.128 radians per second

21 **a** $\frac{\sqrt{3}}{2}\pi$ cm s^{-1} **b** 0 cm s^{-1}

22 **a** **i** $y = -\dfrac{a^2}{4b} + \dfrac{a}{2b}x$ **ii** when $y = 0$, $x = \dfrac{a}{2}$

b **i** $y = -\dfrac{2b}{a}x + b$ **ii** when $x = 0$, $y = b$

iii **Hint:** Let P$''$ be the point on the line $y = -b$ where the distance to P is shortest.
Show that FP = P$''$P.

c **i** **Hint:** Show that $\triangle$FPP$' \cong \triangle$P$''$PP$'$.

ii **Hint:** Show that the tangents meet at $\left(\dfrac{a+c}{2}, \dfrac{ac}{4b}\right)$.

EXERCISE 15A.1

1 **a** **i** 0.6 units^2 **ii** 0.4 units^2 **b** 0.5 units^2

2 **a** 0.737 units^2 **b** 0.653 units^2

3

n	A_L	A_U
10	2.1850	2.4850
25	2.2736	2.3936
50	2.3034	2.3634
100	2.3184	2.3484
500	2.3303	2.3363

converges to $\frac{7}{3}$

4 **a** **i**

n	A_L	A_U
5	0.160 00	0.360 00
10	0.202 50	0.302 50
50	0.240 10	0.260 10
100	0.245 03	0.255 03
500	0.249 00	0.251 00
1000	0.249 50	0.250 50
10 000	0.249 95	0.250 05

ii

n	A_L	A_U
5	0.400 00	0.600 00
10	0.450 00	0.550 00
50	0.490 00	0.510 00
100	0.495 00	0.505 00
500	0.499 00	0.501 00
1000	0.499 50	0.500 50
10 000	0.499 95	0.500 05

iii

n	A_L	A_U
5	0.549 74	0.749 74
10	0.610 51	0.710 51
50	0.656 10	0.676 10
100	0.661 46	0.671 46
500	0.665 65	0.667 65
1000	0.666 16	0.667 16
10 000	0.666 62	0.666 72

iv

n	A_L	A_U
5	0.618 67	0.818 67
10	0.687 40	0.787 40
50	0.738 51	0.758 51
100	0.744 41	0.754 41
500	0.748 93	0.750 93
1000	0.749 47	0.750 47
10 000	0.749 95	0.750 05

b **i** $\frac{1}{4}$ **ii** $\frac{1}{2}$ **iii** $\frac{2}{3}$ **iv** $\frac{3}{4}$ **c** area $= \dfrac{1}{a+1}$

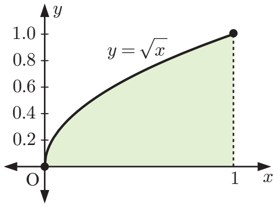
5 **a**

n	Rational bounds for π
10	$2.9045 < \pi < 3.3045$
50	$3.0983 < \pi < 3.1783$
100	$3.1204 < \pi < 3.1604$
200	$3.1312 < \pi < 3.1512$
1000	$3.1396 < \pi < 3.1436$
10 000	$3.1414 < \pi < 3.1418$

b $n = 10\,000$

EXERCISE 15A.2

1 **a**



b

n	A_L	A_U
5	0.5497	0.7497
10	0.6105	0.7105
50	0.6561	0.6761
100	0.6615	0.6715
500	0.6656	0.6676

c $\int_0^1 \sqrt{x}\,dx \approx 0.67$

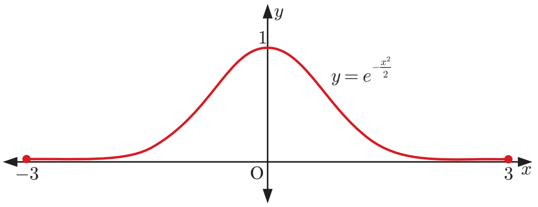
2 **a** $A_L = \dfrac{2}{n}\displaystyle\sum_{i=0}^{n-1}\sqrt{1+{x_i}^3}$, $\quad A_U = \dfrac{2}{n}\displaystyle\sum_{i=1}^{n}\sqrt{1+{x_i}^3}$

b

n	A_L	A_U
50	3.2016	3.2816
100	3.2214	3.2614
500	3.2373	3.2453

c $\int_0^2 \sqrt{1+x^3}\,dx \approx 3.24$

3 **a**



b upper ≈ 1.2506, lower ≈ 1.2493

c upper ≈ 1.2506, lower ≈ 1.2493

d $\int_{-3}^{3} e^{-\frac{x^2}{2}}\,dx \approx 2.4999$ compared to $\sqrt{2\pi} \approx 2.5066$

4 **a** 18 **b** 4.5 **c** 2π

EXERCISE 15B

1 **a** **i** $\dfrac{x^2}{2}$ **ii** $\dfrac{x^3}{3}$ **iii** $\dfrac{x^6}{6}$ **iv** $-\dfrac{1}{x}$

v $-\dfrac{1}{3x^3}$ **vi** $\frac{3}{4}x^{\frac{4}{3}}$ **vii** $2\sqrt{x}$

b The antiderivative of x^n is $\dfrac{x^{n+1}}{n+1}$ $(n \neq -1)$.

2 **a** **i** $\frac{1}{2}e^{2x}$ **ii** $\frac{1}{5}e^{5x}$ **iii** $2e^{\frac{1}{2}x}$ **iv** $100e^{0.01x}$

v $\frac{1}{\pi}e^{\pi x}$ **vi** $3e^{\frac{x}{3}}$

b The antiderivative of e^{kx} is $\dfrac{1}{k}e^{kx}$.

3 **a** $\dfrac{d}{dx}(x^3 + x^2) = 3x^2 + 2x$

$\therefore$ the antiderivative of $6x^2 + 4x = 2x^3 + 2x^2$

b $\dfrac{d}{dx}(e^{3x+1}) = 3e^{3x+1}$

$\therefore$ the antiderivative of $e^{3x+1} = \frac{1}{3}e^{3x+1}$

c $\dfrac{d}{dx}(x\sqrt{x}) = \frac{3}{2}\sqrt{x}$

$\therefore$ the antiderivative of $\sqrt{x} = \frac{2}{3}x\sqrt{x}$

d $\dfrac{d}{dx}(2x+1)^4 = 8(2x+1)^3$

$\therefore$ the antiderivative of $(2x+1)^3 = \frac{1}{8}(2x+1)^4$

EXERCISE 15C

1 **a** $\frac{1}{4}$ units^2 **b** $2\frac{1}{3}$ units^2 **c** $\frac{2}{3}$ units^2

3 **a** $3\frac{3}{4}$ units^2 **b** $24\frac{2}{3}$ units^2 **c** $\frac{-2+4\sqrt{2}}{3}$ units^2

d ≈ 3.48 units^2 **e** 2 units^2

4 **c** **i** $\int_0^1 (-x^2)\,dx = -\frac{1}{3}$, the area between $y = -x^2$ and the x-axis from $x = 0$ to $x = 1$ is $\frac{1}{3}$ units^2.

ii $\int_0^1 (x^2 - x)\,dx = -\frac{1}{6}$, the area between $y = x^2 - x$ and the x-axis from $x = 0$ to $x = 1$ is $\frac{1}{6}$ units^2.

iii $\int_{-2}^{0} 3x\,dx = -6$, the area between $y = 3x$ and the x-axis from $x = -2$ to $x = 0$ is 6 units^2

d $-\pi$

EXERCISE 15D

1 $\dfrac{dy}{dx} = 7x^6$, $\int x^6\,dx = \frac{1}{7}x^7 + c$

2 $\dfrac{dy}{dx} = 3x^2 + 2x$, $\int (3x^2 + 2x)\,dx = x^3 + x^2 + c$

3 $\dfrac{dy}{dx} = 2e^{2x+1}$, $\int e^{2x+1}\,dx = \frac{1}{2}e^{2x+1} + c$

4 $\dfrac{dy}{dx} = 8(2x+1)^3$, $\int (2x+1)^3\,dx = \frac{1}{8}(2x+1)^4 + c$

5 $\dfrac{dy}{dx} = \frac{3}{2}\sqrt{x}$, $\int \sqrt{x}\,dx = \frac{2}{3}x\sqrt{x} + c$

6 $\dfrac{dy}{dx} = -\dfrac{1}{2x\sqrt{x}}$, $\displaystyle\int \frac{1}{x\sqrt{x}}\,dx = -\frac{2}{\sqrt{x}} + c$

7 $\dfrac{dy}{dx} = -2\sin 2x$, $\int \sin 2x\,dx = -\frac{1}{2}\cos 2x + c$

8 $\dfrac{dy}{dx} = -5\cos(1 - 5x)$,

$\int \cos(1 - 5x)\,dx = -\frac{1}{5}\sin(1 - 5x) + c$

9 $\int (2x - 1)(x^2 - x)^2\,dx = \frac{1}{3}(x^2 - x)^3 + c$

11 $\dfrac{dy}{dx} = \dfrac{-2}{\sqrt{1 - 4x}}$, $\displaystyle\int \frac{1}{\sqrt{1 - 4x}}\,dx = -\frac{1}{2}\sqrt{1 - 4x} + c$

EXERCISE 15E.1

1 **a** $\dfrac{x^5}{5} - \dfrac{x^3}{3} - \dfrac{x^2}{2} + 2x + c$ **b** $x^5 - x^4 - 2x^3 - 7x + c$

c $\frac{2}{3}x^{\frac{3}{2}} + e^x + c$ **d** $3e^x + \frac{1}{3}x^3 + c$

e $\frac{2}{5}x^{\frac{5}{2}} - 2x + c$ **f** $-2x^{-\frac{1}{2}} + 2x^2 + c$

g $\frac{1}{8}x^4 - \frac{1}{5}x^5 + \frac{3}{4}x^{\frac{4}{3}} + c$ **h** $\dfrac{x^2}{4} + \frac{1}{3}x^3 - e^x + c$

i $5e^x + \frac{1}{12}x^4 - \frac{2}{3}x^{\frac{3}{2}} + c$

2 **a** $-3\cos x - 2x + c$ **b** $2x^2 - 2\sin x + c$

c $-\cos x - 2\sin x + e^x + c$ **d** $\frac{2}{7}x^3\sqrt{x} + 10\cos x + c$

e $\frac{1}{9}x^3 - \frac{1}{6}x^2 + \sin x + c$ **f** $\cos x + \frac{4}{3}x\sqrt{x} + c$

3 **a** $\frac{1}{3}x^3 + \frac{3}{2}x^2 - 2x + c$ **b** $\frac{2}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$

c $2e^x + \dfrac{1}{x} + c$ **d** $-2x^{-\frac{1}{2}} - 8x^{\frac{1}{2}} + c$

e $\frac{4}{3}x^3 + 2x^2 + x + c$ **f** $\frac{1}{3}x^3 + 2x - \dfrac{1}{x} + c$

g $\frac{4}{3}x^{\frac{3}{2}} - 2x^{\frac{1}{2}} + c$ **h** $2x^{\frac{1}{2}} + 8x^{-\frac{1}{2}} - \frac{20}{3}x^{-\frac{3}{2}} + c$

i $\frac{1}{4}x^4 + x^3 + \frac{3}{2}x^2 + x + c$

4 **a** $\frac{2}{3}x^{\frac{3}{2}} + \frac{1}{2}\sin x + c$ **b** $2e^t + 4\cos t + c$

c $3\sin t + \cos t + c$

5 **a** $y = 6x + c$ **b** $y = \frac{4}{3}x^3 + c$

c $y = \frac{10}{3}x\sqrt{x} - \frac{1}{3}x^3 + c$ **d** $y = -\dfrac{1}{x} + c$

e $y = 2e^x - 5x + c$ **f** $y = x^4 + x^3 + c$

6 **a** $f(x) = x - 2x^2 + \frac{4}{3}x^3 + c$

b $f(x) = \frac{2}{3}x^{\frac{3}{2}} - 4\sqrt{x} + c$ **c** $f(x) = x + \dfrac{5}{x} + c$

EXERCISE 15E.2

1 **a** $f(x) = x^2 - x + 3$ **b** $f(x) = x^3 + x^2 - 7$

c $f(x) = e^x + 2\sqrt{x} - 1 - e$ **d** $f(x) = \frac{1}{2}x^2 - 4\sqrt{x} + \frac{11}{2}$

2 **a** $f(x) = \dfrac{x^3}{3} - 4\sin x + 3$

b $f(x) = 2\sin x + 3\cos x - 2\sqrt{2}$

3 **a** $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{1}{3}$

b $f(x) = 4x^{\frac{5}{2}} + 4x^{\frac{3}{2}} - 4x + 5$

c $f(x) = -\cos x - x + 4$ **d** $f(x) = \frac{1}{3}x^3 - \frac{16}{3}x + 5$

EXERCISE 15F

1 **a** $\frac{1}{8}(2x+5)^4 + c$ **b** $\dfrac{1}{2(3-2x)} + c$

c $\dfrac{-2}{3(2x-1)^3} + c$ **d** $\frac{1}{32}(4x-3)^8 + c$

e $\frac{2}{9}(3x-4)^{\frac{3}{2}} + c$ **f** $-4\sqrt{1-5x} + c$

g $-\frac{3}{5}(1-x)^5 + c$ **h** $-2\sqrt{3-4x} + c$

2 **a** $-\frac{1}{3}\cos(3x) + c$ **b** $-\frac{1}{2}\sin(-4x) + x + c$

c $6\sin\left(\frac{x}{2}\right) + c$ **d** $-\frac{3}{2}\cos(2x) + e^{-x} + c$

e $-\cos\left(2x + \frac{\pi}{6}\right) + c$ **f** $3\sin\left(\frac{\pi}{4} - x\right) + c$

g $\frac{1}{2}\sin(2x) - \frac{1}{2}\cos(2x) + c$

h $-\frac{2}{3}\cos(3x) + \frac{5}{4}\sin(4x) + c$

i $\frac{1}{16}\sin(8x) + 3\cos x + c$

3 $y = \frac{1}{3}(2x-7)^{\frac{3}{2}} + 2$ **4** $(-8, -19)$

5 **a** $\frac{1}{2}(2x-1)^3 + c$ **b** $\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{1}{3}x^3 + c$

c $-\frac{1}{12}(1-3x)^4 + c$ **d** $x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + c$

e $-\frac{8}{3}(5-x)^{\frac{3}{2}} + c$ **f** $\frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x + c$

6 **a** $2e^x + \frac{5}{2}e^{2x} + c$ **b** $\frac{3}{5}e^{5x-2} + c$

c $-\frac{1}{3}e^{7-3x} + c$ **d** $\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$

e $-\frac{1}{2}e^{-2x} - 4e^{-x} + 4x + c$ **f** $\frac{1}{2}x^2 + 5(1-x)^{-1} + c$

7 $y = x - 2e^x + \frac{1}{2}e^{2x} + \frac{11}{2}$

8 $p = -\frac{1}{4}$, $f(x) = \frac{1}{2}\cos(\frac{1}{2}x) + \frac{1}{2}$

10 $f(x) = -e^{-2x} + 4$

11 $f(x) = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{8}e^{-4x} + \frac{1}{8}e^{-4} - \frac{2}{3}$

EXERCISE 15G

1 **a** $\int_1^4 \sqrt{x}\,dx = \frac{14}{3}$, $\int_1^4 (-\sqrt{x})\,dx = -\frac{14}{3}$

b $\int_0^1 x^7\,dx = \frac{1}{8}$, $\int_0^1 (-x^7)\,dx = -\frac{1}{8}$

2 **a** $\frac{1}{3}$ **b** $\frac{7}{3}$ **c** $\frac{8}{3}$ **d** 1

3 **a** -4 **b** 6.25 **c** 2.25 **4** **a** $\frac{1}{3}$ **b** $\frac{2}{3}$ **c** 1

5 **a** $\frac{1}{4}$ **b** $\frac{2}{3}$ **c** $e - 1$ (≈ 1.72) **d** $\frac{1}{2}$

e $1\frac{1}{2}$ **f** $6\frac{2}{3}$ **g** $\ln 3$ (≈ 1.10) **h** $\frac{1}{2}$

i ≈ 1.52 **j** 2 **k** $e - 1$ (≈ 1.72) **l** $\frac{1}{3}$

6 $m = -1$ or $\frac{4}{3}$ **7** **a** $\frac{\pi}{8} + \frac{1}{4}$ **b** $\frac{\pi}{4}$

8 **a** 6.5 **b** -9 **c** 0 **d** -2.5

9 **a** 2π **b** -4 **c** $\frac{\pi}{2}$ **d** $\frac{5\pi}{2} - 4$

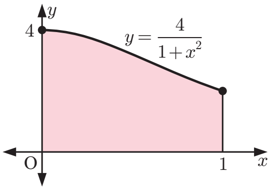
10 **a** $\int_2^7 f(x)\,dx$ **b** $\int_1^9 g(x)\,dx$

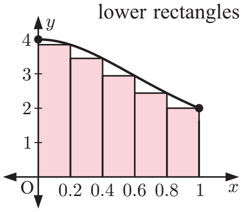
11 **a** -5 **b** 4

12 **a** 4 **b** 0 **c** -8 **d** $k = -\frac{7}{4}$ **13** 0

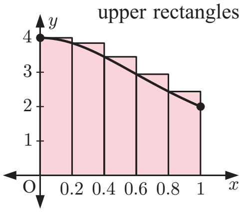
REVIEW SET 15A

1 **a**





lower rectangles



upper rectangles

b

n	A_L	A_U
5	2.9349	3.3349
50	3.1215	3.1615
100	3.1316	3.1516
500	3.1396	3.1436

c $\displaystyle\int_0^1 \frac{4}{1+x^2}\,dx \approx 3.1416$

2 **a** 2π **b** 4

3 **a** $8\sqrt{x} + c$ **b** $-\frac{1}{4}\cos(4x-5) + c$ **c** $-\frac{1}{3}e^{4-3x} + c$

4 **a** $12\frac{4}{9}$ **b** $\sqrt{2}$

5 $\dfrac{dy}{dx} = \dfrac{x}{\sqrt{x^2-4}}$, $\displaystyle\int \frac{x}{\sqrt{x^2-4}}\,dx = \sqrt{x^2-4} + c$

6 $b = \frac{\pi}{4}, \frac{3\pi}{4}$

7 **a** $y = \frac{1}{5}x^5 - \frac{2}{3}x^3 + x + c$ **b** $y = 400x + 40e^{-\frac{x}{2}} + c$

8 $f(x) = 3x^3 + 5x^2 + 6x - 1$ **9** $a = \ln\sqrt{2}$

10 **a** $f(x) = \frac{1}{4}x^4 + \frac{1}{3}x^3 - \frac{10}{3}x + 3$ **b** $3x + 26y = 84$

11 **a** $e^{3x} + 6e^{2x} + 12e^x + 8$ **b** $\frac{1}{3}e^3 + 3e^2 + 12e - 7\frac{1}{3}$

REVIEW SET 15B

1 **a** $A = \frac{17}{4}$, $B = \frac{25}{4}$ **b** $\int_0^2 (4-x^2)\,dx \approx \frac{21}{4}$

2 **a** $-2e^{-x} + 3x + c$ **b** $\frac{2}{3}x\sqrt{x} - 2\sqrt{x} + c$

c $9x + 3e^{2x-1} + \frac{1}{4}e^{4x-2} + c$

3 $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + 2\frac{1}{6}$ **4** $\frac{2}{3}(\sqrt{5} - \sqrt{2})$

5 $\dfrac{d}{dx}(3x^2+x)^3 = 3(3x^2+x)^2(6x+1)$

$\int (3x^2+x)^2(6x+1)\,dx = \frac{1}{3}(3x^2+x)^3 + c$

6 **a** 6 **b** 3 **7** $f\left(\frac{\pi}{2}\right) = 3 - \frac{\pi}{2}$ **8** $e^{-\pi}$

9 $\dfrac{1}{2(n+1)}(2x+3)^{n+1} + c$, $n \neq -1$

10 $a = \frac{1}{3}$, $f'(x) = 2\sqrt{x} + \dfrac{1}{3\sqrt{x}}$ is never 0 as $\sqrt{x} \geqslant 0$ for all x

$\therefore$ $f'(x) > 0$ for all x

11 $a = 0$ or ± 3

EXERCISE 16A

1 **a** 30 units^2 **b** $\frac{9}{2}$ units^2 **c** $\frac{27}{2}$ units^2 **d** 2 units^2

2 **a** $\frac{1}{3}$ units^2 **b** 2 units^2 **c** $63\frac{3}{4}$ units^2
d $(e-1)$ units^2 **e** $20\frac{5}{6}$ units^2 **f** 18 units^2
g $\frac{1}{2}$ units^2 **h** $4\frac{1}{2}$ units^2 **i** $(2e-\dfrac{2}{e})$ units^2

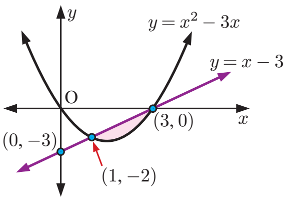
3 $\frac{2}{3}$ units^2

EXERCISE 16B

1 **a** $4\frac{1}{2}$ units^2 **b** $(1+e^{-2})$ units^2 **c** $1\frac{5}{27}$ units^2
d 2 units^2 **e** $2\frac{1}{4}$ units^2 **f** $(\frac{\pi}{2}-1)$ units^2

2 $10\frac{2}{3}$ units^2

3 **a**

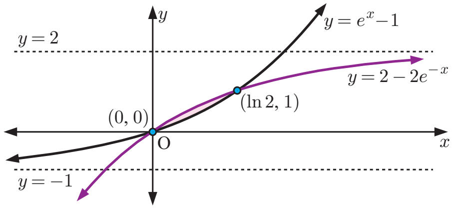


b $(1,-2)$ and $(3,0)$

c $1\frac{1}{3}$ units^2

4 $\frac{1}{3}$ units^2

5 **a**

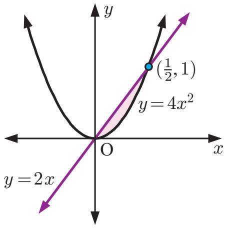


b $(0,0)$ and $(\ln 2, 1)$

c enclosed area $= 3\ln 2 - 2$ (≈ 0.0794) units^2

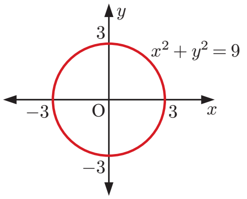
6 $\frac{1}{2}$ units^2

7



enclosed area $= \frac{1}{12}$ units^2

8



a Rearranging $x^2+y^2=9$ gives $y=\pm\sqrt{9-x^2}$. The upper half has $y \geqslant 0$, so $y=\sqrt{9-x^2}$.

b $\frac{9\pi}{4}$

9 **a** $40\frac{1}{2}$ units^2 **b** 8 units^2 **c** 8 units^2

10 **a** C_1 is $y=\sin x$, C_2 is $y=3\sin x$ **b** 4 units^2

11 **a** $\int_3^5 f(x)\,dx = -$ (area between $x=3$ and $x=5$)

b $\int_1^3 f(x)\,dx - \int_3^5 f(x)\,dx + \int_5^7 f(x)\,dx$

12 **a** C_1 is $y=\frac{1}{2}+\frac{1}{2}\cos(2x)$, C_2 is $y=\cos(2x)$

b $A(0,1)$, $B(\frac{\pi}{4},0)$, $C(\frac{\pi}{2},0)$, $D(\frac{3\pi}{4},0)$, $E(\pi,1)$

c Area $= \int_0^{\pi} (\frac{1}{2}+\frac{1}{2}\cos(2x)-\cos(2x))\,dx$

13 If $h(x) \geqslant 0$ on $a \leqslant x \leqslant b$, the area between $y=h(x)$ and the x-axis is $\int_a^b h(x)\,dx$. If $h(x)<0$ on $a \leqslant x \leqslant b$, the area between $y=h(x)$ and the x-axis is $\int_a^b -h(x)\,dx$.

$\therefore$ the area between $y=h(x)$ and the x-axis on $a \leqslant x \leqslant b$ is $\int_a^b |h(x)|\,dx$.

Letting $h(x)=f(x)-g(x)$, the area between $y=f(x)-g(x)$ and the x-axis $y=0$ on $a \leqslant x \leqslant b$ is $\int_a^b |f(x)-g(x)|\,dx$.

Equivalently, the area between $y=f(x)$ and $y=g(x)$ on $a \leqslant x \leqslant b$ is $\int_a^b |f(x)-g(x)|\,dx$.

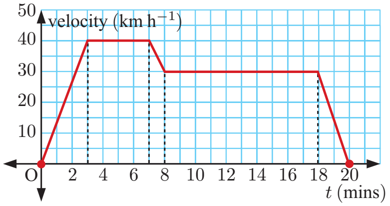
14 $b \approx 1.3104$ **15** $a=\sqrt{3}$

EXERCISE 16C.1

1 110 m

2 **a** **i** travelling forwards
ii travelling backwards (opposite direction)
b 16 km **c** 8 km from starting point (on positive side)

3 **a**



b 9.75 km

EXERCISE 16C.2

1 **a** $s(t)=t-t^2+2$ cm **b** $\frac{1}{2}$ cm **c** 0 cm

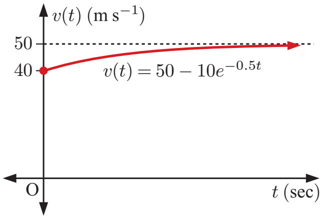
2 **a** $s(t)=\frac{1}{3}t^3-\frac{1}{2}t^2-2t$ cm **b** $5\frac{1}{6}$ cm
c $1\frac{1}{2}$ cm left of its starting point

3 $\frac{\sqrt{3}+2}{4}$ m

4 **a** $s(t)=32t+2t^2+16$ m
b no change of direction
so displacement $= s(t_1)-s(0) = \int_0^{t_1} (32+4t)\,dt$
c acceleration $= 4$ m s^{-2}

5 **a** 41 units **b** 34 units **6** **b** 2 m

7 **a** 40 m s^{-1} **b** 47.8 m s^{-1} **c** 1.39 seconds
d as $t \to \infty$, $v(t) \to 50$ from below
e $a(t)=5e^{-0.5t}$ and as $e^x>0$ for all x, $a(t)>0$ for all t.
f



g ≈ 134.5 m

8 **a** $v(t) = -\dfrac{1}{(t+1)^2}+1$ m s^{-1}

b $s(t)=\dfrac{1}{t+1}+t-1$ m

c The particle is $\frac{4}{3}$ m to the right of the origin, moving to the right at $\frac{8}{9}$ m s^{-1}, and accelerating at $\frac{2}{27}$ m s^{-2}.

9 **a** $v(t) = \dfrac{t^2}{20} - 3t + 45\ \text{m s}^{-1}$

b $\int_0^{60} v(t)\,dt = 900.$ The train travels a total of 900 m in the first 60 seconds.

10 **a** Show that $v(t) = 100 - 80e^{-\frac{1}{20}t}\ \text{m s}^{-1}$ and as $t \to \infty$, $v(t) \to 100\ \text{m s}^{-1}$.

b 370.4 m

REVIEW SET 16A

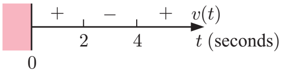
1 $A = \int_a^b [f(x) - g(x)]\,dx + \int_b^c [g(x) - f(x)]\,dx$
$+ \int_c^d [f(x) - g(x)]\,dx$

2 **a** $2 + \pi$ **b** -2 **c** π

3 No, total area shaded $= \int_{-1}^{1} f(x)\,dx - \int_1^3 f(x)\,dx.$

4 $k = \sqrt[3]{16}$ **5** 4.5 units^2

6 **a**



b The particle moves in the positive direction initially, then at $t = 2$, $6\frac{2}{3}$ m from its starting point, it changes direction. It changes direction again at $t = 4$, $5\frac{1}{3}$ m from its starting point, and at $t = 5$, it is $6\frac{2}{3}$ m from its starting point again.

c $6\frac{2}{3}$ m **d** $9\frac{1}{3}$ m

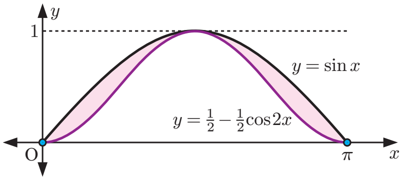
7 $(3 - \ln 4)$ units^2 **8** 2.35 m

REVIEW SET 16B

1 **a** $v(t) = 3t^2 - 30t + 27\ \text{cm s}^{-1}$

b -162 cm (162 cm to the left of the origin)

2 **a**



c $(1 - \frac{\pi}{4})$ units^2

3 $a = \ln 3$, $b = \ln 5$

4 **a** $a(t) = 2 - 6t\ \text{m s}^{-2}$ **b** $s(t) = t^2 - t^3 + c$ m

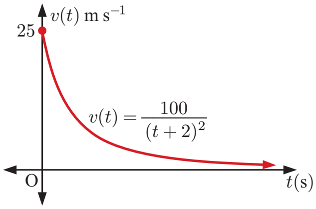
c -4 m (4 m to the left)

5 $k = \frac{4}{3}$ **6** $m = \frac{\pi}{3}$ **7** $(\frac{4}{\pi} - 1)$ units^2

8 **a** $v(0) = 25\ \text{m s}^{-1}$, $v(3) = 4\ \text{m s}^{-1}$

b as $t \to \infty$, $v(t) \to 0$ from above

c



d 3 seconds **e** $a(t) = \dfrac{-200}{(t+2)^3}\ \text{m s}^{-2},\ t \geqslant 0$

f $k = \frac{1}{5}$

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